

The determination of $\alpha_s(M_Z)$ from perturbative analyses of short-distance-sensitive lattice QCD observables revisited

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The determination of $\alpha_s(M_Z)$ via perturbative analyses of short-distance-sensitive lattice observables is revisited, incorporating new lattice data and performing a modified version of the original analysis. The analysis employs two high-intrinsic-scale observables, $\log(W_{11})$ and $\log(W_{12})$, and one lower-intrinsic-scale observable, $\log(W_{12}/u_0^6)$. We find good consistency among the values extracted using the different observables and a final result, $\alpha_s(M_Z) = 0.1192 \pm 0.0011$, in excellent agreement with various recent non-lattice determinations, as well as with the results of a similar, but not identical, re-analysis by the HPQCD collaboration. The relation between the two re-analyses is discussed, focussing on the complementarity of the two approaches.

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1. Introduction and Background

The $n_f = 5$ QCD coupling in the \overline{MS} scheme at the conventionally defined reference scale $\mu = M_Z$ represents one of the fundamental parameters of the Standard Model. The central value of the 2008 PDG assessment [1], $\alpha_s(M_Z) = 0.1176(20)$, remains strongly influenced by the high-precision lattice result, $\alpha_s(M_Z) = 0.1170(12)$, obtained in Ref. [3] from an analysis of UV-sensitive lattice observables using the MILC $a \sim 0.18, 0.12$, and 0.09 fm ensembles. In the last year, a number of independent determinations have appeared, from a number of different sources, yielding typically somewhat higher values. Specifically,

- updates of the global EW fit, taking into account the new 5-loop result for the dimension 0 OPE contributions [4], yield $\alpha_s(M_Z) = 0.1191(27)$ [4, 5];
- the most recent hadronic τ decay determinations [6, 7], whose ranges are encompassed by results of Ref. [7], yield $\alpha_s(M_Z) = 0.1187(16)$ ($\sim 2\sigma$ lower than the earlier result of Ref. [5], for the reasons discussed in Refs. [6, 7]);
- the determination based on 2 to 10.6 GeV R_{had} values yields $\alpha_s(M_Z) = 0.1190^{(+90)}_{(-110)}$ [8];
- an update of the determination from $\frac{\Gamma[\Upsilon(1s) \rightarrow \gamma X]}{\Gamma[\Upsilon(1s) \rightarrow X]}$ yields $\alpha_s(M_Z) = 0.1190^{(+60)}_{(-50)}$ [9];
- recent determinations, $\alpha_s(M_Z) = 0.1198(32)$ [10], $0.1182(45)$ [11], $0.1240(33)$ [12] and $0.1172(22)$ [13], using shape observables in DIS and $e^+e^- \rightarrow hadrons$, if averaged naively, yield a combined result $\alpha_s(M_Z) = 0.1193(15)$.

In view of these results, and, in addition, the availability of the new MILC ensembles with $a \sim 0.15$ and 0.06 fm, it is timely to revisit the earlier lattice analysis. We do so by focussing on three observables, $\log(W_{11})$, $\log(W_{12})$ and $\log(W_{12}/u_0^6)$ which can be convincingly argued to receive only small non-perturbative contributions at the scales of the lattices employed in our analysis. In what follows, we first briefly outline the basics of the method employed in Ref. [3], then describe our implementation of this approach, and finally present our results. We also discuss briefly the differences (and complementarity) between our implementation and that of the other recent similar, but not identical, reanalysis by HPQCD [14]. An expanded discussion of the work reported here may be found in Ref. [15].

The authors of Ref. [3] extracted $\alpha_s(M_Z)$ by studying a large number of UV-sensitive lattice observables, including the three, $\log(W_{11})$, $\log(W_{12})$ and $\log(W_{12}/u_0^6)$ on which we focus below. The perturbative expansion for such an observable, O_k , is written in the form

$$O_k = \sum_{N=1} d_N^{(k)} \hat{\alpha}(Q_k)^N \equiv D_k \hat{\alpha}(Q_k) \sum_{M=0} c_M^{(k)} \hat{\alpha}(Q_k)^M \quad (1.1)$$

