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Nonperturbative Causal Light Cone Field Theory

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We discuss the implementation of Light Cone Field Theory (LCFT) in the causal Bogoliubov-Epstein-Glaser finite S-matrix formulation. The benefits result from the simple vacuum structure of LCFT which make possible calculations of the S-matrix in a perturbative-like manner even in the presence of a nontrivial, nonperturbative vacuum.The interdependence of particle and vacuum properties leads to an iterative scheme for the determination of the vacuum part of the Hamiltonian.

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1. Introduction

The Bogoliubov-Epstein-Glaser Causal Field Theory (BEG-CFT) [1 − 7] is a finite field theory free of divergences at any level of the theory. It has been reviewed in a talk at this conference by Andreas Aste. In conventional, equal time formulation, by construction, CFT is limited to perturbative situations (trivial vacuum). In this contribution we show that such limitations do not exist in Causal Light-Front Field Theory (CLFFT). Finiteness is particularly important in the nonperturbative case where regularization via cutoff is problematic. The reason for the applicability of CLFFT in the nonperturbative domain is the trivial structure of the vacuum-state vector. The signature of nontriviality is the appearance of a vacuum operator structure. This implies good and bad features:

The good news: *the action of Fock-space operators on the vacuum is well defined: the calculation of the S-matrix elements can be performed with perturbative techniques.*

The bad news: *the vacuum sector of the field operators, defined by the projection on the space independent part, is a priori not known. Therefore the Hamiltonian is a priori not known. It must be derived iteratively from constraints and equations of motion which allow to determine the* ***vacuum fields****. Their dynamics is related to correlation functions. They are decisive for critical phenomena.*

Particle fields , defined as the complement of the vacuum part, are determined by the S-matrix. Vacuum fields enter via interaction diagrams into this calculation (medium effect of the vacuum). The induced change of the particle fields acts back on input data for the determination of the vacuum sector fields and so on...This leads, ideally, at the end to a selfconsistent determination of the vacuum field and therefore of the Hamiltonian.

2. Causality Issues

The starting point is the expression for the S-matrix written in the Minkowski-frame, with the necessary caveats concerning distribution splitting

$$S_M = 1 + \sum_{n=1}^{n=\infty} \frac{1}{n!} \int d^4x_1 d^4x_2 \ldots\ldots\ldots d^4x_n g(x_1)\ldots g(x_n) T_n(x_1\ldots\ldots x_n).$$

$$T_n(x_1,..,x_n) = H^{int}(x_1)\Theta(x_1^0 - x_2^0)H^{int}(x_2)\ldots.\Theta(x_{n-1}^0 - x_n^0)H^{int}(x_n).$$

Since the field operators contained in H^{int} are operator valued distributions one has to test the S-matrix with Schwartz space test functions $g(x_i)$ which in addition switch off the interaction at infinity. On the light cone the causal structure is the same as in the Minkowski case: namely for two events at times x_1^0 and x_2^0, separated by a light-like distance, one has: $\Theta(x_2^0 - x_1^0) = \Theta(x_2^+ - x_1^+)$ (where $x_i^+ = x_i^0 + x_i^3$ are the corresponding light cone times) and events with space-like separations can not be connected by causal propagators. Therefore the transition to the light cone $S_M \Rightarrow S_{LC}$ is accomplished by the replacement $x^0 \to x^+$. The building blocks for the construction of the S-matrix are *causal propagators which are defined as commutators.* As an example we show the elementary causal propagator D(x-y) for a scalar field decomposed into positive and negative frequency parts:

$$\begin{aligned} \Phi(x) &= \Phi^{(-)}(x) + \Phi^{+}(x), \\ D(x-y) &= i\left[\Phi^{(-)}(x), \Phi^{(+)}(y)\right] + i\left[\Phi^{(+)}(x), \Phi^{(-)}(y)\right] = D^{(-)}(x-y) + D^{(+)}(x-y). \end{aligned}$$

