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The Landau gauge lattice ghost propagator in stochastic perturbation theory

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We present one- and two-loop results for the ghost propagator in Landau gauge calculated in Numerical Stochastic Perturbation Theory (NSPT). The one-loop results are compared with available standard Lattice Perturbation Theory in the infinite-volume limit. We discuss in detail how to perform the different necessary limits in the NSPT approach and discuss a recipe to treat logarithmic terms by introducing "finite-lattice logs". We find agreement with the one-loop result from standard Lattice Perturbation Theory and estimate, from the non-logarithmic part of the ghost propagator in two-loop order, the unknown constant contribution to the ghost self-energy in the RI'-MOM scheme in Landau gauge. That constant vanishes within our numerical accuracy.

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1. NSPT, Langevin equation, gauge fixing and all that

Numerical Stochastic Perturbation Theory (for a review see Ref. [1]) is a powerful tool to study higher-loop contributions in Lattice Perturbation Theory (LPT). LPT is much more involved than perturbation theory in the continuum, and thus only few results beyond one-loop level are available. There have already been various applications of NSPT in the past: the average plaquette to very high orders in pure Yang-Mills theory to identify the gluon condensate [2], the residual mass for lattice HQEF [3], renormalization factors for bilinear quark operators [4], renormalization factors related to the QCD pressure [5] etc. Relatively new is the application of NSPT to gluon and ghost propagators in Yang-Mills theory [6, 7]. Here we report on first steps towards an NSPT study of the ghost propagator in Landau gauge, in particular at two-loop level.

It is known that the lattice Langevin equation with an additional running "time" t, beyond the four physical dimensions, leads to a distribution of the gauge link fields according to the measure $\exp(-S_G[U])$ in the limit $t\to\infty$. Discretizing the time $t = n\tau$ and using the Euler scheme, the equation can be solved numerically by iteration:

$$U_{x,\mu}(n+1;\eta) = \exp(-F_{x,\mu}[U,\eta])\, U_{x,\mu}(n;\eta) \tag{1.1}$$

with a force containing the gradient of S_G and a Gaussian random noise η,

$$F_{x,\mu}[U,\eta] = \mathrm{i}(\tau\nabla_{x,\mu}S_G[U] + \sqrt{\tau}\,\eta_{x,\mu})\,. \tag{1.2}$$

$\nabla_{x,\mu}$ is the left Lie derivative acting on gauge group-valued variables while S_G is Wilson's one-plaquette gauge action.

In NSPT one rescales $\varepsilon = \beta\tau$ and expands the link fields (and the force) in terms of the bare coupling constant $g \propto \beta^{-1/2}$:

$$U_{x,\mu}(t;\eta) \to 1 + \sum_{l>0}\beta^{-l/2}U^{(l)}_{x,\mu}(t;\eta)\,. \tag{1.3}$$

Then the solution (1.1) transforms into a system of updates $U \to U'$, one for each perturbative component $U^{(l)}$:

$$U^{(1)'} = U^{(1)} - F^{(1)}\,, \quad U^{(2)'} = U^{(2)} - F^{(2)} + \frac{1}{2}(F^{(1)})^2 - F^{(1)}U^{(1)}\,, \quad \ldots \tag{1.4}$$

The random noise η is fed in only through $F^{(1)}$, higher orders become stochastic by propagation of noise through the fields of lower order.

In terms of the (algebra-valued) gauge field variables $A = \log U$,

$$A_{x,\mu}(t;\eta) \to \sum_{l>0}\beta^{-l/2}A^{(l)}_{x,\mu}(t;\eta)\,, \quad A^{(l)}_{x,\mu} = T^a A^{a,(l)}_{x,\mu}\,, \tag{1.5}$$

we are enforcing antihermiticity and tracelessness to all orders in g by requiring

$$A^{(l)\dagger} = -A^{(l)}\,, \qquad \mathrm{Tr}A^{(l)} = 0\,. \tag{1.6}$$

The Landau gauge is achieved by iterative gauge transformations using a perturbatively expanded version of the Fourier-accelerated gauge-fixing method [8] applied to each 50-th configuration in the Langevin process. Only these are evaluated in order to control the autocorrelations. Each Langevin update (1.4) is completed by a stochastic gauge-fixing step and by subtracting zero modes of $A^{(l)}$ as described in Ref. [1].