with $c_0^{(k)} \equiv 1$, $Q_k = d_k/a$ the Brodsky-Lepage-Mackenzie (BLM) scale for the observable O_k , and $\hat{\alpha}$ any coupling having the same expansion to $O(\alpha_s^3)$ (with α_s the usual \overline{MS} coupling) as the usual heavy quark potential coupling, denoted α_V^p below. The coefficients $d_{1,2,3}^{(k)}$ (equivalently, $D_k, c_1^{(k)}$, and $c_2^{(k)}$) have been computed in 3-loop lattice perturbation theory [16] for a number of such observables and, with the corresponding d_k , tabulated in Refs. [3, 14, 16]. They are common for all such

couplings $\hat{\alpha}$. The couplings $\hat{\alpha}$ also share common values for the first three β function coefficients, $\hat{\beta}_0 = 9/4$, $\hat{\beta}_1 = 4$ and $\hat{\beta}_2 = 33.969$, where in our normalization $\mu^2 d\hat{\alpha}(\mu)/d\mu^2 = -\sum_{n=0} \hat{\beta}_n \hat{\alpha}^n(\mu)$ with $\hat{\alpha} \equiv \hat{\alpha}/\pi$. When the expansion of $\hat{\alpha}$ is further specified to $O(\alpha_s^4)$, $\hat{\beta}_3$ is also determined from the known values of the 4-loop \overline{MS} β function coefficients, β_0, \dots, β_3 [17]. The $\hat{\alpha}(Q_k)$ appearing in Eq. (1.1) are determined by the value at a single reference scale, the reference scale value serving as a fit parameter for the analysis. It is, of course, important to remove any non-perturbative contributions to the observable in question in order to make use of Eq. (1.1). The reliability of the analysis will be greatest when such non-perturbative subtractions are small.

Somewhat different choices for $\hat{\alpha}$ are made in the two recent reanalyses. We denote our choice by α_T , and that of Ref. [14] by α_V . The relation between α_V^p and α_s , to $O(\alpha_s^3)$, is of the form [18]

$$\alpha_V^p(q^2) = \alpha_s(\mu^2) [1 + \kappa_1(\mu^2/q^2)\alpha_s(\mu^2) + \kappa_2(\mu^2/q^2)\alpha_s(\mu^2)] \quad (1.2)$$

with the expressions for $\kappa_{1,2}(x)$ given in Ref. [18]. The $n_f = 3$ version of the RHS of Eq. (1.2), with $\mu^2 = q^2$, defines our $\alpha_T(q^2)$. Numerically,

$$\alpha_T(\mu^2) = \alpha_s(\mu^2) [1 + 0.5570\alpha_s(\mu^2) + 1.702\alpha_s^2(\mu^2)] \quad (1.3)$$

This exact (by definition) relation is used to run α_T between different scales using the intermediate α_s coupling, whose running can be reliably performed at 4-loops over the range of scales relevant to the observables considered. The HPQCD coupling, α_V , is defined as follows. Beginning with Eq. (1.2), one takes the RHS, with $\mu^2 = e^{-5/3}q^2$, to define an intermediate coupling, $\alpha'_V(q^2)$. This coupling has a β function, β' , with known values of $\beta'_0, \dots, \beta'_3$, but also non-zero, but unknown, higher order coefficients, $\beta'_{4,5,\dots}$, whose values depend on the presently unknown $\beta_{4,5,\dots}$. The final HPQCD coupling, α_V , is obtained from α'_V by adding terms of $O(\alpha_s^5)$ and higher with coefficients chosen in such a way as to make $\beta_4^V = \beta_5^V = \dots = 0$. Since $\beta_{4,5,\dots}$ are not known, the values of the coefficients needed to implement these constraints are also not known.

Using the expansion parameter α_T , no perturbative uncertainty is encountered in converting the fitted reference scale α_T value to the equivalent reference scale \overline{MS} result. This is not true for the HPQCD parameter α_V . Higher order perturbative uncertainties, however, do remain in our analysis. To see where these occur, and to understand the motivation for the alternate HPQCD choice, let us define $\alpha_0 \equiv \hat{\alpha}(Q_0)$, where Q_0 is the maximum of the BLM scales (corresponding to the finest lattice) for the observable in question. We next expand the couplings at lower BLM scales (coarser lattices) for the same observable, in the standard manner as a power series in α_0 ,

$$\hat{\alpha}(Q_k) = \sum_{N=1} p_N(t_k) \alpha_0^N \quad (1.4)$$

where $t_k = \log(Q_k^2/Q_0^2)$, and the $p_N(t)$ are polynomials in t with coefficients determined by those of $\hat{\beta}$. Substituting this representation into Eq. (1.1), one obtains the following expression, where we replace any occurrences of $\hat{\beta}_0, \dots, \hat{\beta}_2$ with their known numerical values and display only those terms involving one or more of $\hat{\beta}_3$ and the unknown quantities $\hat{\beta}_4, \hat{\beta}_5, \dots, c_3^{(k)}, c_4^{(k)}, \dots$:

