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Supersymmetric theories at finite temperature

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New nonperturbative techniques make possible the calculation of thermodynamic properties of supersymmetric gauge theories. We apply them to supersymmetric QCD at large N_c, with a Chern–Simons term included to give mass to the adjoint partons. The theory is solved nonperturbatively by the technique of supersymmetric discrete light-cone quantization, which uses a discrete momentum grid in light-cone coordinates to convert integral equations for Fock-space wave functions to a supersymmetric matrix representation. The spectral distribution of the representation is computed by Lanczos iteration of the mass-squared eigenvalue problem. Thermodynamic quantities are then computed from an analytic fit to this spectrum.

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1. Introduction

In order to have a well-defined Fock state expansion for field-theoretic eigenstates, we use light-cone coordinates [1] defined by a choice of $x^+ = (t+z)/\sqrt{2}$ as the time direction. Spatial vectors are given by $\underline{x} = (x^-, \vec{x}_\perp)$, with $x^- \equiv (t-z)/\sqrt{2}$ and $\vec{x}_\perp = (x,y)$. The light-cone energy is $p^- = (E-p_z)/\sqrt{2}$ and the light-cone momentum is $\underline{p} = (p^+, \vec{p}_\perp)$ with, $p^+ \equiv (E+p_z)/\sqrt{2}$ and $\vec{p}_\perp = (p_x, p_y)$ The mass-shell condition $p^2 = m^2$ then yields $p^- = \frac{m^2+p_\perp^2}{2p^+}$.

The mass eigenvalue problem $2P^-P^+|\psi\rangle = M^2|\psi\rangle$ can be solved by expanding $|\psi\rangle$ in Fock states, with momentum wave functions as coefficients and diagonalizing a discrete approximation to the resulting coupled system of integral equations. A standard method for discretization is discrete light-cone quantization (DLCQ) [2]. The system is placed in a light-cone box $-L < x^- < L$, $-L_\perp < x,y < L_\perp$, and periodic boundary conditions are applied. This yields a discrete momentum grid, $p_i^+ \to \frac{\pi}{L} n_i$ and $\mathbf{p}_{i\perp} \to (\frac{\pi}{L_\perp} n_{ix}, \frac{\pi}{L_\perp} n_{iy})$, for integers n_i, n_{ix}, and n_{iy}. The integrals in the coupled equations are replaced by discrete sums. The continuum limit $L \to \infty$ is exchanged for a limit in terms of the integer *resolution* $K \equiv \frac{L}{\pi} P^+$ for fixed total momentum P^+.

For supersymmetric theories, this procedure does not automatically preserve supersymmetry. Instead [3], one must discretize the supercharge Q^- and construct P^- from the superalgebra relation $\{Q^-, Q^-\} = 2\sqrt{2}P^-$. This form of DLCQ is known as supersymmetric DLCQ (SDLCQ) [4]. The equivalence with ordinary DLCQ is recovered in the continuum limit.

Most SDLCQ calculations, including those discussed here, are done in the large-N_c limit. This simplifies the calculations by block diagonalizing the mass eigenvalue problem with respect to the number of adjoint strings in the Fock states. We focus on single trace states, which are simple glueballs $\mathrm{Tr}[a^\dagger_{i_1 i_2}(k_1)\ldots b^\dagger_{i_n i_{n+1}}(k_n)]|0\rangle$, and mesons as single strings

$$\bar{f}^\dagger_{i_1}(k_1) a^\dagger_{i_1 i_2}(k_2)\ldots b^\dagger_{i_n i_{n+1}}(k_{n-1})\ldots f^\dagger_{i_p}(k_n)|0\rangle, \tag{1.1}$$

where $\bar{f}^\dagger_i$ and $f^\dagger_i$ create fundamental partons and $a^\dagger_{ij}$ and $b^\dagger_{ij}$ create adjoint partons. Either of these states could be a boson or a fermion.

