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$\mathcal{O}(a^2)$ corrections to the propagator and bilinears of Wilson / clover fermions

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• We present the corrections to the fermion propagator, to second order in the lattice spacing a, in 1-loop perturbation theory. The fermions are described by the clover action and for the gluons we use a 3-parameter family of Symanzik improved actions. Our calculation has been carried out in a general covariant gauge. The results are provided as a polynomial of the clover parameter $c_{\rm SW}$, and are tabulated for 10 popular sets of the Symanzik coefficients (Plaquette, Tree-level Symanzik, Iwasaki, TILW and DBW2 action). • We also study the $O(a^2)$ corrections to matrix elements of fermion bilinear operators that have the form $\overline{\Psi}\Gamma\Psi$, where Γ denotes all possible distinct products of Dirac matrices. These correction terms are essential ingredients for improving, to $O(a^2)$, the matrix elements of the fermion operators. • Our results are applicable also to the case of twisted mass fermions.

A longer write-up of this work, including non-perturbative results, is in preparation together with V. Giménez, V. Lubicz and D. Palao [1].

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1. Introduction

Over the years, many efforts have been made for $\mathcal{O}(a^1)$ improvement in lattice observables, which in many cases is automatic by virtue of symmetries of the fermion action. According to Symanzik's program [2], one can improve the action by adding irrelevant operators. Also, in the twisted mass formulation of QCD [3] at maximal twist, certain observables are $\mathcal{O}(a^1)$ improved, by symmetry considerations.

So far, in the literature there appear two kinds of perturbative evaluations pertaining to the fermion propagator and bilinears of the form $\overline{\Psi}\Gamma\Psi$ (Γ denotes all possible distinct products of Dirac matrices). On the one hand, there are 1-loop computations for $\mathcal{O}(a^1)$ corrections, with an arbitrary fermion mass [4, 5]. On the other hand, there are 2-loop calculations at $\mathcal{O}(a^0)$ level, for massless fermions [6]. 1-loop computations of $\mathcal{O}(a^2)$ corrections did not exist to date; indeed they present some novel difficulties. In particular, extending $\mathcal{O}(a^0)$ calculations up to $\mathcal{O}(a^1)$ does not bring in any novel types of singularities. For instance, terms which were convergent to $\mathcal{O}(a^0)$ may now develop an infrared (IR) logarithmic singularity at worst in 4 dimensions and the way to treat such singularities is well known. In most of the cases, e.g. for $m = 0$, terms which were already IR divergent to $\mathcal{O}(a^0)$ will not contribute to $\mathcal{O}(a^1)$, by parity of loop integration. On the contrary, the IR singularities encountered at $\mathcal{O}(a^2)$ are present even in 6 dimensions, making their extraction more delicate.

2. Description of the calculation

Our calculation is performed for clover fermions, keeping the coefficient $c_{\rm SW}$ as a free parameter. The action describing N_f flavors of degenerate clover (SW) fermions is given in Ref. [6]. We work with massless fermions ($m_{\rm o} = 0$), which simplifies the algebraic expressions, but at the same time requires special treatment for the IR singularities. By taking $m_{\rm o} = 0$, our calculation and results are identical also for the twisted mass action in the chiral limit.

For the gluon part we employ the Symanzik improved action, involving Wilson loops with 4 and 6 links; for the Symanzik coefficients, c_i, multiplying each Wilson loop, we choose 10 sets of values that are widely used in numerical simulations; these are tabulated in Ref. [1].

The Feynman diagrams that enter this computation are shown in Fig. 1; diagrams 1 and 2 contribute to the fermion propagator, while diagram 3 is relevant to the bilinears' improvement.

Fig. 1: Diagrams contributing to the improvement of the propagator (1, 2) and the bilinears (3). A wavy (solid) line represents gluons (fermions). A cross denotes an insertion of Γ (Eq. (4.1)).

