Gapless Dirac Spectrum at High Temperature

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Using the overlap Dirac operator I show that, contrary to some expectations, even well above the critical temperature there is not necessarily a gap in the Dirac spectrum in pure SU(2) gauge theory. This happens when the Polyakov loop and the fermion boundary condition combine to give close to periodic boundary condition for the fermions in the time direction. In this Polyakov loop sector there is a non-vanishing density of Dirac eigenvalues around zero which implies that chiral symmetry is spontaneously broken. I demonstrate this both directly and also by finding good agreement with the random matrix theory prediction for the distribution of the lowest Dirac eigenvalue. I show that the chiral condensate increases with the tempereture therefore it is very unlikely to be explained by topological fluctuations that become rapidly smaller above T_c. Finally I show that it is only a small fraction of the lowest Dirac eigenvalues that decide which Polyakov loop sector is favored by the fermion determinant if dynamical fermions are turned on. This provides a qualitative understanding of how the loss of confinement above T_c implies the restoration of chiral symmetry.

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1. Introduction

It is generally accepted that four dimensional $SU(N)$ gauge theories have a high temperature deconfined phase with the Polyakov loop $Z(N)$ symmetry spontaneously broken. This is in the theory without dynamical fermions. Including massless dynamical fermions will change this picture since the fermionic determinant breaks the $Z(N)$ symmetry explicitly. However, even in this case the transition from the low to the high temperature phase is characterized by a substantial increase of the expectation value of the Polyakov loop.

Another important feature of the theory with massless fermions is that the transition to the high temperature phase is accompanied by the restoration of chiral symmetry that is spontaneously broken at low temperatures. Generally in QCD-like theories the chiral and the deconfinement transition take place at roughly the same temperature regardless of whether the transition is a crossover or a genuine phase transition [1]. How the deconfining and the chiral transition are linked is an important and as yet even qualitatively not understood question.

The study of the interplay between the Polyakov loop and chiral symmetry restoration has a long history. More than 10 years ago Chandrasekharan and Christ noticed that in quenched QCD the chiral condensate vanishes in the high temperature phase only if the phase of the Polyakov loop is real [2]. Inspired by this intriguing result Stephanov used random matrix theory to predict that both in the $SU(2)$ and $SU(3)$ case the chiral restoration temperature depends on the Polyakov loop sector. Moreover, in the SU(2) case chiral symmetry is expected to remain broken at arbitrarily high temperature provided the Polyakov loop is negative [3]. Further support to this scenario was given by calculations in the Nambu-Jona-Lasinio model [4, 5].

More recently direct lattice simulations have also been performed to check whether this really happens. Based on the appearance of a spectral gap above T_c in the $SU(3)$ case, Gattringer et al. concluded that the chiral restoration occurs at the same temperature in all Polyakov loop sectors. They, however, found that the spectral gap above T_c depends on the Polyakov loop sector [6]. In contrast, in the $SU(2)$ case, Bornyakov et al. concluded that in the negative Polyakov loop sector chiral symmetry remains broken up to $T = 2T_c$ [7, 8].

In summary, the somewhat controversial picture is that the Polyakov loop has a strong influence on the chiral condensate, especially in the $SU(2)$ case. A further question in this connection is how a chiral condensate well in the high temperature phase can be understood based on mixing instanton anti-instanton zero modes, given that the topological charge density rapidly drops above T_c.

To shed some more light on these questions, in the present paper I study the low end of the spectrum of the overlap Dirac operator in quenched $SU(2)$ gauge backgrounds well above T_c. In particular I look at how the spectrum is influenced by the Polyakov loop sector in the high temperature phase. I also study how the fermion determinant "selects" a given Polyakov loop sector and how this mechanism connects deconfinement and chiral symmetry restoration.

2. Simulation parameters

Let us first summarize the parameters of the simulations. All of the runs are quenched Wilson action $SU(2)$ lattices at $\beta = 2.60$. This is the critical β for $N_T = 10.4$. In the present study I
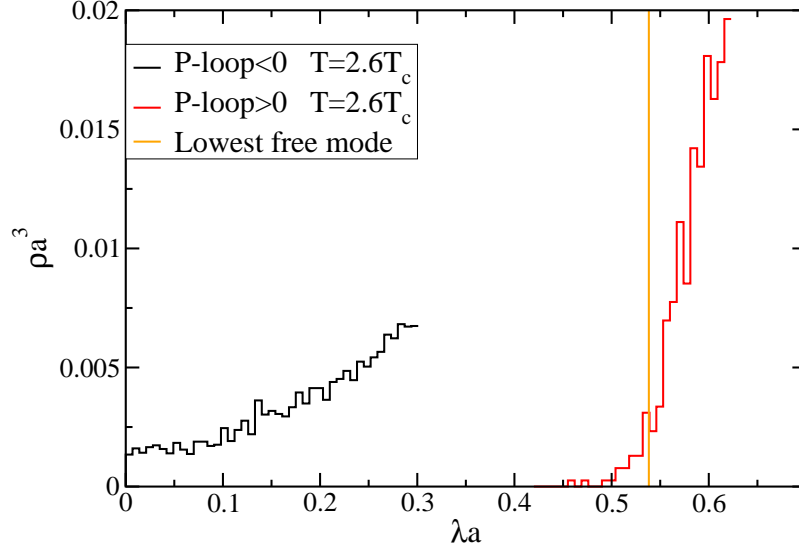
Figure 1: The density of low eigenmodes of the overlap Dirac operator in the two Polyakov loop sectors. The vertical line indicates the lowest Matsubara mode in the given box size.