PoS(LATTICE 2008)217

2. The ghost propagator in NSPT and in standard LPT

The continuum ghost propagator $G(q^2)$ in momentum space is defined as $G^{ab}(q) = \delta^{ab} G(q^2)$. On the lattice it is obtained as the color trace

$$G(aq(k)) = \frac{1}{N_c^2 - 1} G^{aa}(aq(k)) = \frac{1}{N_c^2 - 1} \left\langle \mathrm{Tr}\, M^{-1}(k) \right\rangle_U \tag{2.1}$$

as a function of the lattice momenta $aq_\mu(k) = 2\pi k_\mu a / L_\mu$ associated with plane waves $|k\rangle$ labelled by integers $k_\mu = \left(-L_\mu/2, L_\mu/2\right]$. In Landau gauge, the ghost propagator requires the computation of the inverse of the Faddeev-Popov (FP) operator

$$M = -\partial \cdot D(U)\,, \tag{2.2}$$

with $D(U)$ being the lattice covariant derivative and ∂ the left lattice partial derivative. $M^{-1}(k)$ in (2.1) is the Fourier transform of the inverse FP operator.

The perturbative expansion is based on the mapping

$$\{A_{x,\mu}^{(l)}\} \to \{M^{(l)}\} \to \{\left[M^{-1}\right]^{(l)}\}\,. \tag{2.3}$$

With an expansion of M in terms of $M^{(l)}$ containing $A^{(l)}$, a recursive inversion is possible in coordinate space:

$$\left[M^{-1}\right]^{(0)} = \left[M^{(0)}\right]^{-1}, \quad \left[M^{-1}\right]^{(l)} = -\left[M^{0}\right]^{-1} \sum_{j=0}^{l-1} M^{(l-j)} \left[M^{-1}\right]^{(j)}\,. \tag{2.4}$$

The momentum-space ghost propagator at n-loop order is obtained from even orders $l = 2n$ of M^{-1} sandwiching its foregoing expansion between the plane-wave vectors:

$$G^{(n)}(aq(k)) = \langle k | \left[M^{-1}\right]^{(l=2n)} | k \rangle\,. \tag{2.5}$$

Odd l orders have to vanish numerically. We discuss the results in terms of two forms of the dressing function for one and two loops:

$$J^{(n)}(aq) = (aq)^2\, G^{(n)}(aq(k))\,, \quad \hat{J}^{(n)}(\hat{q}) = \hat{q}^2\, G^{(n)}(aq(k)). \tag{2.6}$$

Here we use the standard notation for hat-variables, *e.g.*

$$\hat{q}_\mu(k_\mu) = \frac{2}{a} \sin\left(\frac{\pi k_\mu}{L_\mu}\right) = \frac{2}{a} \sin\left(\frac{a q_\mu}{2}\right). \tag{2.7}$$

In standard LPT, loop contributions are calculated in the infinite volume and $a \to 0$ limit. In this limit the two dressing functions coincide. The renormalization of the dressing function is performed in the RI'-MOM scheme:

$$J^{\mathrm{RI'}}(q, \mu, \alpha_{\mathrm{RI'}}) = \frac{J(a, q, \alpha_{\mathrm{RI'}})}{Z_{\mathrm{gh}}(a, \mu, \alpha_{\mathrm{RI'}})} \tag{2.8}$$

with the renormalization condition

$$J^{\mathrm{RI'}}(q, \mu, \alpha_{\mathrm{RI'}})|_{q^2 = \mu^2} = 1\,. \tag{2.9}$$

PoS(LATTICE 2008)217

Restricting ourselves to two-loop order, we have *e.g.*

$$J(a,q,\alpha_{\mathrm{RI}'}) = 1 + \sum_{i=1}^{2} \alpha_{\mathrm{RI}'}^{i} \sum_{k=0}^{i} z_{i,k}^{\mathrm{RI}'} \left(\frac{1}{2}\log(aq)^2\right)^k . \tag{2.10}$$

Only the leading coefficients $z_{i,i}^{\mathrm{RI}'}$ are entirely calculable in continuum perturbation theory (PT): $z_{1,1}^{\mathrm{RI}'} = -3N_c/2$, $z_{2,2}^{\mathrm{RI}'} = -35N_c^2/8$. The non-leading coefficients $z_{i,k}^{\mathrm{RI}'}|_{i>k>0}$ are only partly known from PT: $z_{2,1}^{RI'} = \left(-\frac{271}{24} + \frac{35}{6} z_{1,0}^{RI'}\right)$, the $z_{i,0}^{\mathrm{RI}'}$ have to be calculated in LPT. For example, entering $z_{2,1}^{\mathrm{RI}'}$ is $z_{1,0}^{\mathrm{RI}'} = 13.8257$, known from one-loop LPT [9], while $z_{2,0}^{\mathrm{RI}'}$ is unknown.