The finite temperature properties of the theory are computed [5] from the partition function $Z = e^{-p_0/T}$. One does not use the light-cone analog $e^{-p^-/T_{\mathrm{LC}}}$ because it does not correspond to a heat bath at rest. Each mass eigenstate contributes according to its ordinary energy p_0. For bosonic states of mass M_n in one dimension this yields a free-energy contribution of

$$F_B = \frac{VT}{\pi}\sum_{n=1}^{\infty}\int_{M_n}^{\infty} dp_0 \frac{p_0}{\sqrt{p_0^2 - M_n^2}} \ln\left(1 - e^{-p_0/T}\right) \tag{1.2}$$

in a volume V. For fermions we obtain

$$F_F = -\frac{VT}{\pi}\sum_{n=1}^{\infty}\int_{M_n}^{\infty} dp_0 \frac{p_0}{\sqrt{p_0^2 - M_n^2}} \ln\left(1 + e^{-p_0/T}\right). \tag{1.3}$$

In supersymmetric theories, the bosonic and fermionic mass spectra are the same, and we can readily combine these expressions to obtain the total free energy

$$F(T,V) = -(K-1)\pi V T^2 - \frac{2VT}{\pi}\sum_{n=1}^{\infty}\sum_{l=0}^{\infty} M_n \frac{K_1\left((2l+1)\frac{M_n}{T}\right)}{(2l+1)}. \tag{1.4}$$

Here we have expanded the logarithms, integrated over p_0, and included explicitly the contribution of zero-mass states. The sum over l is well approximated by the first few terms. The sum over n can be represented by an integral over a density of states $\int \rho(M)dM$. We then approximate ρ by a continuous function that is fit to the numerical spectrum and compute $\int dM$ by standard numerical techniques. The spectrum is estimated by a Lanczos technique discussed in Sec. 3. A similar approach to the calculation of thermodynamic properties is given by Strauss and Beyer [6], based on earlier work by Elser and Kalloniatis [7].

2. Supersymmetric QCD with a Chern–Simons term

We apply these techniques to $N = 1$ supersymmetric QCD with a Chern–Simons term [8]. The action is

$$S = \int d^3x \mathrm{Tr}\left\{ -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + D_\mu\xi^\dagger D^\mu\xi + i\bar{\Psi}D_\mu\Gamma^\mu\Psi - g\left[\bar{\Psi}\Lambda\xi + \xi^\dagger\bar{\Lambda}\Psi\right] \right. \\ \left. + \frac{i}{2}\bar{\Lambda}\Gamma^\mu D_\mu\Lambda + \frac{\kappa}{2}\varepsilon^{\mu\nu\lambda}\left[A_\mu\partial_\nu A_\lambda + \frac{2i}{3}gA_\mu A_\nu A_\lambda\right] + \kappa\bar{\Lambda}\Lambda \right\}. \tag{2.1}$$

The adjoint fields are the gauge bosons A_μ (gluons) and the Majorana fermion Λ (gluinos); the fundamental fields are the Dirac fermions Ψ (quarks) and the complex scalars ξ (squarks). The Chern–Simons coupling κ provides a mass for adjoint fields and thereby provides a natural limit on the lengths of strings. The covariant derivatives are defined by

$$D_\mu\Lambda = \partial_\mu\Lambda + ig[A_\mu,\Lambda], \quad D_\mu\xi = \partial_\mu\xi + igA_\mu\xi, \; D_\mu\Psi = \partial_\mu\Psi + igA_\mu\Psi. \tag{2.2}$$

The associated supersymmetry transformations are

$$\delta A_\mu = \frac{i}{2}\bar{\varepsilon}\Gamma_\mu\Lambda, \qquad \delta\Lambda = \frac{1}{4}F_{\mu\nu}\Gamma^{\mu\nu}\varepsilon, \qquad \delta\xi = \frac{i}{2}\bar{\varepsilon}\Psi, \qquad \delta\Psi = -\frac{1}{2}\Gamma^\mu\varepsilon D_\mu\xi. \tag{2.3}$$

The supercharge is

$$\bar{\varepsilon}Q = \int dx^- dx^2 \left(\frac{i}{4}\bar{\varepsilon}\Gamma^{\alpha\beta}\Gamma^+ \mathrm{tr}\left(\Lambda F_{\alpha\beta}\right) + \frac{i}{2}D_-\xi^\dagger\bar{\varepsilon}\Psi \right. \\ \left. + \frac{i}{2}\xi^\dagger\bar{\varepsilon}\Gamma^{+\nu}D_\nu\Psi - \frac{i}{2}\bar{\Psi}\varepsilon D^+\xi + \frac{i}{2}D_\nu\bar{\Psi}\Gamma^{+\nu}\varepsilon\xi \right). \tag{2.4}$$