For the algebraic operations involved in manipulating lattice Feynman diagrams, we make use of our symbolic package in Mathematica. Next, we briefly describe the required steps:

• The evaluation of each diagram starts with the contraction among vertices, which is performed automatically once the vertices and the topology of the diagram are specified. The outcome of the

contraction is a preliminary expression for the diagram under study; there follow simplifications of the color dependence, Dirac matrices and tensor structures. We also fully exploit symmetries of the theory to limit the proliferation of the algebraic expressions.

• The above simplifications are followed by the extraction of all functional dependencies on the external momentum p (divergent, convergent terms) and the lattice spacing (terms of order a^0, a^1, a^2). The convergent terms can be treated by naive Taylor expansion in $a\,p$ to the desired order. On the contrary, the isolation of the logarithms and non-Lorentz invariant terms is achieved as described below. As a first task we want to reduce the number of infrared divergent integrals to a minimal set. To do this, we use *iteratively* two kinds of subtractions for the propagator, so that all primitively divergent integrals (initially depending on the fermion and the Symanzik propagator) are expressed in terms of the Wilson propagator. The subtraction for the gluon propagator reads

$$D(q) = D_{plaq}(q) + D_{plaq}(q)\left(D^{-1}_{plaq}(q) - D^{-1}(q)\right)D(q) \tag{2.1}$$

$$D^{\mu\nu}_{plaq}(q) = \frac{\delta_{\mu\nu}}{\hat{q}^2} - (1-\lambda)\frac{\hat{q}_\mu\,\hat{q}_\nu}{(\hat{q}^2)^2}, \quad \hat{q}_\mu = 2\sin(\frac{q_\mu}{2}), \quad \hat{q}^2 = \sum_\mu \hat{q}^2_\mu \tag{2.2}$$

where $D^{\mu\nu}$ is the 4×4 Symanzik propagator. The matrix $(D^{-1}_{plaq}(q) - D^{-1}(q))$, which is $\mathcal{O}(q^4)$, is independent of the gauge parameter, λ, and can be obtained in closed form, as a polynomial in $\hat{q}_\mu$.

• The most laborious part is the computation of the divergent terms, which is performed in a noninteger number of dimensions $D > 4$. Ultraviolet divergences are explicitly isolated à la Zimmermann and evaluated as in the continuum. The remainders are D-dimensional, parameter-free, zero momentum lattice integrals which can be recast in terms of Bessel functions, and finally expressed as sums of a pole part plus numerical constants. We analytically evaluated an extensive basis of superficially divergent loop integrals, which is presented in Ref. [1]. A few of these integrals are very demanding, because they must be evaluated to two further orders in a, beyond the order at which an IR divergence initially sets in. As a consequence, their evaluation requires going to $D > 6$ dimensions, with due care to take into account all possible sources of $\mathcal{O}(a^2)$ corrections. These integrals form a sufficient basis for all integrals which can appear in any $\mathcal{O}(a^2)$ 1-loop calculation.

• The required numerical integrations over loop momenta are performed by highly optimized Fortran programs; these are generated by our Mathematica 'integrator' routine. Each integral is expressed as a sum over the discrete Brillouin zone of finite lattices, with varying size L ($L^4 \leq 128^4$), and evaluated for all values of the Symanzik coefficients which we considered.

• The last part of the evaluation is the extrapolation of the numerical results to infinite lattice size. This procedure entails a systematic error, which is reliably estimated, using a sophisticated inference technique; for 1-loop quantities we expect a fractional error smaller than 10^{-8}.

3. Correction to the fermion propagator

The 1-loop corrections to the fermion propagator arise from the evaluation of diagrams 1 and 2 in Fig. 1. Capitani et al. [5] have calculated the first order terms in the lattice spacing for massive Wilson fermions and the Plaquette action for gluons. We carried out this calculation beyond the first order correction, taking into account all terms up to $\mathcal{O}(a^2)$ and considering a general Symanzik improved gluon action. Our results, to $\mathcal{O}(a^1)$, are in perfect agreement with those of Ref. [5].

