

# Infrared behavior of QCD from the Dyson-Schwinger formalism

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We discuss the properties of two different types of infrared solutions of Landau gauge Yang-Mills theory and argue for one of these (the 'scaling solution'). We furthermore clarify the status of previously obtained results from DSEs on a four-torus. Including quarks we discuss a relation between confinement and dynamical chiral symmetry breaking based on the scaling solution of Yang-Mills theory. An infrared singularity in the quark-gluon vertex allows for a solution of the  $U_A(1)$  problem along the lines of a mechanism suggested by Kogut and Susskind long ago.

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## 1. Introduction

There are (at least) two reasons why the Green's functions of gauge fixed QCD are interesting objects to study. On the one hand they are related to fundamental properties of the theory like confinement and dynamical chiral symmetry breaking. On the other hand they serve as input for calculations of observable quantities like dynamical properties of bound states, as determined e.g. in the framework of Bethe-Salpeter and Faddeev type of equations.

In this talk we are mainly concerned with the first issue. In the framework of covariantly gauge fixed QCD, Kugo and Ojima [1] have developed a confinement scenario that rests on well-defined charges related to unbroken global gauge symmetries. In this framework BRST-symmetry has been used to identify the positive definite space  $\mathcal{H}_{phys}$  of physical states within the total state space  $\mathcal{V}$  of QCD. An unbroken global gauge symmetry is then crucial to show that the states in  $\mathcal{H}_{phys}$  contributing to the physical S-matrix of QCD are indeed colorless. They also argued that this setup guarantees the disappearance of the 'behind-the-moon' problem, i.e. a colorless bound state with colored constituents cannot be delocalized into colored lumps [1].

The well-definedness of global gauge symmetry has been related to the infrared behavior of the propagators of Landau gauge QCD in [1]: Global gauge symmetry is unbroken if in the infrared the ghost propagator is more divergent and the gluon propagator less divergent than a simple pole. For the gluon propagator this means that it is probably at most constant or even vanishing in the infrared. In terms of the dressing functions  $G(p^2)$  and  $Z(p^2)$  of the ghost and gluon propagators

$$D_G(p) = -\frac{G(p^2)}{p^2}, \quad D_{\mu\nu}(p) = \left( \delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right) D(p^2) = \left( \delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right) \frac{Z(p^2)}{p^2}, \quad (1.1)$$

and in terms of a power-law expansion this condition reads

$$Z(p^2) \sim (p^2)^{-\kappa_A}; \quad G(p^2) \sim (p^2)^{-\kappa_C} \quad (1.2)$$

with exponents  $\kappa_A \leq -1$  and  $\kappa_C > 0$ .

Nonperturbative information on the ghost and gluon propagators can be obtained by Dyson-Schwinger equations (DSEs) [2] or functional renormalization group equations (FRGs) [3] in the continuum field theory, or from lattice QCD at finite volume and lattice spacing. In the following we first discuss the two possible types of numerical solutions (named 'scaling' and 'decoupling') in the infinite volume/continuum limit from DSEs. Then we report on various comparisons between solutions from DSEs on a torus and results from lattice QCD in section 3. In the last section we shortly discuss a particular pattern of dynamical chiral symmetry breaking related to the scaling type of behavior of the Yang-Mills sector of QCD.

## 2. Infrared Yang-Mills theory from DSEs

The infrared behavior of the one-particle irreducible (1PI) Green's functions of Yang-Mills theory have been investigated in a number of works. The basic relation  $\kappa_A = -2\kappa_C$  between the dressing functions (1.2) of the gluon and ghost propagator has been extracted in [4, 5] from DSEs. Corresponding results from FRGs have been obtained in [6]. These findings have been generalized to Green's functions with an arbitrary number of legs in [7]. The analysis rests upon a separation

of scales, which takes place in the deep infrared momentum region. Provided there is only one external momentum  $p \ll \Lambda_{\text{QCD}}$  much smaller than  $\Lambda_{\text{QCD}}$  a self-consistent infrared asymptotic solution of the whole tower of Dyson-Schwinger equations for these functions is given by