chose to fix β to avoid renormalization issues for the chiral condensate. I varied the temperature by choosing $N_T = 4$ and 6, which correspond to $T = 2.6T_c$ and $1.7T_c$ respectively. I also varied the spatial size of the box to check the scaling of the low eigenvalue density with the volume.

On these lattices I computed the lowest 16-32 eigenvalues of the overlap Dirac operator [9], which is defined in terms of the Wilson Dirac operator D_w as

$$D_{ov} = 1 - A\left[A^\dagger A\right]^{-\frac{1}{2}}, \qquad A = 1 + s - D_w. \tag{2.1}$$

The real parameter s was chosen to be 0.4 by maximizing the average lowest eigenvalue of the kernel $A^\dagger A$.

3. Chiral condensate

To see how the Polyakov loop influences the chiral condensate in Figure 1 I plotted the low end of the eigenvalue density of the Dirac operator in both Polyakov loop sectors. Here the temperature was chosen to be $T = 2.6T_c$ ($N_T = 4$), well in the high temperature phase. The difference between the two sectors is dramatic. On the one hand, in the $P > 0$ positive Polyakov loop sector there is a sizeable gap in the spectrum and the density of modes at $\lambda = 0$ is clearly zero. On the other hand, in the $P < 0$ sector the eigenvalue density at $\lambda = 0$ is obviously non-vanishing. Through the Banks-Casher relation [10] this implies that at this temperature chiral symmetry is restored only if the Polyakov loop is positive.

Note that the fermion boundary condition in the time direction is anti-periodic and in the $SU(2)$ case a change from periodic to antiperiodic boundary condition is exactly equivalent to flipping the sign of the Polyakov loop. Figure 1 shows that the gap in the spectrum appears if the anti-periodic boundary condition and the Polyakov loop combine to give an effective anti-periodic boundary
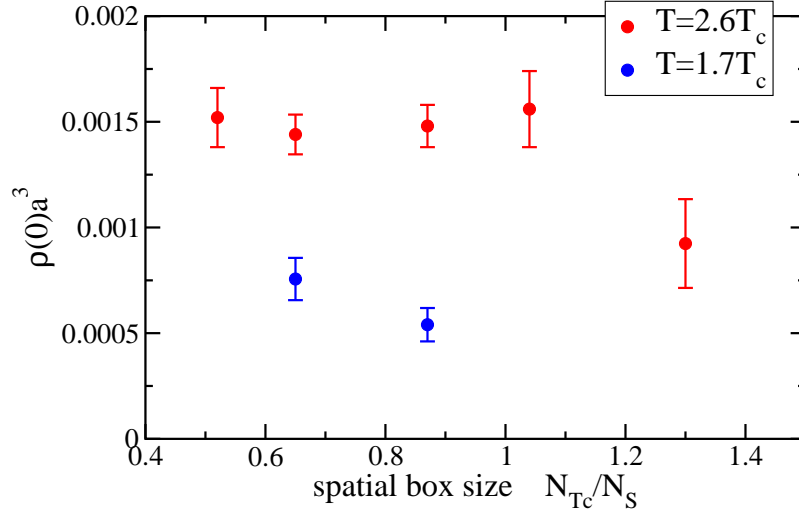
Figure 2: $\rho(0)$, the density of modes of the overlap Dirac operator normalized by the four-volume in spatial boxes of various sizes in the negative Polyakov loop sector.

condition to the fermions. Moreover, the gap is roughly equal to the lowest Matsubara frequency of the fermions at $N_T = 4$, which I also indicated in the same Figure. I also checked that this also happens for $N_T = 6$ (not shown in the Figure).

In order to establish the presence of a non-zero chiral condensate in the negative Polyakov loop sector one should check that the eigenvalue density around zero scales with the three-volume. In Figure 2 I show the eigenvalue density normalized by the volume in spatial boxes of different sizes. The size of the lattice in the spatial directions is given in units of N_{T_c}, the length scale corresponding to the critical temperature, i.e. the confinement scale. The eigenvalue density appears to be constant down to spatial sizes of roughly the confinement scale. This shows that indeed, the chiral condensate persists even at these temperatures provided the Polyakov loop is negative.