From the relation [10] $\alpha_{\mathrm{RI}'} = \alpha_0 + (-(22/3)\,N_c \log(a\mu) + 73.9355)\,\alpha_0^2 + \ldots$, with the bare coupling $\alpha_0 = N_c/(8\pi^2\beta)$, we get for the two-loop dressing function:

$$J^{2-\mathrm{loop}}(a,q,\beta) = 1 + \frac{1}{\beta}\left(J_{1,1}\log(aq)^2 + J_{1,0}\right) + \frac{1}{\beta^2}\left(J_{2,2}\log^2(aq)^2 + J_{2,1}\log(aq)^2 + J_{2,0}\right) \tag{2.11}$$

with

$$J_{1,1} = -0.0854897\,, \quad J_{1,0} = 0.525314\,, \quad J_{2,2} = 0.0215195\,, \quad J_{2,1} = -0.358423 \tag{2.12}$$

and the unknown finite two-loop finite constant $J_{2,0}$ or $z_{2,0}^{\mathrm{RI}'}$,

$$J_{2,0} = 1.47572 + 0.00144365\, z_{2,0}^{\mathrm{RI}'} . \tag{2.13}$$

3. Results

The aim of this first investigation of the ghost propagator in NSPT was the confirmation of the known $J_{1,0}$, and a prediction of the unknown $J_{2,0}$. We concentrate ourselves on an analysis of $\hat{J}^{(n)}(\hat{q})$.

As an example of the measured ghost propagator we show the one- and two-loop results $\hat{J}^{(1)}$ and $\hat{J}^{(2)}$ for the dressing function in Fig. 1 together with $\hat{J}^{(n=3/2)}$ that is bound to vanish.

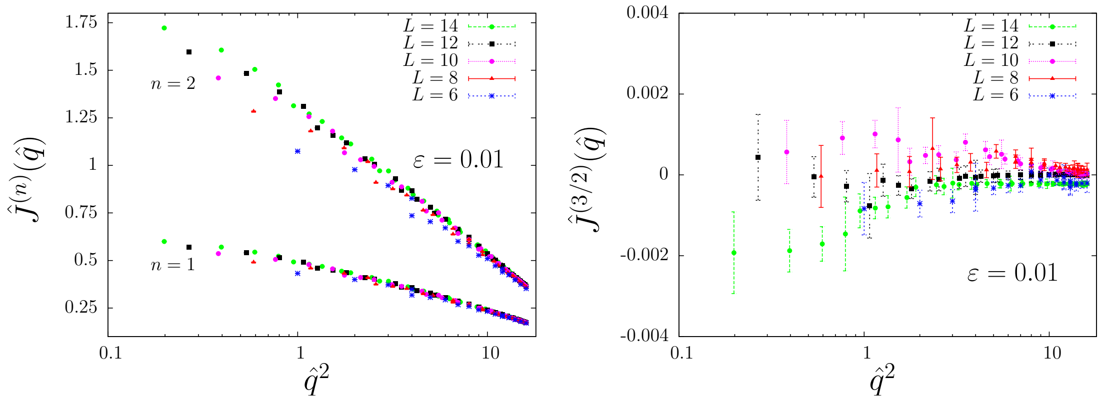

Figure 1: Measured ghost dressing function $\hat{J}(\hat{q})$ vs. $\hat{q}^2$ for all inequivalent lattice momentum 4-tuples (k_1,k_2,k_3,k_4) - see (2.2) - near the diagonal ones for lattice sizes $L = 6,\ldots,14$ and for the time step $\varepsilon = 0.01$. Left: The one-loop ($\propto \beta^{-1}$) and two-loop ($\propto \beta^{-2}$) contributions, right: the vanishing contribution $\propto \beta^{-3/2}$.

3.1 The limits to be taken

- The limit $\varepsilon \to 0$: We solved the Langevin equations for different step sizes $\varepsilon = 0.07, \dots, 0.01$ and obtained the Langevin result for each chosen momentum set of the propagator at fixed lattice size L and $\varepsilon = 0$ by extrapolation as shown in Fig. 2.