The following equation is the total expression for the inverse propagator S^{-1} as a function of the external momentum p, the coupling constant g, the number of colors N, the clover coefficient $c_{\rm SW}$, and the gauge parameter λ. The quantities $\varepsilon^{(i,j)}$ appearing in our results for S^{-1} are numerical coefficients depending on the Symanzik parameters, calculated for each action we have considered; they are tabulated in Ref. [1]. In Eq. (3.2) and Eqs. (4.5), (4.8) we present the values of $\varepsilon^{(i,j)}$ for the Plaquette and Iwasaki actions (top and bottom numbers, respectively); only 5 decimal points are shown, due to lack of space

$$
\begin{aligned}
S^{-1}(p) \;=\; & i\not{p} + \frac{a}{2}p^2 - i\frac{a^2}{6}\not{p}^3 - i\not{p}\tilde{g}^2\Big[\varepsilon^{(0,1)} - 4.79201\,\lambda + \varepsilon^{(0,2)}c_{\rm SW} + \varepsilon^{(0,3)}c_{\rm SW}^2 + \lambda\ln(a^2p^2)\Big] \\
& - a\,p^2\,\tilde{g}^2\Big[\varepsilon^{(1,1)} - 3.86388\,\lambda + \varepsilon^{(1,2)}c_{\rm SW} + \varepsilon^{(1,3)}c_{\rm SW}^2 - \frac{1}{2}\,(3 - 2\lambda - 3c_{\rm SW})\ln(a^2p^2)\Big] \\
& - i a^2\,\not{p}^3\,\tilde{g}^2\Big[\varepsilon^{(2,1)} + 0.50700\,\lambda + \varepsilon^{(2,2)}c_{\rm SW} + \varepsilon^{(2,3)}c_{\rm SW}^2 + \left(\frac{101}{120} - \frac{11}{30}C_2 - \frac{\lambda}{6}\right)\ln(a^2p^2)\Big] \\
& - i a^2\,p^2\,\not{p}\tilde{g}^2\Big[\varepsilon^{(2,4)} + 1.51605\,\lambda + \varepsilon^{(2,5)}c_{\rm SW} + \varepsilon^{(2,6)}c_{\rm SW}^2 \\
& \qquad + \left(\frac{59}{240} + \frac{c_1}{2} + \frac{C_2}{60} - \frac{1}{4}\left(\frac{3}{2}\lambda + c_{\rm SW} + c_{\rm SW}^2\right)\right)\ln(a^2p^2)\Big] \\
& - i a^2\,\not{p}\,\frac{\sum_\mu p_\mu^4}{p^2}\,\tilde{g}^2\Big[-\frac{3}{80} - \frac{C_2}{10} - \frac{5}{48}\lambda\Big]
\end{aligned}
\tag{3.1}
$$

We define $\tilde{g}^2 \equiv g^2 C_F/(16\pi^2)$, $C_F = (N^2-1)/(2N)$, $C_2 = c_1 - c_2 - c_3$ and $\not{p}^3 = \sum_\mu \gamma_\mu p_\mu^3$; the specific values $\lambda = 1\,(\lambda = 0)$ correspond to the Feynman (Landau) gauge. We observe that the $\mathcal{O}(a^1)$ logarithms are independent of the Symanzik coefficients c_i ; on the contrary $\mathcal{O}(a^2)$ logarithms have a mild dependence on c_i .

$$
\begin{aligned}
&\varepsilon^{(0,1)} = \begin{matrix}16.64441\\8.11657\end{matrix}, \quad \varepsilon^{(0,2)} = \begin{matrix}-2.24887\\-1.60101\end{matrix}, \quad \varepsilon^{(0,3)} = \begin{matrix}-1.39727\\-0.97321\end{matrix}, \quad \varepsilon^{(1,1)} = \begin{matrix}12.82693\\7.40724\end{matrix} \\
&\varepsilon^{(1,2)} = \begin{matrix}-5.20234\\-3.88884\end{matrix}, \quad \varepsilon^{(1,3)} = \begin{matrix}-0.08173\\-0.06103\end{matrix}, \quad \varepsilon^{(2,1)} = \begin{matrix}-4.74536\\-3.20180\end{matrix}, \quad \varepsilon^{(2,2)} = \begin{matrix}0.02029\\0.08250\end{matrix} \\
&\varepsilon^{(2,3)} = \begin{matrix}0.10349\\0.04192\end{matrix}, \quad \varepsilon^{(2,4)} = \begin{matrix}-1.50481\\-0.62022\end{matrix}, \quad \varepsilon^{(2,5)} = \begin{matrix}0.70358\\0.55587\end{matrix}, \quad \varepsilon^{(2,6)} = \begin{matrix}0.53432\\0.41846\end{matrix}
\end{aligned}
\tag{3.2}
$$