$$\begin{aligned} \frac{O_k}{D_k} = & \dots + \alpha_0^4 (c_3^{(k)} + \dots) + \alpha_0^5 (c_4^{(k)} - 0.01027\hat{\beta}_3 - 2.865c_3^{(k)}t_k + \dots) + \alpha_0^6 (c_5^{(k)} - 0.00327\hat{\beta}_4t_k \\ & - 3.581c_4^{(k)}t_k + [0.02573t_k^2 - 0.02053c_1^{(k)}t_k]\hat{\beta}_3[5.129t_k^2 - 1.621t_k]c_3^{(k)} + \dots) \end{aligned}$$

$$\begin{aligned}
& +\alpha_0^7 \left(c_6^{(k)} - 0.001040 \hat{\beta}_5 t_k + [0.009361 t_k^2 - 0.006536 c_1^{(k)} t_k] \hat{\beta}_4 \right. \\
& \quad + [-0.04213 t_k^3 + (0.01664 + 0.06617 c_1^{(k)}) t_k^2 - 0.03080 c_2^{(k)} t_k] \hat{\beta}_3 - 4.297 c_5^{(k)} t_k \\
& \quad \left. + [7.694 t_k^2 - 2.026 t_k] c_4^{(k)} + [-7.347 t_k^3 + 6.386 t_k^2 - 4.382 t_k] c_3^{(k)} + \dots \right) + \dots \quad (1.5)
\end{aligned}$$

Running the \overline{MS} coupling numerically using the 4-loop-truncated β function is equivalent to keeping terms involving β_0, \dots, β_3 to all orders, and setting $\beta_4 = \beta_5 = \dots = 0$. The neglect of β_4, β_5, \dots also alters $\hat{\beta}_4, \hat{\beta}_5, \dots$, and hence produces a “distortion” of the true t_k -dependence, beginning at $O(\alpha_0^6)$. Since it is the scale-dependence of O_k which allows one to fit the unknown coefficients $c_{3,4,\dots}^{(k)}$, as well as α_0 , it follows that the 4-loop truncation forces compensating changes in at least the coefficients $c_{4,5,\dots}^{(k)}$. A shift in the values of $c_4^{(k)}$, however, leads also to a shift in the $O(\alpha_0^5)$ coefficient which, in general, will necessitate a compensating shift in $c_3^{(k)}$ as well. This in turn will necessitate a shift in α_0 . Since the \overline{MS} β function is known only to 4-loop order, such truncated-running effects are unavoidable at some level. From Eq. (1.5), however, it follows that their size can be minimized by taking Q_0 as large as possible (achieved by working with the observable with the highest intrinsic BLM scale) and keeping t_k from becoming too large (achieved by restricting one’s attention, if possible, to a subset of finer lattices). Note that, by defining α_V in such a way that the 4-loop-truncated β function β^V is exact, the HPQCD expansion parameter choice, by definition, avoids these truncated-running problems. The price paid is the unknown relation between α_V and α_s beyond $O(\alpha_s^4)$. The impact of this uncertainty on $\alpha_s(M_Z)$ cannot be controlled, either through the choice of observable or through the restriction to a subset of finer lattices.

From the discussion above we see that the two different coupling choices lead to complementary analyses. If the impact of the neglect of higher order perturbative corrections in both cases is small, the two approaches should give compatible results for analyses based on the same observables, providing a form of mutual cross-check. Good agreement is, indeed, found (see Ref. [15] for details).

2. Results

We now turn to the results of our analysis. We employ data on our observables from the MILC $a \sim 0.06, 0.09, 0.12, 0.15$ and 0.18 fm ensembles [19]. To minimize incompletely incorporated higher order perturbative contributions, we perform our main analysis using the three finest lattices, expanding to full 5-fold fits to test the stability of our solutions. The physical scales for the various ensembles are determined using the measured values of r_1/a and the recent MILC assessment, $r_1 = 0.318(7)$ [20]. The uncertainties on the extracted α_s values associated with those on r_1/a and r_1 are added linearly to arrive at an “overall scale uncertainty” contribution to the total error.