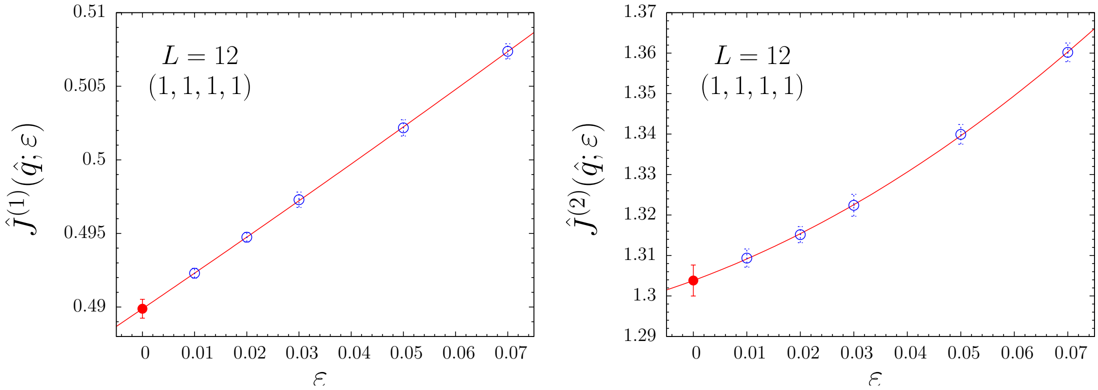

Figure 2: Linear plus quadratic correction extrapolation to $\varepsilon = 0$ of the one-loop (left) and two-loop (right) ghost dressing function for the momentum tuple $(1,1,1,1)$ on a lattice of size 12^4.

- The limits $L \to \infty$ and $a \to 0$: In order to make contact with standard LPT both limits have to be performed. To extract the non-logarithmic constants in those limits we make the following ansatz for the dressing function taking into account hypercubic symmetry (one-loop example; here we use the standard notation for hypercubic invariants)

$$\hat{J}^{(1)}(\hat{q}) = J_{1,1}\text{“}\log \hat{q}^2\text{”} + \hat{J}_{1,0;L}(\hat{q})\,, \tag{3.1}$$

$$\hat{J}_{1,0;\,L}(\hat{q}) = \hat{J}_{1,0;\,L} + c_1\,\hat{q}^2 + c_2\,\frac{\hat{q}^4}{\hat{q}^2} + c_3\,\hat{q}^4 + c_4\,(\hat{q}^2)^2 + c_5\,\frac{\hat{q}^6}{\hat{q}^2} + c_6\,(\hat{q}^2)^3 + \cdots \tag{3.2}$$

The problem arising here is how to represent – on finite lattices – the logs that appear in the $L \to \infty$ regime. Our proposal here is to replace the divergent lattice integrals, that give rise to the logarithms, by finite lattice sums and use these expressions in the fits at fixed L.

3.2 Handling the lattice logs encountered

We illustrate the procedure by the example of a typical one-loop divergent integral

$$A(aq) = (4\pi)^2 \int_{-\pi/a}^{\pi/a} \frac{d^4k}{(2\pi)^4}\, \frac{1}{\hat{k}^2\,\widehat{(k+q)}^2}\,. \tag{3.3}$$

In the limit $aq \to 0$ [11] one gets

$$A(aq) = -\log(aq)^2 + a_1\,, \qquad a_1 = 2 + F_0 - \gamma_E = 5.79201\,. \tag{3.4}$$

On a lattice with finite L we calculate the corresponding lattice sums:

$$A(i^q, L) = \frac{1}{L^4} \sum_{i_1,i_2,i_3,i_4} \frac{1}{\left[\sum_{\mu=1}^4 \sin^2\left(\frac{\pi}{L} i_\mu\right)\right]\left[\sum_{\nu=1}^4 \sin^2\left(\frac{\pi}{L}(i_\nu - i_\nu^q)\right)\right]} \tag{3.5}$$

with $ak_\mu = \frac{2\pi i_\mu}{L}$, $aq_\mu = \frac{2\pi i^q_\mu}{L}$, $\{i_\mu, i^q_\mu\} \in \left(-\frac{L}{2}, \frac{L}{2}\right]$. This leads – for each L – to the replacement:

$$J_{1,0}\log(aq)^2 \longrightarrow 2\,J_{1,0}\left(A(i^q,L) - a_1\right)\,. \tag{3.6}$$

This also results in a reshuffling of irrelevant terms. The result is a flattening of the data with the log-terms subtracted (see Fig. 3). This then allows to extract the $V \to \infty$ limit fitting the remaining