4. Improved operators

In the context of this work we also compute the contributions up to $\mathcal{O}(a^2)$ to the forward matrix elements of local fermion operators that have the form $\overline{\Psi}\Gamma\Psi$. Γ corresponds to the following set of products of the Dirac matrices

$$
\Gamma = \hat{1}\ \text{(scalar)}, \quad \gamma^5\ \text{(pseudoscalar)}, \quad \gamma_\mu\ \text{(vector)}, \quad \gamma^5\gamma_\mu\ \text{(axial)}, \quad \frac{1}{2}\gamma^5[\gamma_\mu,\gamma_\nu]\ \text{(tensor)} \tag{4.1}
$$

The $\mathcal{O}(a^2)$ correction terms are derived from the evaluation of diagram 3 shown in Fig. 1. One may improve the local bilinears by the addition of higher-dimension operators

$$
\left(\mathcal{O}^\Gamma\right)^{\rm imp} = \overline{\Psi}\Gamma\Psi + a\,g_0^2\sum_{i=1}^{n} k_i^\Gamma\,\overline{\Psi}Q_i^\Gamma\Psi + a^2\,g_0^2\sum_{i=1}^{\tilde{n}}\tilde{k}_i^\Gamma\,\overline{\Psi}\tilde{Q}_i^\Gamma\Psi \tag{4.2}
$$

where the first term is the unimproved operator and $\Psi Q_i^\Gamma \Psi$ ($\Psi \tilde{Q}_i^\Gamma \Psi$) are operators with the same symmetries as the original ones, but with dimension higher by one (two) units. To achieve $\mathcal{O}(a^2)$ improvement, we must choose k_i^Γ and $\tilde{k}_i^\Gamma$ appropriately, in order to cancel out all $\mathcal{O}(a^1, a^2)$ contributions in matrix elements.

In the rest of this section we show our results for the 1-loop corrections to the amputated 2-point Green's function (diagram 3 of Fig. 1), at momentum p: $\Lambda^\Gamma(p) = \langle \Psi \left(\overline{\Psi}\Gamma\Psi\right) \overline{\Psi} \rangle_{(p)}^{amp}$. The values of all the Symanzik dependent coefficients, $\varepsilon_S, \varepsilon_P, \varepsilon_V, \varepsilon_A, \varepsilon_T$, with their systematic errors, can be found in Ref. [1]. We begin with the $\mathcal{O}(a^2)$ corrected expressions for $\Lambda^S(p)$ and $\Lambda^P(p)$; including the tree-level term, we obtain