Quark-mass-dependent non-perturbative contributions are found, using the data for different mass combinations am_ℓ/am_s , to be very linear in $2am_\ell + am_s$, allowing these contributions to be fitted and removed with good reliability. Because of details of the analysis not discussed here, the uncertainty in this subtraction appears as part of “the overall scale uncertainty” discussed above [15].

Mass-independent non-perturbative contributions are assumed dominated by the $D = 4$ gluon condensate contribution. The corresponding leading order contribution to the $m \times n$ Wilson loop

W_{mn} , denoted by $\delta_g W_{mn}$, is known from Ref. [21],

$$\delta_g W_{mn} = \frac{-\pi^2}{36} m^2 n^2 a^4 \left\langle \frac{\alpha_s}{\pi} G^2 \right\rangle. \quad (2.1)$$

As central input for the condensate we employ the result of the updated charmonium sum rule analysis of Ref. [22], $\langle \frac{\alpha_s}{\pi} G^2 \rangle = (0.009 \pm 0.007) \text{ GeV}^4$. Since the error is already close to 100%, we take the difference between results obtained with and without the resulting subtraction as a measure of the associated uncertainty. The neglect of mass-independent non-perturbative contributions with $D > 4$, for which no pre-existing constraints are available, should be safe so long as the estimated $D = 4$ gluon condensate subtraction is small.

The shifts associated with the central gluon condensate subtraction grow with increasing lattice spacing a and, for the observables considered here, are (i) -0.01% , -0.02% and -0.08% for $a \sim 0.06$ fm, (ii) -0.1% , -0.4% and -1.3% for $a \sim 0.12$ fm, and (iii) -0.5% , -1.8% and -5.6% for $a \sim 0.18$ fm, where in each case the values quoted correspond to the observables $\log(W_{11})$, $\log(W_{12})$ and $\log(W_{12}/u_0^6)$, in that order. The corrections, as claimed, are small, making the mass-independent non-perturbative subtraction safe for both the central 3-fold and extended 5-fold fits [23]. The subtractions are particularly small for the three finest lattices and for the plaquette observable, $\log(W_{11})$.

In line with the results of Ref. [3], we find that, even for the highest-scale observables and three finest lattices, the known terms in the perturbative expansion of the O_k are insufficient to provide a description of the observed scale-dependence. When $c_3^{(k)}$ is added to the fit, however, very good fit qualities are found, with $\chi^2/dof < 1$ (very significantly so for the 3-fold fits). It follows that, with our current errors, it is not possible to sensibly fit additional coefficients in the expansions of the O_k . This raises concerns about possible associated truncation uncertainties. Since the relative weight of higher order relative to lower order terms grows with decreasing scale, the comparison of the results of the 3-fold and 5-fold fits provides one handle on such a truncation uncertainty. If higher order terms which have been neglected are in fact *not* negligible, then the growth with decreasing scale of the resulting fractional error should show up as an instability in the values of the parameters extracted using the different fits. We see no signs for such an instability within the errors of our fits, but nonetheless include a component equal to the difference of central values obtained from the 3-fold and 5-fold fits as part of our error estimate. This “stability component” is added in quadrature with the overall scale uncertainty, the gluon condensate subtraction uncertainty, and the small uncertainty associated with varying the $c_2^{(k)}$ (and, if relevant, $c_1^{(k)}$) within the errors in their numerical evaluations to arrive at the total error on our results.

After converting our result for the reference scale $n_f = 3$ α_T coupling to the corresponding $n_f = 3$ \overline{MS} value, we run the result to M_Z using the usual self-consistent combination of 4-loop running and 3-loop matching [24], taking the flavor thresholds to lie at $rm_c(m_c)$ and $rm_b(m_b)$, with r varying between 1 and 3, $m_c(m_c) = 1.286 \pm 0.013$ GeV and $m_b(m_b) = 4.164 \pm 0.025$ GeV [25]. The evolution to M_Z produces an additional 0.0003 uncertainty on $\alpha_s(M_Z)$ [4].