This scenario is further supported by comparing the distribution of the lowest Dirac eigenvalue with the prediction of random matrix theory (RMT). Since at this high temperature most of the configurations belong to the topological sector $Q = 0$ I do the comparison only in this sector. In Figure 3 I show the cumulative distribution of the lowest Dirac mode compared to the analytically known RMT prediction for the chiral orthogonal ensemble [11] that is supposed to describe the $SU(2)$ theory. This comparison involves one adjustable parameter, ΣV, where Σ is the value of the chiral condensate and V is the volume. ΣV is used to rescale the eigenvalues to fit the eigenvalue density to the universal RMT curve. If the spatial volume is large enough, the distribution of the rescaled eigenvalues agrees well with the RMT prediction lending further support to the presence of a non-zero condensate.

The non-zero Dirac eigenvalue density around $\lambda = 0$ and as a consequence the chiral condensate are usually attributed to mixing instanton anti-instanton would-be zero modes. Since the topological charge scales with the volume these would-be zero modes are in principle capable of providing an eigenvalue density that is also proportional to the volume. There are also speculations that the chiral condensate in the high temperature phase might also be due to topological charge
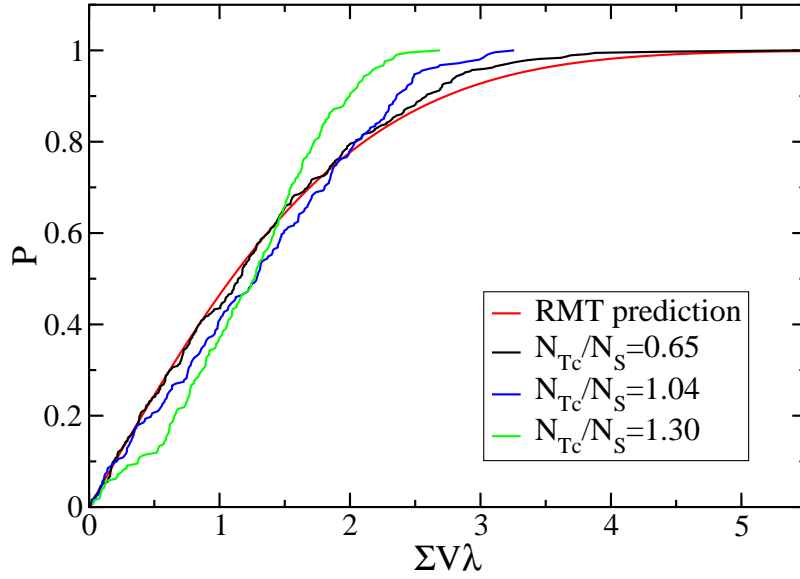
Figure 3: The cumulative distribution of the rescaled lowest Dirac eigenvalue in the $Q = 0$ topological sector compared to the prediction of random matrix theory for the chiral orthogonal ensemble.

fluctuations, in particular calorons. On the other hand, as the system is heated above the critical temperature, the topological susceptibility starts to drop sharply. If the understanding of low Dirac eigenvalues based on topological fluctuations continues to work above the critical temperature one expects the eigenmode density to drop with the topological susceptibility. In sharp contrast with that in Figure 1 the eigenmode density can be seen to increase with the temperature going up. The opposite behavior of the topological susceptibility and the eigenmode density strongly suggests that above T_c topological fluctuations become less and less responsible for low eigenmodes and as the temperature goes further up some other yet unknown mechanism takes over.

4. Dynamical fermions: how they select the "correct" Polyakov sector?

We have seen that in the negative Polyakov loop sector there is a chiral condensate that even increases with the temperature. At this point one could ask the question how in the real world chiral symmetry is restored above T_c. The answer is that the fermion determinant breaks the Polyakov loop symmetry explicitly. Our experience is that above T_c both in the $SU(2)$ and the $SU(3)$ case the fermion determinant "favors" the sector where the Polyakov loop lies along the positive real axis.

This can be qualitatively understood based on the above discussion of how the gap in the Dirac spectrum is connected to the lowest Matsubara frequency. The positive real Polyakov loop is exactly the one that combines with the anti-periodic fermion boundary condition to give the largest possible effective twist to the fermions in the time direction and the largest first Matsubara frequency. This in turn results in fewer low modes and a larger value of the determinant on average.