Contractions are defined by:

$$i\overbrace{\Phi(x)\Phi(y)} = D^{(-)}(x-y) = -D^{(+)}(x-y),$$

with momentum space representations

$$D^{(\pm)}(p) = \pm i\delta(p^2-m^2)\Theta(\pm p^-)$$

Technical remark 1: $D^{(+)}$and $D^{(-)}$ are not causal separately!!; only the sum is causal.
Technical remark 2: For c-number fields - present in vacuum fields - contractions are defined by Poisson brackets of fields and their conjugate momenta. One gets $\left\{\phi^{(-)}(\overrightarrow{k}), \phi^{(+)}(\overrightarrow{k'})\right\} = iN(\overrightarrow{k})\delta(\overrightarrow{k}-\overrightarrow{k'})$, where $N(\overrightarrow{k})$ is the mean density of particles with momentum $\overrightarrow{k}$ and energy $k^-(\overrightarrow{k})$. The c-number propagator becomes $D_{cl}^{(\pm)}(k) = \pm i\delta(k^2-m^2)N(\overrightarrow{k})\Theta(\pm k^-)$.
The causality condition allows to calculate T-matrix elements iteratively starting from the first order term $T_1(x) =: H^{int}(x)$: up to order n [1]; for a time ordering $x_1^0 > x_2^0 > ......x_n^0$ one gets $T_n(x_1, x_2, ..., x_n) = T_1(x_1)T_1(x_2).......T_1(x_n)$.

3. Construction of the total field

The total field is decomposed as $\Psi(x) = \Phi(x) + \Omega(x)$; here $\Phi(x)$ is the total operator valued field which is obtained as a sum of normal ordered products of free field operators $\varphi_1(x)$ via the Haag series [8]. It contains particle sector parts - obtainable through multiple action of H^{int} on $\varphi_1(x)$ - and nonperturbative vacuum sector parts. $\Omega(x)$ is the c-number part of the nonperturbative vacuum field containing static and dynamic zero modes. It is not accessible via perturbation theory. Classical c-number fields are essential for the description of phase transitions and fluctuations of order parameters near critical points. $\Omega(x)$ represents a medium effect for the propagation of perturbative fields. $\Omega(x)$ and the nonperturbative part of $\Phi(x)$ have to be determined from vacuum matrix elements of constraints and equations of motion.

3.1 Formal construction via Haag series in terms of free fields

An expansion of $\Phi(x)$ is performed in terms of products of free field $\varphi_1(x) = \varphi_1^{(+)}(x) + \varphi_1^{(-)}(x)$. Denoting by $\varphi_n(x)$ with n=1,2,3.. the contribution involving n free field we have

$$\Phi(x) = \varphi_1(x) + \varphi_2(x) + \varphi_3(x) + ... = \varphi_1(x) + \int g_2(x_1-x, x_2-x, x) : \varphi_1(x_1)\varphi_1(x_2) : d^dx_1dx_2^d + \int g_3(x_1-x, x_2-x, x_3-x, x) : \varphi_1(x_1)\varphi_1(x_2)\varphi_1(x_3) : dx_1^d dx_2^d dx_3^d + ...$$

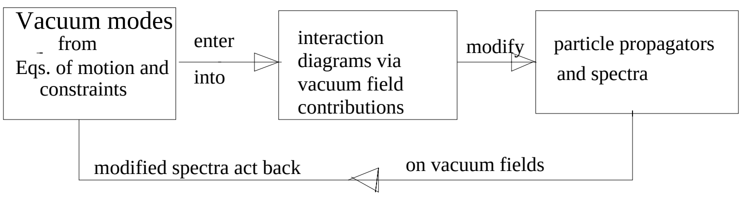
Since each factor φ_i is a sum of a positive and negative frequency part, all kinds of products of creation and annihilation operators appear in the normal products. Originally, the normal order prescription served to ensure that the vacuum expectation value of $\Phi(x)$ vanishes, if the vacuum is perturbative. In the Minkowski case, in a nonperturbative situation with a nontrivial vacuum, this prescription becomes ineffective. In the light cone case one can maintain the normal order

[1] The symbol :A: means normal order of the operator A

prescription for the above definition of selfinteracting fields; however in models with different types of interacting fields products of these fields will appear in the Haag series; in this case a normal order prescription would kill all interaction effects in matrix elements of vacuum to vacuum or vacuum to particle state types. As we shall discuss in the examples below the Weyl order prescription is more appropriate to construct bosonic and fermionic fields.