$$\begin{aligned}
\Lambda^S(p) \;=\; & 1 + \tilde{g}^2 \Big[\varepsilon_S^{(0,1)} + 5.79201\,\lambda + \varepsilon_S^{(0,2)}\, c_{\rm SW} + \varepsilon_S^{(0,3)}\, c_{\rm SW}^2 - \ln(a^2 p^2)\,(3+\lambda) \Big] \\
& + a\, i \not{p} \tilde{g}^2 \Big[\varepsilon_S^{(1,1)} - 3.93576\,\lambda + \varepsilon_S^{(1,2)}\, c_{\rm SW} + \varepsilon_S^{(1,3)}\, c_{\rm SW}^2 + \left(\frac{3}{2} + \lambda + \frac{3}{2} c_{\rm SW} \right) \ln(a^2 p^2) \Big] \\
& + a^2\, p^2\, \tilde{g}^2 \Big[\varepsilon_S^{(2,1)} - 2.27359\,\lambda + \varepsilon_S^{(2,2)}\, c_{\rm SW} + \varepsilon_S^{(2,3)}\, c_{\rm SW}^2 + \left(-\frac{1}{4} + \frac{3}{4}\lambda + \frac{3}{2} c_{\rm SW} \right) \ln(a^2 p^2) \Big] \\
& + a^2\, \frac{\sum_\mu p_\mu^4}{p^2}\, \tilde{g}^2 \Big[\frac{13}{24} + \frac{C_2}{2} - \frac{\lambda}{8} \Big]
\end{aligned} \tag{4.3}$$

$$\begin{aligned}
\Lambda^P(p) \;=\; & \gamma^5 + \gamma^5 \tilde{g}^2 \Big[\varepsilon_P^{(0,1)} + 5.79201\,\lambda + \varepsilon_P^{(0,2)}\, c_{\rm SW}^2 - \ln(a^2 p^2)\,(3+\lambda) \Big] \\
& + a^2\, p^2\, \gamma^5 \tilde{g}^2 \Big[\varepsilon_P^{(2,1)} - 0.83810\,\lambda + \varepsilon_P^{(2,2)}\, c_{\rm SW}^2 + \left(-\frac{1}{4} + \frac{1}{4}\lambda \right) \ln(a^2 p^2) \Big] \\
& + a^2\, \frac{\sum_\mu p_\mu^4}{p^2}\, \gamma^5 \tilde{g}^2 \Big[\frac{13}{24} + \frac{C_2}{2} - \frac{\lambda}{8} \Big]
\end{aligned} \tag{4.4}$$

$\Lambda^P(p)$ is free of $\mathcal{O}(a^1)$ terms and all contributions linear in $c_{\rm SW}$ vanish. The values of the numerical coefficients ε_S and ε_P for the Wilson and Iwasaki gluon action are

$$\begin{aligned}
&\varepsilon_S^{(0,1)} = \begin{matrix} 0.30800 \\ 0.74092 \end{matrix}, \quad \varepsilon_S^{(0,2)} = \begin{matrix} 9.98678 \\ 6.90168 \end{matrix}, \quad \varepsilon_S^{(0,3)} = \begin{matrix} 0.01689 \\ -0.29335 \end{matrix}, \quad \varepsilon_S^{(1,1)} = \begin{matrix} 0.65863 \\ -0.05097 \end{matrix}, \quad \varepsilon_S^{(1,2)} = \begin{matrix} -4.20299 \\ -2.88571 \end{matrix} \\
&\varepsilon_S^{(1,3)} = \begin{matrix} -1.28605 \\ -0.90950 \end{matrix}, \quad \varepsilon_S^{(2,1)} = \begin{matrix} 2.60041 \\ 2.02123 \end{matrix}, \quad \varepsilon_S^{(2,2)} = \begin{matrix} -4.15080 \\ -3.23460 \end{matrix}, \quad \varepsilon_S^{(2,3)} = \begin{matrix} 0.17641 \\ 0.23450 \end{matrix}, \quad \varepsilon_P^{(0,1)} = \begin{matrix} 9.95103 \\ 6.55611 \end{matrix} \\
&\varepsilon_P^{(0,2)} = \begin{matrix} 3.43328 \\ 2.25383 \end{matrix}, \quad \varepsilon_P^{(2,1)} = \begin{matrix} 0.84420 \\ 0.66991 \end{matrix}, \quad \varepsilon_P^{(2,2)} = \begin{matrix} -0.25823 \\ -0.30221 \end{matrix}
\end{aligned} \tag{4.5}$$