Is it really only the lowest modes that decide which Polyakov loop sector is favored by the determinant? Modes higher in the spectrum can also depend on the Polyakov loop and since there are far more of those, in principle they can also have a sizeable influence. To decide this I computed
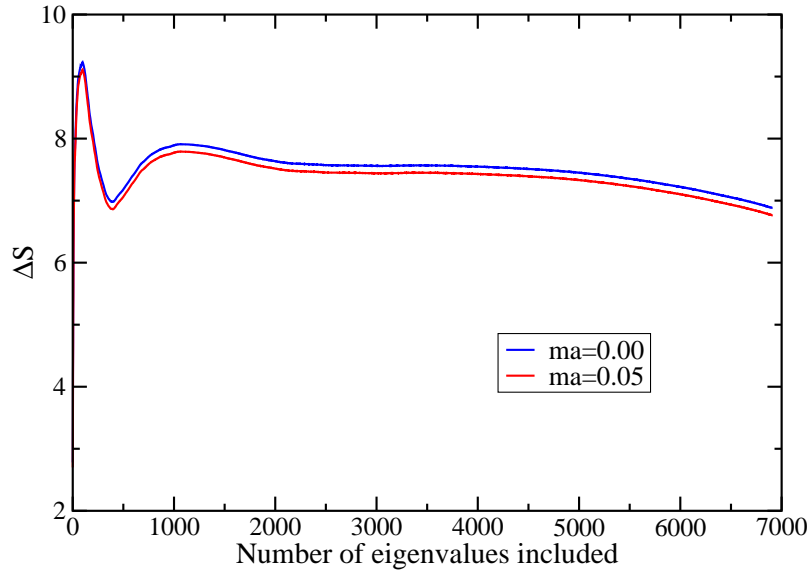
Figure 4: The difference in fermion action between periodic and anti-periodic boundary conditions on a single $6^3 \times 4$ configuration at Wilson $\beta = 2.40$. Flipping the boundary condition is exactly equivalent to transforming the configuration to the other Polyakov loop sector. The horizontal axis depicts the number of smallest eigenvalues included when computing the action difference. The two curves both correspond to a single flavor of fermion, but with different masses.

all the eigenvalues of the overlap Dirac operator on small, but physically relevant lattices. If all the eigenvalues are available in both Polyakov loop sectors one can build up the difference in the fermion action starting from the low end of the spectrum.

In Figure 4 I plotted how the action difference between the two sectors depends on the number of eigenvalues included in the determinant. This was done on a single $6^3 \times 4$ configuration at Wilson $\beta = 2.40$ by flipping the boundary condition to reach the other Polyakov loop sector. Looking at several configurations the pattern seems to be the same. Most of the action difference comes from a tiny fraction ($< 1\%$) of the eigenvalues at the low end of the spectrum. The same behavior can be confirmed also on larger configurations and higher β's where computing all the eigenvalues would be prohibitively expensive. In this case I computed only part of the spectrum and checked that after including the same small fraction of eigenvalues the action difference rapidly stabilizes. Thus it is indeed the difference in the lowest eigenvalues between the Polyakov sectors that is responsible for selecting the "correct" Polyakov sector by suppressing all other sectors through the fermion determinant.

5. Conclusions

We have seen that in the high temperature phase of the pure $SU(2)$ gauge theory the density of Dirac eigenmodes around zero strongly depends on the Polyakov loop. In the quenched theory the Polyakov loop $Z(2)$ symmetry is spontaneously broken above T_c and I showed that in the $P < 0$ sector the low Dirac mode density and as a consequence the chiral condensate remain non-zero up

to $T = 2.6T_c$, the highest temperature considered in the present study. Since up to this point the eigenvalue density increases with the temperature, it is very likely that the condensate remains non-zero at arbitrarily hight temperatures. I gave further evidence for a non-vanishing chiral condensate by finding good agreement for the distribution of the lowest Dirac mode in the $Q = 0$ topological sector with the analytically known prediction of random matrix theory.

In the other Polyakov loop sector above T_c, where $P > 0$, a gap appears in the spectrum. I showed that the gap is governed by the lowest Matsubara mode determined by the effective boundary condition given by the Polyakov loop combined with the thermal anti-periodic boundary condition. Note that in contrast to the $SU(2)$ case, if the gauge group is $SU(3)$, the Polyakov loop cannot exactly cancel the anti-periodic boundary condition. Therefore the gap is expected to be present in all sectors, but it is still governed by the lowest free Dirac eigenvalue, as was seen in Ref. [6].

I also showed that contrary to common belief [8, 12], topological charge fluctuations are very unlikely to explain the chiral condensate above T_c since the topological susceptibility decreases, while the chiral condensate increases with the temperature. It would be interesting to see what other mechanism could take over the role of topological charge in this context.

Finally I showed that by far the most important contribution to the fermion determinant is given by a tiny fraction of the lowest eigenvalues. These are responsible for the fact that dynamical fermions suppress all Polyakov loop sectors except for the one with the fewest small eigenvalues. It might be possible to develop this simple qualitative picture into a quantitative understanding of how the loss of confinement above T_c implies the restoration of chiral symmetry. Further work in this direction is in progress.

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