

$\pi^+\pi^+$, K^+K^+ and BB Interactions

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The most recent calculations of $\pi^+\pi^+$ and K^+K^+ scattering by the NPLQCD collaboration using domain-wall valence quarks on staggered MILC configurations are presented. In addition, a quenched calculation of the potentials between two B-mesons is discussed.

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1. Introduction

Pion-pion scattering at low energies is the simplest and best-understood hadron-hadron scattering process. Its simplicity and tractability follow from the fact that the pions are identified as the pseudo-Goldstone bosons associated with the spontaneous breaking of the approximate $SU(2)_L \otimes SU(2)_R$ chiral symmetry of QCD. For this reason, the low-momentum interactions of pions are strongly constrained by the approximate chiral symmetry, more so than other hadrons. The scattering lengths for $\pi\pi$ scattering in the s-wave are uniquely predicted at leading order (LO) in chiral perturbation theory (χ -PT) [1]:

$$m_\pi a_{\pi\pi}^{I=0} = 0.1588 \quad ; \quad m_\pi a_{\pi\pi}^{I=2} = -0.04537 \quad , \quad (1.1)$$

at the physical charged pion mass. Subleading orders in the chiral expansion of the $\pi\pi$ amplitude give rise to perturbatively-small deviations from the tree level, and contain both calculable non-analytic contributions and analytic terms with new coefficients that are not determined by chiral symmetry alone [2, 3, 4]. In order to have predictive power at subleading orders, these coefficients must be obtained from experiment or computed with Lattice QCD. While the perturbative expansion of the $\pi\pi$ scattering amplitude is expected to converge rapidly, the KK amplitude is expected to receive sizable contributions from higher orders. Naive expectations suggest that perturbative corrections to the KK scattering amplitude are set by m_K^2/Λ_χ^2 .

Recently, we (NPLQCD) have performed the first $n_f = 2 + 1$ flavor QCD calculation of the $\pi^+\pi^+$ and K^+K^+ scattering lengths [5, 6, 7]. The $\pi^+\pi^+$ scattering length has been calculated with percent level precision. Domain-wall valence quarks were computed on various ensembles of MILC lattices (staggered sea quarks), and mixed-action chiral perturbation theory [8, 9] was used to eliminate the leading effects of the finite lattice-spacing. The results of the lattice calculations are found to be consistent with tree-level chiral perturbation theory, even at large pion and kaon masses, within the uncertainties of the calculations. We have also performed a quenched calculation of the potentials between two B-mesons. As the effective field theory that gives rise to these potentials is the same as that describing the interactions between nucleons (up to the values of the counterterms), these potentials provide insight into the interactions between two or more nucleons.

2. Hadronic Interactions, the Maiani-Testa Theorem and Lüscher's Method

Extracting hadronic interactions from Lattice QCD calculations is far more complicated than the determination of the spectrum of stable particles. This is encapsulated in the Maiani-Testa theorem [10], which states that S-matrix elements cannot be extracted from infinite-volume Euclidean-space Green functions except at kinematic thresholds¹. Of course, it is clear from the statement of this theorem how it can be evaded, one computes Euclidean-space correlation functions at finite volume to extract S-matrix elements, the formulation of which was known for decades in the context of non-relativistic quantum mechanics [11] and extended to quantum field theory by Lüscher [12, 13]. Lüscher showed that the energy of two particles in a finite volume depends in a calculable way upon their elastic scattering amplitude and their masses for energies below the inelastic threshold. As a concrete example consider $\pi^+\pi^+$ scattering. A $\pi^+\pi^+$ correlation function

¹An infinite number of infinitely precise calculations would allow one to circumvent this theorem.

in the A_1 representation of the cubic group [14] (that projects onto the s-wave state in the continuum limit) is

$$C_{\pi^+\pi^+}(p,t) = \sum_{|\mathbf{p}|=p} \sum_{\mathbf{x},\mathbf{y}} e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} \langle \pi^-(t,\mathbf{x}) \pi^-(t,\mathbf{y}) \pi^+(0,\mathbf{0}) \pi^+(0,\mathbf{0}) \rangle . \quad (2.1)$$