Fermionic quantities are written in terms of components as

$$\Lambda = \left(\lambda, \tilde{\lambda}\right)^T, \qquad \Psi = (\psi, \tilde{\psi})^T, \qquad Q = \left(Q^+, Q^-\right)^T. \tag{2.5}$$

In light-cone gauge ($A^+ = 0$), the nondynamical fields satisfy the following constraints:

$$\partial_-\tilde{\lambda} = -\frac{ig}{\sqrt{2}}\left([A^2,\lambda] + i\xi\psi^\dagger - i\psi\xi^\dagger\right), \tag{2.6}$$

$$\partial_-\tilde{\psi} = -\frac{ig}{\sqrt{2}}A^2\psi + \frac{g}{\sqrt{2}}\lambda\xi - \kappa\lambda/\sqrt{2}, \; \partial_-^2 A^- = gJ, \tag{2.7}$$

with

$$J \equiv i[A^2, \partial_- A^2] + \frac{1}{\sqrt{2}}\{\lambda, \lambda\} + \kappa \partial_- A^2 - ih\partial_- \xi \xi^\dagger + i\xi \partial_- \xi^\dagger + \sqrt{2}\psi\psi^\dagger \,. \tag{2.8}$$

The (2+1)-dimensional theory is reduced to 1+1 dimensions by assuming fields to be independent of the transverse coordinate x^2. The reduced supercharge is

$$\begin{aligned} Q^- = g \int dx^- \Bigg\{ & 2^{3/4} \left(\mathrm{i}[A^2, \partial_- A^2] - \kappa \partial_- A^2 + \frac{1}{\sqrt{2}}\{\lambda, \lambda\} \right) \frac{1}{\partial_-} \lambda \\ & - \frac{1}{\sqrt{2}} \left(i\sqrt{2}\xi \partial_- \xi^\dagger - i\sqrt{2}\partial_- \xi \xi^\dagger + 2\psi\psi^\dagger \right) \frac{1}{\partial_-} \lambda \\ & - 2\left(\xi^\dagger A^2 \psi + \psi^\dagger A^2 \xi \right) \Bigg\} \,. \end{aligned} \tag{2.9}$$

The mode expansions for the dynamical fields are

$$A^2_{ij}(0, x^-) = \frac{1}{\sqrt{4\pi}} \sum_{k=1}^{\infty} \frac{1}{\sqrt{k}} \left(a_{ij}(k) e^{-ik\pi x^-/L} + a^\dagger_{ji}(k) e^{ik\pi x^-/L} \right) , \tag{2.10}$$

$$\lambda_{ij}(0, x^-) = \frac{1}{2^{\frac{1}{4}}\sqrt{2L}} \sum_{k=1}^{\infty} \left(b_{ij}(k) e^{-ik\pi x^-/L} + b^\dagger_{ji}(k) e^{ik\pi x^-/L} \right) , \tag{2.11}$$

$$\xi_i(0, x^-) = \frac{1}{\sqrt{4\pi}} \sum_{k=1}^{\infty} \frac{1}{\sqrt{k}} \left(c_i(k) e^{-ik\pi x^-/L} + \tilde{c}^\dagger_i(k) e^{ik\pi x^-/L} \right) , \tag{2.12}$$

$$\psi_i(0, x^-) = \frac{1}{2^{\frac{1}{4}}\sqrt{2L}} \sum_{k=1}^{\infty} \left(d_i(k) e^{-ik\pi x^-/L} + \tilde{d}^\dagger_i(k) e^{ik\pi x^-/L} \right) . \tag{2.13}$$

The creation and annihilation operators satisfy commutation relations, which for finite N_c are

$$\left[a_{ij}, a^\dagger_{kl} \right] = \left(\delta_{il}\delta_{kj} - \frac{1}{N_c}\delta_{ij}\delta_{kl} \right) , \ \left\{ b_{ij}, b^\dagger_{kl} \right\} = \left(\delta_{il}\delta_{kj} - \frac{1}{N_c}\delta_{ij}\delta_{kl} \right) , \tag{2.14}$$

$$\left[c_i, c^\dagger_j \right] = \delta_{ij} \,, \quad \left[\tilde{c}_i, \tilde{c}^\dagger_j \right] = \delta_{ij} \,, \ \left\{ d_i, d^\dagger_j \right\} = \delta_{ij} \quad \left\{ \tilde{d}_i, \tilde{d}^\dagger_j \right\} = \delta_{ij} \,. \tag{2.15}$$

In addition to supersymmetry there is a Z_2 symmetry [9] $a_{ij}(k, n^\perp) \to -a_{ji}(k, n^\perp)$, $b_{ij}(k, n^\perp) \to -b_{ji}(k, n^\perp)$. This further block-diagonalizes the Hamiltonian by dividing states between even and odd numbers of gluons.