3.2 Coupling of perturbative and vacuum fields

The interaction $H^{int}(x)$ couples $\Phi(x)$ and $\Omega(x)$. As a result interaction diagrams for $\Phi(x)$ contain as internal lines contributions from $\Omega(x)$. The result is a change of spectrum in Φ-sector due to vacuum contributions. In turn again there is a change of constraints and equations of motion due to modified propagators. In turn a modification of Ω follows due to modified constraints and equations of motion - and so on. Hence the following iterative scheme:



Schematic scenario for interdependence of vacuum and particle modes

4. Example Φ^4_{1+1}

The Lagrangian L and equation of motion (EQM) are

$$L = \frac{1}{2}\partial^+\Phi(x)\partial_-\Phi(x) - \frac{1}{2}m^2\Phi^2(x) - \frac{g}{4!}\Phi^4(x); \quad (\partial^+\partial^- + m^2)\Phi(x) + \frac{g}{4!}\Phi^3(x) = 0.$$

In addition, as a consequence of the singular nature of the LC-Lagrangian, there is a constraint. It is identical to the projection of the EQM on the vacuum sector:

$$\Theta_3 = m^2\Phi(x) + \frac{g}{3!}\Phi^3(x) = 0.$$

For the construction of the field operator we use the Haag series up to φ_2:
$\Psi^3 = (\varphi_1+\varphi_2+\Omega)^3 = \varphi_1^3 + \underline{\varphi_1^2(\varphi_2+\Omega)+\varphi_1(\varphi_2+\Omega)\varphi_1+(\varphi_2+\Omega)\varphi_1^2}$ + terms of higher order in φ_2. The underlined terms are non-perturbative and generate vacuum zero modes - but only if the Weyl ordering is used. The nonperturbative fields $\Omega(x)$, $\varphi_2(x)$, $\varphi_4(x)$,.... are non zero only if the order parameter of the broken phase $<\Psi(x)>= \Phi_0 \neq 0$ i.e. if the coupling strength exceeds a critical value $g > g_{cr}$. The nonperturbative field $\varphi_2(x)$ has the following explicit form [9]:

$$\begin{aligned}\varphi_2(x) \;=\; & \frac{1}{4\pi}\int\int \frac{\Theta(k_1^+)\Theta(k_2^+)}{\sqrt{k_1^+k_2^+}} f(k_1^+)f(k_2^+)\{G_2^{+-}(k_1^+,-k_2^+;x)\\ & a^+(k_1^+)a(k_2^+)e^{\frac{i}{2}(k_1^+-k_2^+)x} + \text{terms for } G_2^{++} \text{and } G_2^{--}\}dk_1^+dk_2^+.\end{aligned}$$

The testfunctions $f(k_i^+)$ come in via the Fock expansion of the free field where they are required to give a meaning to the operator valued distribution $\varphi_1(x)$. The dependence of G_2 on x is induced by a nonperturbative, x-dependent back-ground field $\Omega(x) = \widetilde{\Omega(k)} e^{\frac{i}{2}kx}$ [9].

