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Lattice Chirality and the Decoupling of Mirror Fermions

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With LHC commissioned just a few weeks ago, all sorts of ideas about physics beyond the standard model are being explored. A strong-coupling chiral theory appearing at TeV scale remains a possibility but also a very hard scenario to study. When it comes to strongly coupled theories, lattice regularization is by far the most reliable method. But defining exact chiral gauge theory on the lattice remains a difficult problem on its own. We show that the idea to use additional non-gauge, high-scale mirror-sector dynamics to decouple the mirror fermions without breaking the gauge symmetry might lead to a practically manageable solution. We demonstrate, using the exact lattice chirality, that partition functions of lattice gauge theories with vector like fermion representations can be split into "light" and "mirror" parts, each containing a chiral representation. We show that such a splitting is only well defined when both sectors are separately anomaly free. We also prove that, the generating function and therefore the spectrum of an arbitrary chiral gauge theory is a smooth function of the background gauge field, if and only if the anomaly free condition is satisfied. We reached this conclusion by proving some very general properties of an arbitrary chiral gauge theory on lattice, and the results should be of importance for further studies in this field.

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1. Introduction

Most popular scenarios proposed for LHC-scale physics beyond standard model are weakly coupled theories, giving the best chance of analytic predictions. It, however, remains possible that some strongly coupled dynamics is at work at the TeV scale, which may be responsible for many interesting physical problems such as the electro-weak symmetry breaking. The available methods suitable for analyzing strongly coupled dynamics are limited. An incomplete list may include 't Hooft anomaly matching, supersymmetry, large-N expansion, and most recently, AdS/CFT correspondence. Often they do not represent a first principle approach and some do not work well for chiral theories. At this moment, the lattice formulation remains the most reliable non-perturbative description of a strongly coupled quantum field theory.

Besides all the motivations for physics beyond standard model, just to define the Standard Model on a lattice is also a theoretically curious question that requires the construction of chiral QFT on the lattice since the Standard Model is a chiral theory.

It is well known that defining a chiral theory on the lattice is problematic due to the fermion doubling problem. Defining vector-like theory, on the other hand, is less difficult. It, therefore, seems to be a reasonable idea to consider the possibility that one could start with a vector-like theory on the lattice and then deform the theory in such a way that half of the chiral sector, often referred to as the "mirror sector", decouples from the infrared *without* breaking the gauge symmetry, which would then automatically give rise to a chiral theory whose degrees of freedom consist of only the other chiral half of the theory.

Such an idea has been proposed and studied long ago [1]. Due to the lack of an exact chiral symmetry on the lattice, this method has, for a long time, remained fruitless. Since the work of Ginsparg and Wilson [2], it has been realized that exact chiral symmetry can be defined on a lattice with finite spacing. The idea of decoupling of the mirror fermions to realize a chiral gauge theory on the lattice is thereafter resurrected [3]. In this proceeding we report some of the recent work in this direction and some theoretical development on lattice chiral gauge theory using the Ginsparg-Wilson (overlap) chiral fermions.

2. Strong-coupling symmetric phase and the 1-0 toy model

One essential ingredient that makes possible the proposal of "decoupling of the mirror fermions" without breaking the gauge symmetry in the infrared is the existence of the strong-coupling symmetric phase in lattice gauge theories.

A simply example is the gauged-XY model, in which such a phase is known to exist. The action of this theory is given by:

$$-S_{\kappa} = \sum_{\mathbf{x}} \left(\frac{\beta}{2} \prod_{\text{plaq}} U + \frac{\kappa}{2} \sum_{\hat{\mu}} \phi^*_{\mathbf{x}} U_{\mathbf{x},\mathbf{x}+\hat{\mu}} \phi_{\mathbf{x}+\hat{\mu}} \right) + \text{h.c.} \tag{2.1}$$

where U is the Wilson-line of a $U(1)$ gauge field, denoted as $U_{x,x+\hat{\mu}}$, around each plaquette on a lattice, and $\phi_x = e^{i\eta_x}$ is a unitary field. This theory has been studied by many authors. See [4] for an example. Recently some numerical study of such a theory on a 2-d lattice was presented in [5]. Besides clarifying some of the remaining issues of this model, plots that demonstrate clearly a transition from a Higgs phase to a strong-coupling symmetric phase while κ varies from above to below 1 were presented there. In the symmetric phase, the only dynamical degree of freedom ϕ is heavier than the cutoff and one is left with in the low energy limit a pure $U(1)$ gauge theory that is not broken.