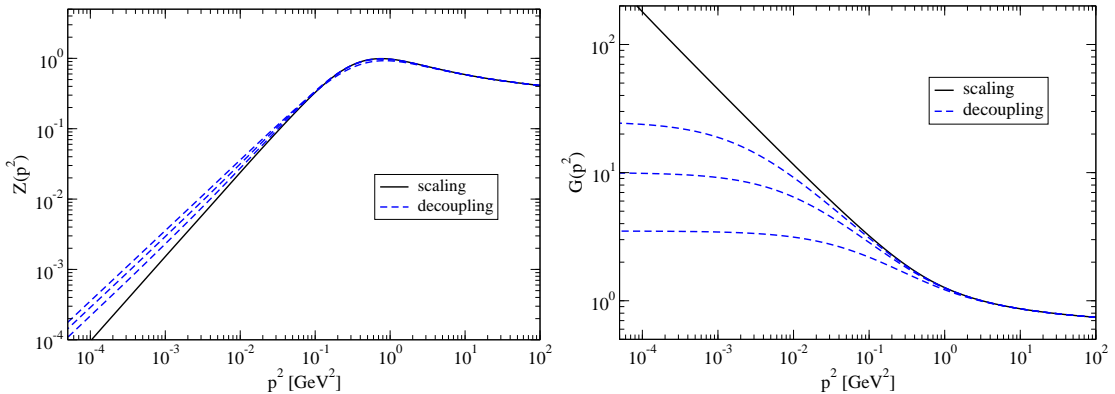
$$\Gamma^{n,m}(p^2) \sim (p^2)^{(n-m)\kappa}. \quad (2.1)$$

Here  $\Gamma^{n,m}(p^2)$  denotes the dressing function of the infrared leading tensor structure of the 1PI-Green's function with  $2n$  external ghost legs and  $m$  external gluon legs. This solution agrees with the Slavnov-Taylor identities and is the unique scaling solution in the infrared [8]. Here 'scaling' denotes the fact that *all* Green's functions obey nontrivial power laws in the infrared with an anomalous dimension  $\kappa > 0$  [9]. For the ghost and gluon dressing functions (1.2) this scaling type of solution yields the abovementioned power law  $\kappa = \kappa_C = -\kappa_A/2$ .

The absence of scaling implies the decoupling of (some) degrees of freedom. A solution of this type has been discussed e.g. in [10, 11, 12] and is given by  $\kappa_C = 0$  and  $\kappa_A = -1$ . We refer to this type of solution as the 'decoupling solution'.

Both types of infrared behavior can also be obtained as numerical solutions for the coupled systems of ghost and gluon DSEs. In [5, 11] the infrared boundary condition  $G(0)$ , i.e. the value of the ghost dressing function at zero momentum, has been identified as a parameter that allows to switch between these two types. Clearly,  $G(0)^{-1} = 0$  corresponds to an infrared diverging ghost dressing function implementing the scaling solution, whereas  $G(0) = \text{const.}$  produces an infrared finite ghost by construction. The gluon propagator is then either massive in the sense that  $D(0) = \lim_{p^2 \rightarrow 0} Z(p^2)/p^2 = \text{const.}$  for decoupling, or has the power like behavior (1.2) with  $\kappa = \kappa_C = (93 - \sqrt{1201})/98 \approx 0.595353$  [5] in the case of scaling. The corresponding numerical solutions of the coupled ghost and gluon DSEs have been determined in [13] and are shown in fig. 1.

The decoupling type of solution contains an arbitrary and unfixed parameter: the value of the ghost at zero momentum and correspondingly the finite value  $D(0)$  of the gluon. If the gluon were a massive, physical particle this value could be fixed from experiment. However, even for decoupling



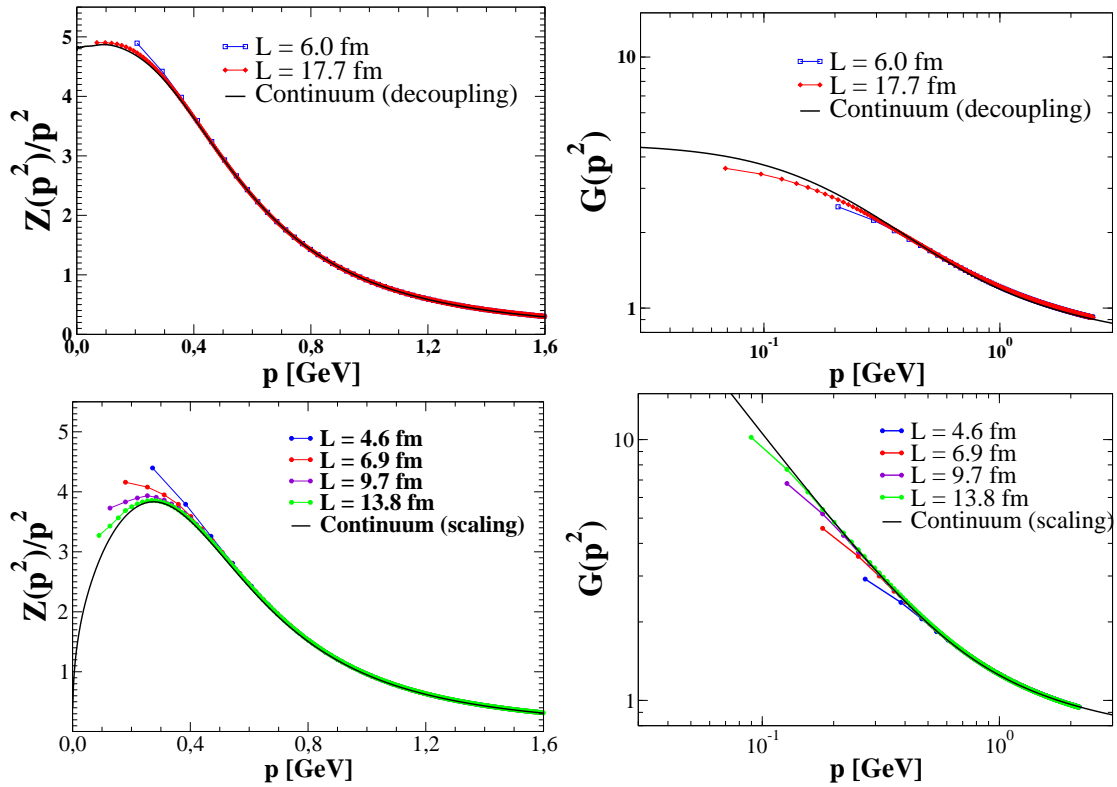
**Figure 1:** Numerical solutions for the ghost and gluon dressing function with two different boundary conditions  $G(0)$ . The results displayed here are obtained within the truncation scheme introduced in [13]. Differences to the scheme defined in [14] are, however, only very small and would not be visible in the plots.