The $\mathcal{O}(a^2)$ corrected expressions for $\Lambda^V(p)$, $\Lambda^A(p)$ and $\Lambda^T(p)$ are very complicated, in the sense that there is a variety of momentum contributions and therefore many Symanzik dependent coefficients, as can be seen from Eqs. (4.6) - (4.7). In fact, we relegate our result for $\Lambda^T(p)$ to the longer write up [1]. We also list the coefficients ε_V and ε_A for the Wilson and Iwasaki actions in Eq. (4.8).

$$
\begin{aligned}
\Lambda^V(p) \;=\; & \gamma_\mu + \frac{\not{p}\, p_\mu}{p^2}\,\tilde{g}^2\Big[-2\lambda\Big] + \gamma_\mu\,\tilde{g}^2\Big[\varepsilon_V^{(0,1)} + 4.79201\,\lambda + \varepsilon_V^{(0,2)}\,c_{\rm SW} + \varepsilon_V^{(0,3)}\,c_{\rm SW}^2 - \lambda\ln(a^2p^2)\Big] \\
& + a\,i\,p_\mu\,\tilde{g}^2\Big[\varepsilon_V^{(1,1)} - 0.93576\,\lambda + \varepsilon_V^{(1,2)}\,c_{\rm SW} + \varepsilon_V^{(1,3)}\,c_{\rm SW}^2 + (-3+\lambda+3\,c_{\rm SW})\ln(a^2p^2)\Big] \\
& + a^2\,\gamma_\mu\,p_\mu^2\,\tilde{g}^2\Big[\varepsilon_V^{(2,1)} + \frac{\lambda}{8} + \varepsilon_V^{(2,2)}\,c_{\rm SW} + \varepsilon_V^{(2,3)}\,c_{\rm SW}^2 + \left(-\frac{53}{120} + \frac{11}{10}C_2\right)\ln(a^2p^2)\Big] \\
& + a^2\,\gamma_\mu\,p^2\,\tilde{g}^2\Big[\varepsilon_V^{(2,4)} - 0.81104\,\lambda + \varepsilon_V^{(2,5)}\,c_{\rm SW} + \varepsilon_V^{(2,6)}\,c_{\rm SW}^2 \\
& \qquad + \left(\frac{11}{240} - \frac{c_1}{2} - \frac{C_2}{60} + \frac{\lambda}{8} - \frac{5}{12}\,c_{\rm SW} + \frac{c_{\rm SW}^2}{4}\right)\ln(a^2p^2)\Big] \\
& + a^2\,\not{p}\,p_\mu\,\tilde{g}^2\Big[\varepsilon_V^{(2,7)} + 0.24364\,\lambda + \varepsilon_V^{(2,8)}\,c_{\rm SW} + \varepsilon_V^{(2,9)}\,c_{\rm SW}^2 \\
& \qquad + \left(-\frac{149}{120} - c_1 - \frac{C_2}{30} + \frac{\lambda}{4} + \frac{c_{\rm SW}}{6} + \frac{c_{\rm SW}^2}{2}\right)\ln(a^2p^2)\Big] \\
& + a^2\,\gamma_\mu\,\frac{\sum_\rho p_\rho^4}{p^2}\,\tilde{g}^2\Big[\frac{3}{80} + \frac{C_2}{10} + \frac{5}{48}\,\lambda\Big] + a^2\,\frac{\not{p}^3\,p_\mu}{p^2}\,\tilde{g}^2\Big[-\frac{101}{60} + \frac{11}{15}C_2 + \frac{\lambda}{3}\Big] \\
& + a^2\,\frac{\not{p}\,p_\mu^3}{p^2}\,\tilde{g}^2\Big[-\frac{1}{60} + \frac{2}{5}C_2 + \frac{\lambda}{12}\Big] + a^2\,\frac{\not{p}\,p_\mu\,\sum_\rho p_\rho^4}{(p^2)^2}\,\tilde{g}^2\Big[-\frac{3}{40} - \frac{C_2}{5} - \frac{5}{24}\,\lambda\Big] \qquad (4.6)
\end{aligned}
$$