Coming back to the topic of "decoupling of the mirror fermion", in [3], a toy model, the so called 1-0 model was studied. The action is given by

$$\begin{aligned} S =& S_{\text{light}} + S_{\text{mirror}} \\ S_{\text{light}} =& (\bar{\psi}_+, \mathrm{D}_1 \psi_+) + (\bar{\chi}_-, \mathrm{D}_0 \chi_-) \\ S_{\text{mirror}} =& (\bar{\psi}_-, \mathrm{D}_1 \psi_-) + (\bar{\chi}_+, \mathrm{D}_0 \chi_+) \\ &+ y\{(\bar{\psi}_-, \phi^* \chi_+) + (\bar{\chi}_+, \phi \psi_-) \\ &+ h[(\psi_-^T, \phi \gamma_2 \chi_+) - (\bar{\chi}_+, \gamma_2 \phi^* \bar{\psi}_-^T)]\} \\ S_\kappa =& \frac{\kappa}{2} \sum_{\mathbf{x},\hat{\mu}} [2 - (\phi_{\mathbf{x}}^* U_{\mathbf{x},\mathbf{x}+\hat{\mu}} \phi_{\mathbf{x}+\hat{\mu}} + \text{h.c.})] \,. \end{aligned} \tag{2.2}$$

Here $\phi_{\mathbf{x}} = e^{i\eta_{\mathbf{x}}}$ is again a unitary field, which is usually referred to as the "unitary Higgs field". $\psi(\bar{\psi})$ and $\chi(\bar{\chi})$ are Dirac fermions in the (anti)fundamental representations of charge 1 and 0 respectively. $(\psi, \chi) = \sum_{\mathbf{x}} \psi_{\mathbf{x}} \cdot \chi_{\mathbf{x}}$ and $\pm$ denotes the Ginsparg-Wilson chiral components of the spinors. Strong-coupling symmetric phases of lattice Yukawa models or multi-fermion-interaction theories were discussed earlier [1, 6]. Now, taking advantage of the exact chiral symmetry on the lattice defined in the Ginsparg-Wilson (overlap fermion) formalism, the authors of [3] studied the 1-0 model numerically at zero gauge field background, and found strong evidence for a symmetric phase appearing when y is large enough and $h > 1$. In this phase, all the excitations of the mirror fermions as well as the unitary higgs field ϕ are heavy than the lattice cutoff. Assuming this is true, and remains so while gauge field is turned on, one may conclude that in the low energy, we would end up with a $U(1)$ chiral gauge theory on the lattice, circumventing all the known difficulties of defining such a theory explicitly.

Whether the mirror parts in the 1-0 model (2.2) are all heavy and whether the continuum limit in the infrared of this model is well defined remain to be further investigated [7]. Some immediate puzzles already arise at this stage. Most curiously, the light fermion content is anomalous described by the action

$$S_{\text{light}} = (\bar{\psi}_+, \mathrm{D}_1 \psi_+) + (\bar{\chi}_-, \mathrm{D}_0 \chi_-). \tag{2.3}$$

This immediately implies that something must go wrong once the gauge field is turned on. Would it always introduce massless degree of freedom so that, in the low energy, the theory is automatically anomaly free and therefore vector-like? If so, how does this happen? It is therefore important to understand what role the gauge anomaly plays in this setup.

In chiral lattice gauge theories, gauge anomaly is intriguingly related to the definition of fermion measure and has been extensively studied by various authors (see [8, 9] and the references therein), but all the previous study was focused on fermion bilinear theories only. The current model of our interest, the 1-0 model, is much messier. One usually expects that the gauge anomaly depends on the fermion content only and is independent to the details of the action. To understand how such an expectation is realized on the lattice in the Ginsparg-Wilson formalism, we need to generalize the known knowledge on this topic, discovered by earlier authors, to a much more general case. By learning exactly how the gauge anomaly affects the existence of the lattice chiral gauge theory, we hope to be able to make some educated guess on whether the properties of the spectrum found in [3] would remain true or be totally spoiled in the more complete models where both the mirror and light sectors are anomaly free. To this end, we turn to some general analysis of chiral gauge theory on the lattice below.