In relatively large lattice volumes the energy difference between the interacting and non-interacting two-meson states is a small fraction of the total energy, which is dominated by the masses of the mesons. In order to extract this energy difference the ratio of correlation functions, $G_{\pi^+\pi^+}(p,t)$, can be formed, where

$$G_{\pi^+\pi^+}(p,t) \equiv \frac{C_{\pi^+\pi^+}(p,t)}{C_{\pi^+}(t)C_{\pi^+}(t)} \rightarrow \sum_{n=0}^{\infty} \mathcal{A}_n e^{-\Delta E_n t} , \quad (2.2)$$

and the arrow denotes the large-time behavior of $G_{\pi^+\pi^+}$. The single pion correlation function is $C_{\pi^+}(t)$. The energy eigenvalue, E_n , and its deviation from the sum of the rest masses of the particle, ΔE_n , are related to the center-of-mass momentum p_n by $\Delta E_n \equiv E_n - 2m_\pi = 2\sqrt{p_n^2 + m_\pi^2} - 2m_\pi$. To obtain $p \cot \delta(p)$, where $\delta(p)$ is the phase shift, the square of the center-of-mass momentum, p , is extracted from this energy shift and inserted into [11, 12, 13, 15]

$$p \cot \delta(p) = \frac{1}{\pi L} \mathbf{S} \left(\left(\frac{pL}{2\pi} \right)^2 \right) , \quad (2.3)$$

which is valid below the inelastic threshold. The regulated three-dimensional sum is [16]

$$\mathbf{S}(x) \equiv \sum_{\mathbf{j}}^{\|\mathbf{j}\| < \Lambda} \frac{1}{\|\mathbf{j}\|^2 - x} - 4\pi\Lambda , \quad (2.4)$$

where the summation is over all triplets of integers \mathbf{j} such that $\|\mathbf{j}\| < \Lambda$ and the limit $\Lambda \rightarrow \infty$ is implicit. Therefore, by measuring the energy-shift, ΔE_n , of the two particles in the finite lattice

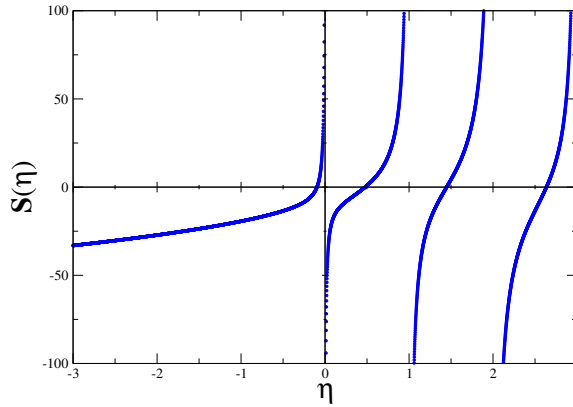


Figure 1: The function $\mathbf{S}(\eta)$ vs. η , defined in Eq. (2.4), has poles only for $\eta \geq 0$.

volume, the scattering phase-shift is determined at ΔE_n .

3. $\pi^+\pi^+$ Scattering

The prediction for the physical value of the $I = 2$ $\pi\pi$ scattering length from our mixed-action calculation is $m_\pi a_{\pi\pi}^{I=2} = -0.04330 \pm 0.00042$ [5, 6], which agrees within uncertainties with the (non-lattice) determination of CGL [17]. In Table 1 and Fig. 2 we offer a comparison between various determinations².

	$m_\pi a_{\pi\pi}^{I=2}$
χ -PT (Tree Level)	-0.04438
NPLQCD (2007)	-0.04330 ± 0.00042
E 865 (2003)	-0.0454 ± 0.0031
NPLQCD (2005)	-0.0426 ± 0.0018
MILC (2006)*	-0.0432 ± 0.0006
MILC (2004)*	-0.0433 ± 0.0009
CGL (2001)	-0.0444 ± 0.0010

Table 1: A compilation of the various calculations and predictions for the $I = 2$ $\pi\pi$ scattering length.