the gluon is *not* massive in this sense [13] and it is therefore hard to see how  $D(0)$  could be fixed unambiguously. This problem is absent for the scaling type of solutions.

Although both types of solutions can be obtained from the DSEs, their status is certainly different. From the discussion in the introduction we note that only the scaling type of solutions agrees with the Kugo-Ojima scenario in the sense that it corresponds to an unbroken global gauge symmetry. On the other hand, a broken global gauge symmetry is a clear signal for a system in the Higgs phase. We are therefore led to the conclusion that the scaling solution represents the confined phase of Yang-Mills theory, whereas the decoupling type of solutions represents something like a Higgs phase. These arguments and additional ones related to the breaking of BRST symmetry in the decoupling case are discussed in detail in [13].

### 3. DSEs on a torus: finite volume effects

In general there are some caveats in comparing results from the continuum Dyson-Schwinger approach to those of lattice calculations (see [15] and refs. therein). The quantitative aspects of the continuum solutions depend on the details of the chosen truncation scheme, whereas the lattice calculations are *ab initio*. Gauge fixing is different in the two approaches and the effects of Gribov copies have to be taken into account. Furthermore, lattice calculations are carried out on a compact manifold, and therefore one has to deal with effects due to finite volume and lattice spacing.



**Figure 2:** Numerical solutions for the ghost and gluon dressing function in the continuum and on tori with different volumes. In the upper panel we display solutions of the decoupling type, whereas on the lower panel scaling solutions are shown.

To quantify the 'plain' volume effects (i.e. those not connected to the gauge fixing procedure) we formulated the DSEs on a torus without changing the truncation scheme. Scaling solutions on a torus have been found in [16], whereas solutions of decoupling type have been produced in [17].

In general, one would expect to see differences to the corresponding continuum solutions for small volumes, which disappear continuously when the volume is chosen larger and larger. This is indeed the case as shown in fig. 2. For both types of solutions we obtain a smooth infinite volume/continuum limit as the volumes are increased<sup>1</sup>.

It is apparent from the results of fig. 2 that volumes of  $V = (15\text{fm})^4$  and more are necessary to observe signals of the infinite volume/continuum behavior of the dressing functions also on a torus. As discussed in detail in [16] the technical reason for this is that one needs a range of momenta  $p$  with  $\frac{\pi}{L} \ll p \ll \Lambda_{QCD}$  to observe this behavior; these three scales need to be widely separated. In addition it is worth noting that the infrared behavior of the Green's functions does not reflect dynamical properties of the theory. These play a role at momenta of the order of or larger than  $\Lambda_{QCD}$  and are not plagued by volume effects of this magnitude. Scaling or decoupling on the other hand are phenomena that occur due to the absence of dynamics in the deep infrared momentum region. They are characteristic of the global properties of the theory as e.g. the conservation or breaking of global gauge symmetries. Scaling is also related to the dominance of the Faddeev-Popov determinant represented by the ghost degrees of freedom in the infrared. This dominance allows for the formulation of an infrared effective theory where the Yang-Mills part of the Lagrangian can be neglected [19].