3. General definition of chiral lattice theory and the "splitting-theorem"

We start by giving the most general definition of a chiral theory on the lattice. A chiral theory is described by an action S as a functional of the spinors which satisfies the property that

$$S[X,Y^{\dagger},O] = S[PX,Y^{\dagger},O] = S[X,Y^{\dagger}\hat{P},O]. \tag{3.1}$$

Here X and Y denote the fundamental and anti-fundamental spinors and should be simply understood as a $2N^d$-dimensional vectors, where N is the lattice size and d is the lattice dimension. P and $\hat{P}$ are two projection operators that satisfy $P^2 = P$ and $\hat{P}^2 = \hat{P}$, which may or may not depend on the background gauge field. If they do not, the theory never has gauge anomaly and therefore necessarily has a fermion doubling problem. In the case of the overlap fermions, $\hat{P}$ is defined through the Ginsparg-Wilson operator which does depend on the gauge field background while P is just the normal chiral projection operator defined through γ_5 matrix. Here O represent collectively any other local operators the theory may depend on.

To define the partition function, one has to choose a set of orthonormal basis $\{u_i,\, v_i\}$ consisting of the eigenvectors of the projection operator P and $\hat{P}$ as

$$Pu_i = u_i, \quad v_i^{\dagger}\hat{P} = v_i^{\dagger}. \tag{3.2}$$

The partition function is defined as the usual Grassmann integral

$$Z = \int \prod_{i,j} \mathrm{d}c_i \mathrm{d}\bar{c}_j\, e^{S\left[\sum_i c_i u_i,\, \sum_j \bar{c}_j v_j^{\dagger},\, O\right]}. \tag{3.3}$$

Such a partition function is not uniquely defined. If we choose a different orthonormal basis with $v_i' = \mathscr{U}_{ij} v_j$, where $\mathscr{U}$ is a unitary matrix, the partition function defined using the new vectors differs from the previous one by a factor of $\det \mathscr{U}$. This is usually not a problem as such a phase factor is an arbitrary constant that bears no physical significance. In the case of overlap fermion, however, it becomes a serious issue as the vectors v_i are usually gauge field dependent, and so is this ambiguous phase factor.

Lüscher proved [8] that, in the case of the Abelian gauge theory, such a phase ambiguity is uniquely fixed if one demands that

- the partition function $Z[U]$ is a smooth local functional of the gauge field U,
- and $Z[U]$ is gauge invariant,

provided that the fermion content is anomaly free. The proof was given for fermion bilinear theory

$$S = \sum_{\mathbf{x}} \bar{\psi}\hat{P}_{+}\mathrm{D}P_{+}\psi \tag{3.4}$$

only. To generalize it, in particular, to the much more complicated 1-0 model mentioned earlier, we proved in [10] a simple but powerful theorem given below, which we call the "splitting theorem".

For any chiral action that satisfies property (3.1) and has a partition function defined by (3.3), under arbitrary variations of the vectors

$$u_i \to u_i + \delta u_i, \quad v_j \to v_j + \delta v_j \tag{3.5}$$

and the operators

$$O \to O + \delta O \tag{3.6}$$

we proved that the variation of the partition function Z always splits into two parts as

$$\delta \log Z = \mathscr{J} + \left\langle \frac{\delta S}{\delta O}\delta O \right\rangle, \tag{3.7}$$

where

$$\mathscr{J} = \sum_i (u_i^\dagger \cdot \delta u_i) + \sum_i (\delta v_i^\dagger \cdot v_i). \tag{3.8}$$

$\mathscr{J}$ depends only on the choice of the eigenvectors and is totally independent on the details of the Lagrangian. It is often referred to as the "measure term", despite that it is not quite a Jacobian of the variation of some measure in the rigorous sense. $\langle O \rangle$ denotes the expectation value of operator O. $\frac{\delta S}{\delta O}\delta O$ denotes the variation of S due to the variations of operator O only. It is easily verified that the last piece in equation (3.7) is basis independent. This theorem is the manifestation on the lattice of the fact that the gauge anomaly depends solely on the fermion representations and not the details of the action.