$$
\begin{aligned}
\Lambda^A(p) \;=\; & \gamma^5\gamma_\mu + \frac{\gamma^5\,\not{p}\,p_\mu}{p^2}\,\tilde{g}^2\Big[-2\lambda\Big] \\
& + \gamma^5\gamma_\mu\,\tilde{g}^2\Big[\varepsilon_A^{(0,1)} + 4.79201\,\lambda + \varepsilon_A^{(0,2)}\,c_{\rm SW} + \varepsilon_A^{(0,3)}\,c_{\rm SW}^2 - \lambda\ln(a^2p^2)\Big] \\
& + a\,i\,\gamma^5\,\left(\gamma_\mu\,\not{p} - p_\mu\right)\,\tilde{g}^2\Big[\varepsilon_A^{(1,1)} - 2.93576\,\lambda + \varepsilon_A^{(1,2)}\,c_{\rm SW} + \varepsilon_A^{(1,3)}\,c_{\rm SW}^2 + \lambda\ln(a^2p^2)\Big] \\
& + a^2\,\gamma^5\,\gamma_\mu\,p_\mu^2\,\tilde{g}^2\Big[\varepsilon_A^{(2,1)} + \frac{\lambda}{8} + \varepsilon_A^{(2,2)}\,c_{\rm SW} + \varepsilon_A^{(2,3)}\,c_{\rm SW}^2 + \left(-\frac{53}{120} + \frac{11}{10}C_2\right)\ln(a^2p^2)\Big] \\
& + a^2\,\gamma^5\,\gamma_\mu\,p^2\,\tilde{g}^2\Big[\varepsilon_A^{(2,4)} - 1.74652\,\lambda + \varepsilon_A^{(2,5)}\,c_{\rm SW} + \varepsilon_A^{(2,6)}\,c_{\rm SW}^2 \\
& \qquad + \left(-\frac{109}{240} - \frac{c_1}{2} - \frac{C_2}{60} + \frac{5}{8}\lambda + \frac{7}{12}\,c_{\rm SW} - \frac{c_{\rm SW}^2}{4}\right)\ln(a^2p^2)\Big] \\
& + a^2\,\gamma^5\,\not{p}\,p_\mu\,\tilde{g}^2\Big[\varepsilon_A^{(2,7)} + 1.11462\,\lambda + \varepsilon_A^{(2,8)}\,c_{\rm SW} + \varepsilon_A^{(2,9)}\,c_{\rm SW}^2 \\
& \qquad + \left(\frac{91}{120} - c_1 - \frac{C_2}{30} - \frac{3}{4}\lambda - \frac{5}{6}\,c_{\rm SW} - \frac{c_{\rm SW}^2}{2}\right)\ln(a^2p^2)\Big] \\
& + a^2\,\gamma^5\,\gamma_\mu\,\frac{\sum_\rho p_\rho^4}{p^2}\,\tilde{g}^2\Big[\frac{3}{80} + \frac{C_2}{10} + \frac{5}{48}\,\lambda\Big] + a^2\,\gamma^5\,\frac{\not{p}^3\,p_\mu}{p^2}\,\tilde{g}^2\Big[-\frac{101}{60} + \frac{11}{15}C_2 + \frac{\lambda}{3}\Big] \\
& + a^2\,\gamma^5\,\frac{\not{p}\,p_\mu^3}{p^2}\,\tilde{g}^2\Big[-\frac{1}{60} + \frac{2}{5}C_2 + \frac{\lambda}{12}\Big] + a^2\,\gamma^5\,\frac{\not{p}\,p_\mu\,\sum_\rho p_\rho^4}{(p^2)^2}\,\tilde{g}^2\Big[-\frac{3}{40} - \frac{C_2}{5} - \frac{5}{24}\,\lambda\Big] \qquad (4.7)
\end{aligned}
$$