#### 4. Dynamical chiral symmetry breaking

In the quark Dyson-Schwinger equation the central object responsible for dynamical chiral symmetry breaking is the quark-gluon vertex as the sole carrier of quark-gluon interactions. Based on the scaling type of infrared solutions (2.1), one can derive the analytical infrared behavior of this vertex [20]. To this end one has to carefully distinguish the cases of broken or unbroken chiral symmetry. Whereas in the broken case the full quark-gluon vertex  $\Gamma_\mu$  can consist of up to twelve linearly independent Dirac tensors, these reduce to a maximum of six when chiral symmetry is realized in the Wigner-Weyl mode. Correspondingly, a broken symmetry induces two tensor structures in the quark propagator, whereas only one is left when chiral symmetry is restored. In a similar way, chiral symmetry breaking reflects itself in every Green's function with quark content.

The presence or absence of the additional tensor structures turns out to be crucial for the infrared behavior of the quark-gluon vertex. When chiral symmetry is broken (either explicitly or dynamically with a valence quark mass  $m > \Lambda_{QCD}$ ) one obtains a self-consistent solution of the vertex-DSE which behaves like [20]

$$\lambda^{D\chi SB} \sim (p^2)^{-1/2-\kappa}. \quad (4.1)$$

Here  $\lambda$  denotes generically any dressing of the twelve tensor structures. If, however, chiral symmetry is unbroken one obtains the weaker singularity

$$\lambda^{\chi S} \sim (p^2)^{-\kappa}. \quad (4.2)$$

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<sup>1</sup>A corresponding comparison in refs. [17, 18] is misleading since in these works decoupling solutions on a torus have been compared with the scaling solution in the continuum.

As a consequence the running coupling from the quark-gluon vertex either is infrared divergent ('infrared slavery') or develops a fixed point:

$$\alpha^{qg}(p^2) = \alpha_\mu [\lambda(p^2)]^2 [Z_f(p^2)]^2 Z(p^2) \sim \begin{cases} \frac{1}{p^2} \frac{\text{const}_{qg}^{D\chi SB}}{N_c} & : D\chi SB \\ \frac{\text{const}_{qg}^{\chi S}}{N_c} & : \chi S \end{cases} \quad (4.3)$$

(Here we use that the quark propagator is constant in the infrared, i.e.  $Z_f(p^2) \sim \text{const}$  [21].) Note that in all couplings the irrational anomalous dimensions ( $\sim \kappa$ ) of the individual dressing functions cancel in the RG-invariant products.

Besides the divergence (4.2) of the quark-gluon vertex with all momenta going to zero there also exists a soft collinear-like divergence dependent only upon the external gluon momentum  $k^2$  [20]:

$$\Gamma \sim (k^2)^{-\kappa-1/2} \quad (4.4)$$

This additional divergence has two interesting consequences. First, one can analyze the behavior of the quark four-point function  $H(k^2)$  which includes the (static) quark potential. With (4.1) and (4.2), one obtains  $H(k^2) \sim 1/k^4$  in the Nambu-Goldstone and  $H(k^2) \sim 1/k^2$  in the Wigner-Weyl realization of chiral symmetry. This leads to a quark-antiquark potential of

$$V(\mathbf{r}) = \frac{1}{(2\pi)^3} \int d^3k e^{i\mathbf{k}\mathbf{r}} H(\mathbf{k}^2) \sim \begin{cases} |r| & : D\chi SB \\ \frac{1}{|r|} & : \chi S \end{cases} \quad (4.5)$$

which establishes a link between dynamical chiral symmetry breaking and confinement [20].

The second consequence concerns the  $U_A(1)$ -problem. A confinement driven mechanism for the generation of the topological mass of the  $\eta'$  in the chiral limit has been suggested by Kogut and Susskind many years ago [22]. It involves the calculation of a certain type of diagram ('diamond diagram'), which generates such a mass in the presence of an infrared divergent gluon propagator  $D(k) \sim 1/k^4$  for  $k^2 \rightarrow 0$ . Today we have excellent evidence that the gluon propagator cannot be that singular. However, there is the above-mentioned singularity in the quark-gluon vertex. Indeed, the combination of a gluon propagator and two dressed vertices appearing in the diamond diagram gives precisely a singularity of necessary strength:

$$\Gamma(k^2) \frac{Z(k^2)}{k^2} \Gamma(k^2) \xrightarrow{k^2 \rightarrow 0} (k^2)^{-1/2-\kappa} \frac{(k^2)^{2\kappa}}{k^2} (k^2)^{-1/2-\kappa} \sim 1/k^4 \quad (4.6)$$

One then obtains the masses  $m_\eta, m_{\eta'}$  and the singlet-octet mixing angle  $\theta$  of [23]

$$\theta = -23.2, \quad m_\eta = 479 \text{ MeV}, \quad m_{\eta'} = 906 \text{ MeV} \quad (4.7)$$

in the chiral limit. These values demonstrate that the Kogut-Susskind mechanism works in principle. Via the Witten-Veneziano relation one obtains the topological susceptibility  $\chi^2$  of

$$\chi^2 = (169 \text{ MeV})^4, \quad (4.8)$$

in qualitative agreement with lattice results [24].

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