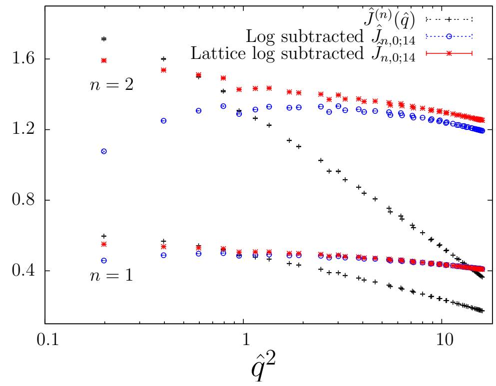

Figure 3: Original and remaining "non-logarithmic" contributions to $\hat{J}$ using logarithms and lattice logarithms at one-loop and two-loop level as function of $\hat{q}^2$ for a lattice 14^4.

non-logarithmic data (at present no momentum cuts on the data are used) with the ansatz (3.2). In a similar spirit, a log-squared behavior in a two-loop contribution is modeled by using the following expression as a discretized version of [11]

$$E(aq) = (4\pi)^4 \int_{-\pi/a}^{\pi/a} \frac{d^4k}{(2\pi)^4}\,\frac{1}{\hat{k}^2\,\widehat{(k+q)}^2}A(ak) \to \frac{1}{2}\log^2(aq)^2 - (a_1+1)\log(aq)^2 + 28.0086 \tag{3.7}$$

where $A(ak)$ and a_1 are defined in (3.3) and (3.4).

3.3 Results based on the outlined fitting procedure

The results for $\hat{J}_{1,0;\,L}$ and $\hat{J}_{2,0;\,L}$ as function of $1/L^4$ are shown in Fig. 4. A linear fit for $L = 10, 12, 14$ leads to the one-loop result

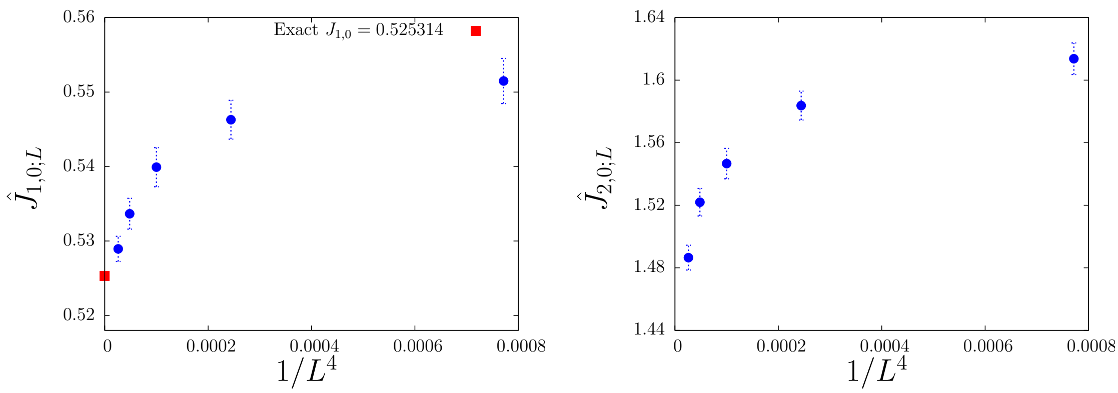

Figure 4: The $V \to \infty$ limit of the constant $\hat{J}_{n,0;L}$.

$$\hat{J}_{1,0}^{\text{Fit}} = 0.5255(24) \tag{3.8}$$

in agreement with the expectations. A linear fit as in the one-loop case would lead to a preliminary two-loop value $\hat{J}_{2,0}^{\text{Fit}} = 1.47(2)$. This results in the non-logarithmic contribution $z_{2,0}^{\text{RI}'}$ to the two-loop ghost self-energy in the RI'-MOM scheme in Landau gauge being compatible with zero.

4. Summary

- We have performed the first two-loop calculation of the lattice ghost propagator in Landau gauge.
- The one-loop constant $J_{1,0}$ agrees with the known $V \to \infty$ result.
- The two-loop constant $J_{2,0}$ has been estimated for the first time.
- A detailed analysis of all necessary limits has been performed.
- A proposal about how to mimic the usual logarithmic terms on finite lattices is made. An alternative procedure outlined in Ref. [7] is under development.
- A detailed comparison for a finite volume and a set of lattice momenta with Monte Carlo data would be desirable in order to separate out the nonperturbative effects on the ghost propagator.

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