3. Lanczos algorithm for density of states

To compute the free energy, we need the density of states for the mass spectrum of the theory, which is generally given by $\rho(M^2) = \sum_n d_n \delta(M^2 - M_n^2)$ where d_n is the degeneracy of the mass eigenvalue M_n. It is the derivative of the cumulative distribution function (CDF) $N(M^2) = \int^{M^2} dM^2 \rho(M^2)$.

The density can be written in the form of a trace over $e^{-iP^-x^+}$

$$\begin{aligned} \rho(M^2) &= \frac{1}{2P^+} \sum_n d_n \delta(M^2/2P^+ - P^-_n) \qquad (3.1) \\ &= \frac{1}{4\pi P^+} \int_{-\infty}^{\infty} e^{iM^2 x^+/2P^+} \sum_n d_n e^{-iP^-_n x^+} dx^+ \\ &= \frac{1}{4\pi P^+} \int_{-\infty}^{\infty} e^{iM^2 x^+/2P^+} \mathrm{Tr} e^{-iP^- x^+} dx^+ . \qquad (3.2) \end{aligned}$$

The trace can be estimated by an average over a random sample of vectors [10]. Define a local density for a single vector $|s\rangle$ as

$$\rho_s(M^2) = \frac{1}{4\pi P^+}\int_{-\infty}^{\infty} e^{iM^2x^+/2P^+}\langle s|e^{-iP^-x^+}|s\rangle dx^+, \tag{3.3}$$

so that the average can be written

$$\rho(M^2) \simeq \frac{1}{S}\sum_{s=1}^{S}\rho_s(M^2). \tag{3.4}$$

The sample eigenstates $|s\rangle$ can be chosen as random phase vectors [11], where the coefficient of each Fock state in the basis is a random number of modulus one.

The matrix element $\langle s|e^{-iP^-x^+}|s\rangle$ is estimated by Lanczos iteration [12]. Let D be the length of $|s\rangle$, and define $|u_1\rangle = \frac{1}{\sqrt{D}}|s\rangle$ as the initial Lanczos vector. Then

$$\rho_s(M^2) = \frac{D}{4\pi P^+}\int e^{iM^2x^+/2P^+}\langle u_1|e^{-iP^-x^+}|u_1\rangle dx^+, \tag{3.5}$$

and $\langle u_1|e^{-iP^-x^+}|u_1\rangle$ can be approximated by the $(1,1)$ element of the exponentiation of the Lanczos tridiagonalization of P^-.

To construct the exponentiation, let P_s^- be the tridiagonal Lanczos matrix and solve

$$P_s^-\vec{c}_n^s = \frac{M_{sn}^2}{2P^+}\vec{c}_n^s. \tag{3.6}$$

A diagonal matrix $\Lambda_{ij} = \delta_{ij}\frac{M_{sn}^2}{2P^+}$ is related by the usual similarity transformation $P_s^- = U\Lambda U^{-1}$, where $U_{ij} = (c_j^s)_i$. The $(1,1)$ element is given by

$$\left(e^{-iP_s^-x^+}\right)_{11} = \sum_n |(c_n^s)_1|^2 e^{-iM_{sn}^2x^+/2P^+}. \tag{3.7}$$

We note that $w_{sn} \equiv D|(c_n^s)_1|^2$ is the weight of each Lanczos eigenvalue in the spectrum, and the local density is

$$\begin{aligned}\rho_s(M^2) &\simeq \frac{D}{4\pi P^+}\int e^{iM^2x^+/2P^+}\sum_n |(c_n^s)_1|^2 e^{-iM_{sn}^2x^+/2P^+}dx^+ \\ &\simeq \frac{D}{4\pi P^+}\sum_n |(c_n^s)_1|^2 2\pi\delta(M^2/2P^+ - M_{sn}^2/2P^+) \simeq \sum_n w_{sn}\delta(M^2 - M_{sn}^2),\end{aligned} \tag{3.8}$$

Although only the extreme Lanczos eigenvalues are good approximations to eigenvalues of the original P^-, the other Lanczos eigenvalues provide a smeared representation of the full spectrum.