4.1 Equations for $\Omega(x)$ and $\varphi_2(x)$.

Equations of motion and the constraint Θ_3 yield equations for the unknown amplitudes Ω, G_2^{+-}, G_2^{--} (constrained and dynamical zero modes). They are obtained [9] by taking appropriate matrix elements of the equations of motion according to:

$< 0|EQM|0 > - > \Omega$, $<q_1|EQM|q_1 > - > G_2^{+-}, < q_1 q_2|EQM|0 > - > G_2^{++}$

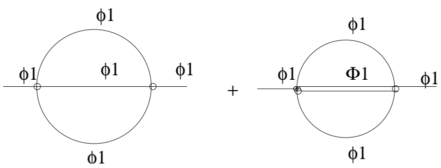
Due to the interaction term Φ^3 this is a system of nonlinear equations. In the long wavelength limit ($k^+ - > 0$) the amplitudes become small and the equations can be linearized. They are given explicitely in [9]. Their symbolic structures are:

$$\begin{aligned}
\Delta_\Phi^{-1} + K_\Phi \otimes \left[G_2^{+-} \oplus G_2^{++}\right] &= 0 \quad (A),\\
\Delta^{-1}_{\varphi_2^{+-}} G_2^{+-} + K_{\varphi_2^{+-}} \otimes \left[G_2^{+-} \oplus G_2^{++}\right] &= g\Omega \quad (B),\\
\Delta^{-1}_{\varphi_2^{++}} G_2^{++} + K_{\varphi_2^{++}} \otimes \left[G_2^{+-} \oplus G_2^{++}\right] &= g\Omega \quad (C).
\end{aligned}$$

The $\Delta_\Phi^{-1}, \Delta^{-1}_{\varphi_2^{+-}}, \Delta^{-1}_{\varphi_2^{++}}$ are inverse propagators for the fields Ω, G_2^{+-}, G_2^{++}. The $K_\Phi, K_{\varphi_2^{+-}}, K_{\varphi_2^{++}}$ are integral interaction kernels. The strategy for the solution is: 1) Solve (B) and (C) in terms of Ω. 2) Insert solution of (B) and (C) in (A). The result is a homogenous equation for Ω i.e. a dispersion relation for the c-number field Ω from which the critical coupling and the β-function can be extracted.

4.2 Vacuum field contributions to interaction diagrams

Here we give an example for the action of the lowest order contributions of the nonperturbative fields Ω and φ_2 in selfenergy diagrams of order g^2; in the diagram we denote these fields generically by Φ_1.



Manifestation of nonperturbative vacuum effects in interaction diagrams.

In the perturbative diagram each vertex is made of 4 field operators like $\varphi_1^{(-)}\varphi_1^{(+)}\varphi_1^{(+)}\varphi_1^{(+)}$, where 3 of them are contracted pairwise between 2 vertices. In the corresponding nonperturbative diagram the vertex factor is $\varphi_1^{(-)}\varphi_1^{(+)}\Phi_1^{(+)}\varphi_1^{(+)}$ where $\Phi_1^{(+)}$ can either come from Ω or from the nonpertutbative part of φ_2. Near the phase transition the nonperturbative fields are small. Therefore diagrams with two Φ_1 - lines are omitted. It is clearly seen that the calculations, though being non-perturbative, use perturbative techniques, the nonperturbative element residing in the construction of Ω.

5. Example with bosonic and fermionic fields

We consider the Lagrangian of the Yukawa model

$$\mathscr{L} = \overline{\psi}(i\gamma_\mu \partial^\mu - m)\psi + 2\partial_-\Phi\partial_+\Phi - \frac{1}{2}\partial_\perp^2\Phi - \frac{\mu^2}{2}\Phi^2 - g\Phi\overline{\psi}\psi$$