4. Gauge invariance and the smoothness of chiral partition functions

The splitting theorem can be used to derive the gauge variation of $\log Z$ for an arbitrary chiral theory. An chiral theory $S[X, Y^\dagger, O]$ is said to have a gauge symmetry if it remains invariant under the following gauge transformation

$$\delta_\omega X = i\omega X, \ \delta_\omega Y = i\omega Y, \ \delta_\omega O = i[\,\omega\,,O\,], \tag{4.1}$$

where $\omega = \omega(x)$ is an arbitrary scalar field on the lattice. By the "splitting-theorem", one can derive that the variation of $\log Z$ is given by

$$\delta_\omega \log Z = \mathscr{J}_\omega + i\mathrm{Tr}\omega(P - \hat{P}). \tag{4.2}$$

It is easily verified that $\mathrm{Tr}\omega(P - \hat{P}) = 0$ for any ω if and only if fermion content is anomaly free.

Lüscher has proven that $\mathscr{J}$ can be chosen uniquely as a smooth function of the gauge field U provided that anomaly free condition is satisfied, in which case $\log Z$ is gauge invariant for fermion bilinear theories [8]. Using this fact and the "splitting-theorem", we immediately conclude that, as long as $\mathscr{J}$ is so chosen, $\log Z$ is gauge invariant for an arbitrary action S provided that it satisfies property (3.1), since the gauge variation of $\log Z$ given by (4.2) is identical for all chiral gauge theories completely independent to the details of S.

We can also prove the smoothness of the chiral partition function with respect to the gauge field U by using the splitting theorem recursively. Assuming that the action S by itself has no poles and therefore $\left\langle \frac{\delta S}{\delta O} \delta O \right\rangle < \infty$, it can be quickly verified that this quantity by itself can be considered as a partition function of a new "chiral action" $S^{(1)}$ [10]. One can then apply the splitting theorem one more time while calculating the second derivatives of $\log Z$. Following this logic and applying the "splitting-theorem" iteratively, one can prove that the derivatives of $\log Z$ to arbitrary order $n \in \mathbb{Z}$ are finite and therefore prove that $\log Z$ is a smooth function of the gauge field U. We emphasize that even though the measure term $\mathscr{J}$ can be chosen as a smooth function of U when anomaly free condition is satisfied, some of the eigenvectors v_i always become singular at certain places in the gauge field configuration space, which is why without knowing the details of S, it is necessary to apply the "splitting-theorem" iteratively to prove the smoothness of the chiral partition function.

5. Conclusion and outlook

We now arrive at the partial answer to the question raised in section 2. We understand that the splitting of the original vector-like theory into $+$ and $-$ chiral sectors in the Ginsparg-Wilson formalism can be made smoothly with respect to the gauge field U *if and only if* each chiral sectors satisfy the anomaly free condition. While this condition is violated, one faces the difficulty of defining the fermion measure of each chiral sector and no matter how one is to fix the phase ambiguity, always some singularities appear in the gauge field configuration space. This is explicitly demonstrated in [10] for the simple case where only homogeneous Wilson lines are turned on. In fact, the vectors chosen in the numerical simulations for the zero-gauge field background in [3] sit right on this kind of singularities.

On the other hand, the obstacle of defining a sensible chiral gauge theory with anomalous fermion content due to the singularities is known to be topological, meaning that one can always avoid it locally in the gauge field configuration space by tuning the boundary conditions of the fermions. Consequently, it appears reasonable to expect that the local properties[1] (namely if perturbative expansion of the theory with respect to the gauge coupling can be trusted) won't be seriously affected. Therefore the spectrum found in the 1-0 model, despite the defect that each chiral sector is anomalous and singular, may still reveal some useful information, as long as the gauge coupling is kept sufficiently small, about the mirror fermion mass in more complete models where anomaly free condition is respected.

Of course, it remains to be verified, either numerically or analytically, if such an expectation is justified. Interestingly, using the "splitting-theorem" and the anomaly equations, one is able to derive a family of fermion two-point functions at the zero gauge field background and predict its long range behavior analytically independent to the details of the action S. Therefore, those objects might be useful to probe the spectrum of any model proposed for "decoupling of the mirror fermions". We hope to report more results on this topic in the near future [7]. We also hope to

[1] Here, "local" refers to the locality in the gauge field configuration spaces.

understand exactly how the continuum limit of the 1-0 model behaves, in particular if unitarity is preserved. Again, we hope to report more on that in following work.

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