The contribution to the CDF is $N_s(M^2) \equiv \int^{M^2} dM^2\rho(M^2) \simeq \sum_n w_{sn}\theta(M^2 - M_{sn}^2)$. The full CDF is then approximated by the average $N(M^2) \simeq \frac{1}{S}\sum_s N_s(M^2)$. The theta functions are averaged, starting with the first sample as a template for the values M_{1n}^2 at which to evaluate N. The contributions to N of the other samples are estimated at these values by linear interpolation in cases where the Lanczos eigenvalues M_{sn}^2 are not the same as those in the first set. In cases where duplicate eigenvalues are generated by the Lanczos iterations, only one is included in the template, and the associated weights are added together.

The convergence of the approximation is dependent on the number of Lanczos iterations per sample, as well as the number S of samples. Comparisons with explicit diagonalizations showed that the recommended value [10] of 20 samples is sufficient. The number of Lanczos iterations needs to be on the order of 1000 per sample; using only 100 leaves errors on the order of 1-2%.

A sample of results obtained for the free energy are given in Figs. 1 and 2. Additional results can be found in [5].

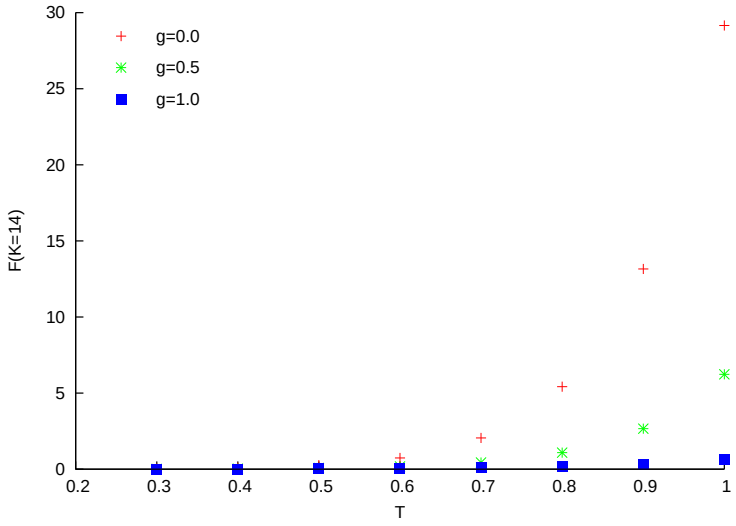

Figure 1: Free energy at fixed Yang–Mills coupling g as a function of temperature T for resolution $K = 14$.

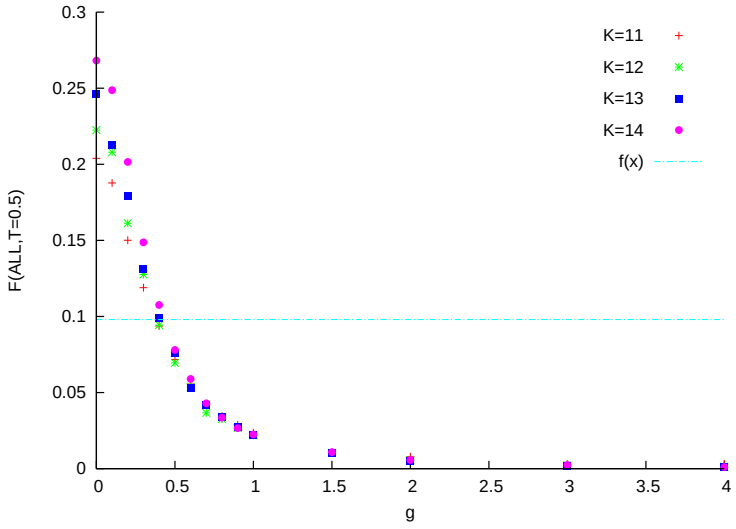

Figure 2: Free energy at fixed temperature $T = 0.5\kappa$ as a function of the Yang–Mills coupling g for a sequence of resolutions K.

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