In this case the fermionic fields contribute to nonperturbative vacuum fields in the form of composite fields built from fermion fields coupled to spinless scalars. As usual, if the fermion field is written in terms of the upper ψ_1 and lower ψ_2 component, the EQM splits into two coupled equations: the first one is a genuine equation of motion, where the light cone time derivative of ψ_1 is coupled to ψ_2; the second expresses the longitudinal space derivative of ψ_2 through ψ_1, which means that ψ_2 is a dependent quantity. Conventionally ψ_1 and ψ_2 are called independent and dependent fields respectively. The Lagrangian being singular one has to go through the Dirac-Bergmann algorithm [10] to find all the constraints which are all related to the constrained nature of bosonic and fermionic momenta. Here we list the canonical momenta and the corresponding constraints [2]:
$\pi_{\psi_1} = 0$; $\Theta_1 = \pi_{\psi_1} \simeq 0$; $\pi_{\psi_1^+} = 0$; $\Theta_2 = \pi_{\psi_1^+} \simeq 0$; $\pi_{\psi_2} = -i\psi_2^+$; $\Theta_3 = \pi_{\psi_2} \simeq -i\psi_2^+ \simeq 0$;
$\pi_{\psi_2^+} = -i\psi_2$; $\Theta_4 = \pi_{\psi_2^+} \simeq -i\psi_2 \simeq 0$; $\pi_\Phi = 2\partial_-\Phi$; $\Theta_5 = \pi_\Phi - 2\partial_-\Phi \simeq 0$
The vacuum field $\Omega(x)$ is composed of a bosonic field Φ_{vac} and a fermionic contribution $(\overline{\psi}\psi)_{vac}$ and eventually more complicated combinations of them. The equations of motion for the fermions and bosons are respectively

$$(i\gamma_\mu \partial^\mu - m)\psi = g\Phi\psi \qquad (i\gamma_\mu \partial^\mu + m)\overline{\psi} = g\Phi\overline{\psi}$$

$$4\partial_+\partial_-\Phi - \partial_\perp^2\Phi + \mu^2\Phi - g\overline{\psi}\psi = 0$$

with solutions (inspired by the Haag series):

$$\psi = gG_{\psi^0}\Phi\psi + \psi^0 \qquad G_{\psi^0} = \frac{1}{i\gamma_\mu \partial^\mu - m}.$$

$$\overline{\psi} = gG_{\overline{\psi^0}}\Phi\overline{\psi} + \overline{\psi}^0 \qquad G_{\overline{\psi^0}} = \frac{1}{i\gamma_\mu \partial^\mu + m}.$$

$$\Phi = gG_{\varphi_1}\overline{\psi}\psi + \varphi_1 + \Omega \qquad G_{\varphi_1} = \frac{1}{4\partial_+\partial_- - \partial_\perp^2 + \mu^2}$$

In the last equation φ_1 is the free bosonic field (the first term in the Haag series; see section 3.1). The free fermion field is denoted by $\psi_i^0, i = 1,2$. The zero mode Ω is obtained by projection of the EQM for Φ on the vacuum sector:

$$\mu^2\Omega = g(\overline{\psi}\psi)_{vac}, \quad i.e. \quad \mu^2\Omega_0 = g < 0|\overline{\psi}\psi|0 >$$

Ω_0 is given entirely in terms of $< 0|\overline{\psi}\psi|0 >$. In lowest order one finds

$$< 0|\overline{\psi}\psi|0 >= -\frac{1}{4\pi^3} m \int \frac{dp^3}{p_+} f^2(p_+, \overrightarrow{p}_\perp)$$

[2]For the meaning of the weak equality $\simeq$ see [10]

In the iterative solution of these equations the Weyl ordering of products of operators must be used in order to keep track of the effects of the zero modes. Iteration of the equations of motion changes the functional dependence of Ω_0 on $<0|\overline{\psi}\psi|0>$ and the value of $<0|\overline{\psi}\psi|0>$ itself, but the fact that the bosonic zero mode is determined by the fermionic condensate remains valid.

Things would be different in the Yukawa-Higgs model where the bosonic mexican hat potential yields as a first approximation an independent fixing of the bosonic zero mode Ω_0. The coupling term $g\Omega_0\overline{\psi}\psi$ acts then as a mass term (mass generation in the standard model via Higgs mechanism).

6. Concluding remarks

The combination of the description of the signature of nontrivial vacua by zero modes of the fields with the techniques of the causal approach has highly nontrivial and remarkable consequences:

1. The inclusion of nonperturbative vacuum effects in the calculation of the S-matrix.
2. The possibility to construct a mathematically sound theory which avoids all divergences related to the use of ill-defined distributions in the standard approach.

These two properties qualify the Causal Light Front Field Theory as an ideal candidate for studying nonperturbative physics

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