Our results for $\alpha_s(M_Z)$, combining all errors as indicated above, are 0.1192(11) from the fit based on $\log(W_{11})$, and 0.1193(11) from those based on $\log(W_{12})$ and $\log(W_{12}/u_0^6)$. The consistency represents an improvement over that of the corresponding results quoted in Ref. [3]. In line with the arguments above, we believe the most reliable analysis to be that obtained using the three

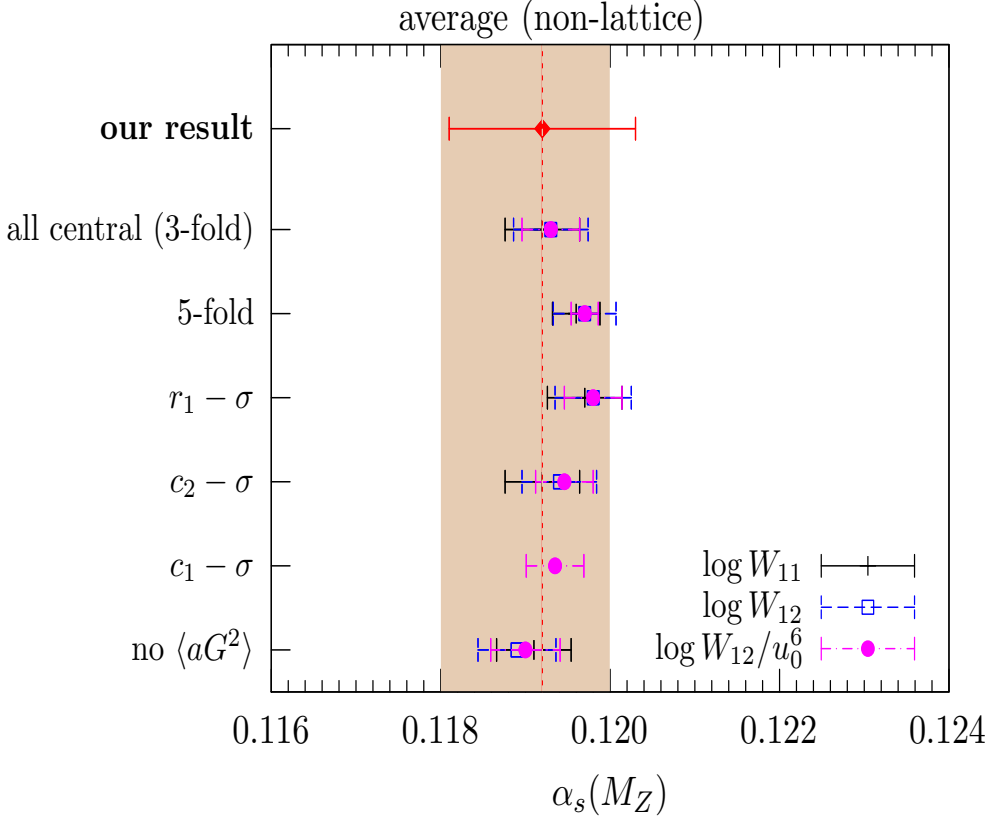


Figure 1: Contributions to the errors on $\alpha_s(M_Z)$. Shown are the results for $\alpha_s(M_Z)$ obtained using (i) the 3-fold fit strategy, with central values for all input (the “central” case), (ii) the alternate 5-fold fit strategy, still with central values for all input, and (iii) the 3-fold fit strategy, with, one at a time, each of the input quantities shifted from its central values by 1σ , retaining central values for the remaining inputs. The error bars shown in each such case are those associated with the uncertainties in r_1/a for the various ensembles.

finest lattices, and the observable, $\log(W_{11})$, having both the highest intrinsic scale and smallest gluon condensate subtractions. Our final determination, based on this case, is thus

$$\alpha_s(M_Z) = 0.1192(11) . \quad (2.2)$$

The dominant contribution, 0.0009, to the total error is that associated with the overall scale uncertainty. A graphical depiction of the various components of the error is given in Figure 1 where, for clarity, only one-sided errors are shown. The results are in excellent agreement with those of the independent determinations mentioned above, whose average is shown in the figure by the shaded band.

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- [23] The reader should bear in mind that the fractional $D = 4$ gluon condensate corrections can be rather large for a number of the Creutz ratios and tadpole-rescaled ratios considered in Ref. [14] (see Ref. [15] for numerical details). The HPQCD analysis in fact allows the gluon condensate to be determined independently for each observable considered, using zero central value and a width given by the conventional SVZ value as priors in the full Bayesian fit. The variation in the values of the gluon condensate across the different observables (as reported to us privately by the HPQCD collaboration) is likely to be due to the presence of neglected mass-independent $D > 4$ contributions. An update by the HPQCD collaboration is in progress to incorporate such higher D contributions into their fit form. The degree to which the output gluon condensate values corresponding to different observables are in agreement will provide a useful test of the consistency of this approach.
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