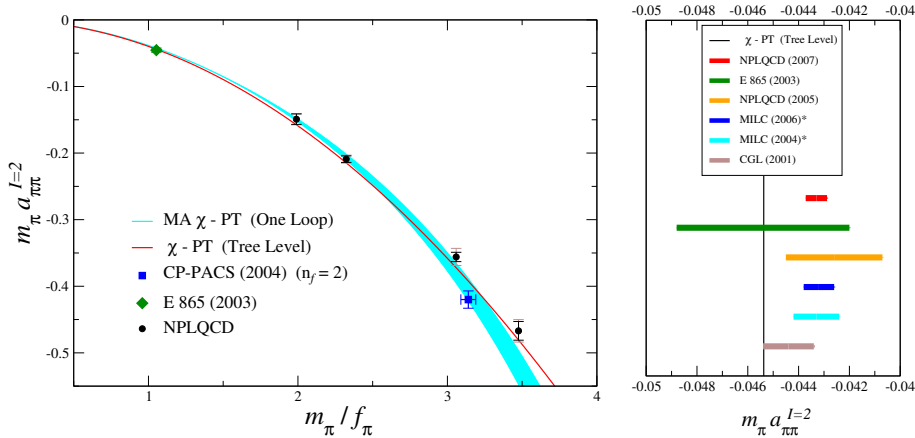


Figure 2: The left panel shows $m_\pi a_{\pi\pi}^{I=2}$ vs. m_π/f_π (ovals) [5, 6]. Also shown are the experimental value from Ref. [18] (diamond) and the lowest quark mass result of the $n_f = 2$ calculation of CP-PACS [19] (square). The blue band corresponds to a fit to the lightest three data points using the one-loop MA χ -PT formula, and the red line is the tree-level χ -PT result. The right panel shows a bar chart of the various determinations of the $I = 2$ $\pi\pi$ scattering length tabulated in Table 1. See footnote 2.

4. K^+K^+ Scattering

The K^+K^+ scattering length is calculated in the same way as the $\pi^+\pi^+$ scattering length, but requires strange quark propagators. The results of the lattice calculation of K^+K^+ scattering are extrapolated to the physical values of $m_{\pi^+}/f_{K^+} = 0.8731 \pm 0.0096$, $m_{K^+}/f_{K^+} = 3.088 \pm 0.018$ and

²The stars on the MILC results indicate that these are not lattice calculations of the $I = 2$ $\pi\pi$ scattering length but rather a hybrid prediction which uses MILC's determination of various low-energy constants together with the Roy equations), and the Roy equation determination of Ref. [17] (CGL (2001)).

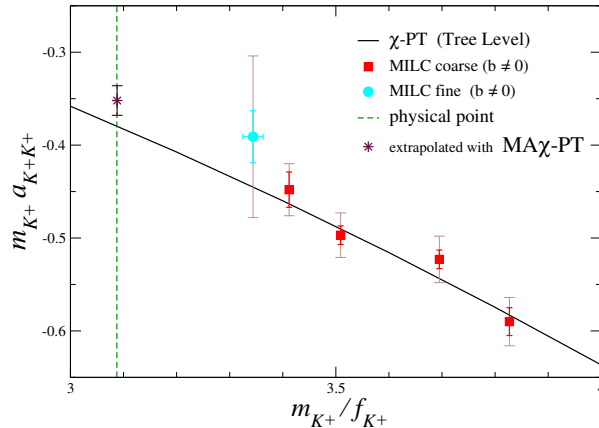


Figure 3: $m_{K^+} a_{K^+K^+}$ versus m_{K^+}/f_{K^+} [7]. The points with error-bars are the results of this lattice calculation (not extrapolated to the continuum). The solid curve corresponds to the tree-level prediction of χ -PT. The point denoted by a star and its associated uncertainty is the chiral extrapolation to the physical meson masses and to the continuum.

$m_\eta/f_{K^+} = 3.425 \pm 0.0019$ assuming isospin symmetry, and the absence of electromagnetism. Considering the systematic uncertainties in the chiral extrapolation of the results shown in Fig. 3, along with the statistical uncertainties, gives $m_{K^+} a_{K^+K^+} = -0.352 \pm 0.016$ [7], where the statistical and systematic errors have been combined in quadrature.

5. BB Potentials

Energy-independent potentials can be rigorously defined and calculated for systems composed of two (or more) hadrons containing a heavy quark in the heavy-quark limit, $m_Q \rightarrow \infty$. This is interesting for more than academic reasons as the light degrees of freedom (dof) in the B-meson have the same quantum numbers as the nucleon, isospin- $\frac{1}{2}$ and spin- $\frac{1}{2}$. As such, the EFT describing the interactions between two B-mesons has the same form as that describing the interactions between two nucleons, but the counterterms that enter into each EFT are different. Therefore, a deeper understanding of the EFT description of nuclear physics can be gained by Lattice QCD calculations of the potentials between B-mesons. We computed the potential between two B-mesons in the four possible spin-isospin channels (neglecting $B_d^0 - \bar{B}_d^0$ mixing) in relatively small volume DBW2 lattices with $L \sim 1.6$ fm, with a pion mass of $m_\pi \sim 403$ MeV, and lattice-spacing of $b \sim 0.1$ fm [20]. The calculation was quenched and the naive Wilson action was used for the quarks. At this relatively fine lattice spacing, much finer than previous calculations, we were able to extract a non-zero potential, but the small volume meant that the contributions to the potential from image B-mesons (periodic BC's) were visible.