$$\begin{array}{llllllllllllll}
\varepsilon_V^{(0,1)} = \begin{array}{r} 3.97338 \\ 2.98283 \end{array}, & \varepsilon_V^{(0,2)} = \begin{array}{r} -2.49670 \\ -1.72542 \end{array}, & \varepsilon_V^{(0,3)} = \begin{array}{r} 0.85410 \\ 0.63680 \end{array}, & \varepsilon_V^{(1,1)} = \begin{array}{r} 2.71098 \\ 0.90743 \end{array}, & \varepsilon_V^{(1,2)} = \begin{array}{r} -1.84814 \\ -0.80352 \end{array} \\
\varepsilon_V^{(1,3)} = \begin{array}{r} -0.39053 \\ -0.35601 \end{array}, & \varepsilon_V^{(2,1)} = \begin{array}{r} 1.55410 \\ 1.45731 \end{array}, & \varepsilon_V^{(2,2)} = \begin{array}{r} 0.32907 \\ 0.08590 \end{array}, & \varepsilon_V^{(2,3)} = \begin{array}{r} -0.00602 \\ 0.07935 \end{array}, & \varepsilon_V^{(2,4)} = \begin{array}{r} 0.25007 \\ -0.26685 \end{array} \\
\varepsilon_V^{(2,5)} = \begin{array}{r} 0.88599 \\ 0.71279 \end{array}, & \varepsilon_V^{(2,6)} = \begin{array}{r} -0.30036 \\ -0.25078 \end{array}, & \varepsilon_V^{(2,7)} = \begin{array}{r} 1.27888 \\ 0.76263 \end{array}, & \varepsilon_V^{(2,8)} = \begin{array}{r} 0.27776 \\ 0.29755 \end{array}, & \varepsilon_V^{(2,9)} = \begin{array}{r} -0.35475 \\ -0.18427 \end{array} \\
\varepsilon_A^{(0,1)} = \begin{array}{r} -0.84813 \\ 0.07524 \end{array}, & \varepsilon_A^{(0,2)} = \begin{array}{r} 2.49670 \\ 1.72542 \end{array}, & \varepsilon_A^{(0,3)} = \begin{array}{r} -0.85410 \\ -0.63680 \end{array}, & \varepsilon_A^{(1,1)} = \begin{array}{r} 1.34275 \\ 0.26850 \end{array}, & \varepsilon_A^{(1,2)} = \begin{array}{r} -1.71809 \\ -1.23802 \end{array} \\
\varepsilon_A^{(1,3)} = \begin{array}{r} 0.13018 \\ 0.11867 \end{array}, & \varepsilon_A^{(2,1)} = \begin{array}{r} 0.38791 \\ 0.05918 \end{array}, & \varepsilon_A^{(2,2)} = \begin{array}{r} 1.85117 \\ 1.57070 \end{array}, & \varepsilon_A^{(2,3)} = \begin{array}{r} -0.09309 \\ -0.13933 \end{array}, & \varepsilon_A^{(2,4)} = \begin{array}{r} 1.63504 \\ 0.68458 \end{array} \\
\varepsilon_A^{(2,5)} = \begin{array}{r} -1.59946 \\ -1.24801 \end{array}, & \varepsilon_A^{(2,6)} = \begin{array}{r} 0.33390 \\ 0.26827 \end{array}, & \varepsilon_A^{(2,7)} = \begin{array}{r} 0.41759 \\ 1.05772 \end{array}, & \varepsilon_A^{(2,8)} = \begin{array}{r} 0.39585 \\ 0.18672 \end{array}, & \varepsilon_A^{(2,9)} = \begin{array}{r} 0.31972 \\ 0.17429 \end{array}
\end{array} \tag{4.8}$$

5. Conclusions

Our results show clearly that $\mathcal{O}(a^2)$ effects are quite pronounced in the Green's functions we have considered. The $\mathcal{O}(a^2)$ contributions which we have calculated can be directly used in order to construct improved operators, bringing the chiral limit within reach. Possible follow-ups to the present work include:

- Extending to the case of nonzero renormalized mass.
- Improvement of higher-dimension bilinear operators, such as those involved in hadronic form factors, and of 4-fermi operators.

A comparison with non-perturbative estimates of matrix elements, coming from numerical simulations, will be presented in Ref. [1].

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