Constructing the t-channel potentials, defined via the quantum numbers of the exchange particles, in keeping with nuclear physics tradition, isolated statistical fluctuations into the channel

associated with the “ σ ”-meson, leaving the channels with the quantum numbers of the π , ρ and ω with relatively small statistical errors. The potentials are shown in Fig. 4.

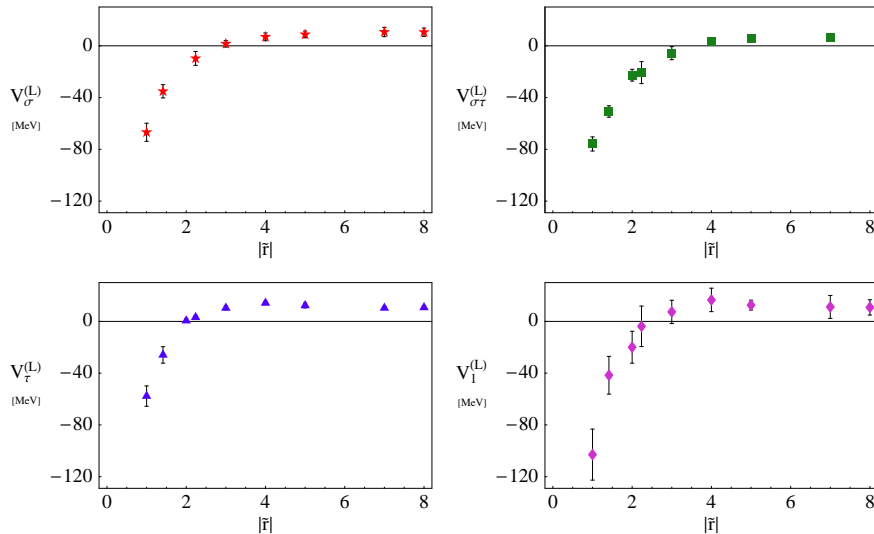


Figure 4: The potentials between B-mesons in the finite lattice volume. $V_{\sigma}^{(L)}$, $V_{\tau}^{(L)}$, $V_{\sigma\tau}^{(L)}$, and $V_1^{(L)}$ correspond to the potentials in the exchange-channels with spin-isospin of $(J, I) = (1, 0)$, $(0, 1)$, $(1, 1)$ and $(0, 0)$.

Given the uncertainties in the potentials, and the number of counterterms that appear in the EFT describing the long- and medium- distance interactions between the B-mesons, it was possible to make only a parameterization of each potential beyond the leading light-meson contribution. Since only the longest range contribution to the potential in each channel can be identified, we fit our results at large separations, $|\mathbf{r}| > \Lambda_{\chi}^{-1}$, using the finite-volume versions of the simplified infinite-volume potentials. The short distance forms are entirely model dependent and are the simplest forms that we could find that provide a reasonable description of the data. Using the measured values and uncertainties of m_{π} and m_{ρ} and the physical value of f_{π} we first determine the light-meson couplings by fitting the finite-volume potentials at the two largest separations.³ The resulting $BB\pi$ coupling is found to be $g = 0.57 \pm 0.06$.

6. Conclusions

I have presented the results of recent calculations by the NPLQCD collaboration of the $\pi^+\pi^+$ and K^+K^+ scattering lengths, and the potentials between two B-mesons. Percent level precision predictions for the $\pi^+\pi^+$ scattering length were made possible by recent theoretical progress in describing mixed-action calculations with chiral perturbation theory and by the large number of domain-wall propagators ($\sim 2.5 \times 10^4$) that were calculated on the coarse MILC lattices ensembles.

The lattice results for meson-meson scattering pose an interesting puzzle. The $\pi^+\pi^+$ scattering length tracks the current algebra result up to pion masses that are expected to be at the edge

³Simple fits using the infinite-volume long range behavior were considered in Ref. [21].

of the chiral regime in the two-flavor sector. While in the two flavor theory one expects fairly good convergence of the chiral expansion and, moreover, one expects that the effective expansion parameter is small in the channel with maximal isospin, the lattice calculation clearly imply a cancellation between chiral logs and counterterms (evaluated at a given scale). The same phenomenon occurs in K^+K^+ scattering (Fig. 3) where the chiral expansion is governed by the strange quark mass and is therefore expected to be more slowly converging. The π^+K^+ scattering length exhibits similar behavior [22]. This mysterious cancellation between chiral logs and counterterms for the meson-meson scattering lengths begs for an explanation.

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