Chiral perturbation theory, $K \to \pi\pi$ decays and 2+1 flavor domain wall QCD

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We present a calculation of the low energy constants describing the real and imaginary parts of the $K \to \pi\pi$ decay amplitudes A_0 and A_2. Leading and next leading order chiral perturbation theory is used and its applicability assessed. A combination of statistical and systematic errors limits the precision of the results. The apparent limitations of chiral perturbation theory raise doubts about the accuracy of a possible extrapolation to physical $K \to \pi\pi$ kinematics.

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Quantitative understanding of the two pion decay of the K meson has been an outstanding problem in particle physics for the past fifty years. While much progress has been made in explaining the large size of the $\Delta I = 1/2$ amplitude relative to that with $\Delta I = 3/2$, a precise prediction of this "$\Delta I = 1/2$ rule", has not been achieved. Of even greater interest is determining if the direct CP violation observed in $K_L \rightarrow \pi\pi$ decay can be computed from the single CP violating phase present in the CKM matrix or whether new physics is required.

Lattice QCD offers the promise of a first-principles calculation of these quantities. However, two serious difficulties must be overcome. The first arises from the complexity of the low energy weak Lagrangian whose matrix elements determine the needed decay amplitudes. There are seven independent four-quark operators. In an environment with accurate flavor and chiral symmetries, these seven operators mix in groups of one, two and four and also with operators of lower dimension. Without flavor and chiral symmetry, a much larger class of operators must be studied.

The second difficulty comes from the two-pion final state. The two-pion state isolated at asymptotically large time in a Euclidean lattice QCD calculation will be the lowest energy state with the given quantum numbers. Thus, the $I = 2$ state will be two pions at nearly zero relative momentum, not the physical value of $p \approx 205$ MeV. For the $I = 0$ case, the lowest energy state will be the vacuum.

The first difficulty appears to now be largely overcome. By using a chiral fermion formulation, *e.g.* domain wall fermions, and the Rome-Southampton RI/MOM normalization procedure it is possible to isolate and properly normalize the seven relevant weak operators, accurately including the effects of operator mixing. This was developed and shown to be practical in our earlier quenched calculation seven years ago [1], a reference defining the conventions used here.

In this talk, we attempt to avoid the problems created by the two-pion final state by using chiral perturbation theory (ChPT) to relate the $K \rightarrow \pi\pi$ amplitude of interest to $K \rightarrow \pi$ and $K \rightarrow |0\rangle$ matrix elements which are much easier to evaluate using lattice methods [2]. This method was used in Ref. [1]. However, the series of quark masses used in that calculation were too large in size and too few in number to provide a test of chiral perturbation theory.

The most apparent difficulty in that earlier calculation is the use of the quenched approximation. As discovered by Golterman and Pallente [3], quenched chiral perturbation theory for the (8,1) amplitudes reveals significant, unphysical logarithms not present in full QCD calculation and subsequent numerical studies of these terms suggested that they could be large [4].

The present calculation, reported here, removes this problem by using the 2+1 flavor lattice ensembles of the RBC and UKQCD collaborations. By including both unitary and partially quenched quark masses and working with somewhat lighter masses, the present calculation also provides information about the validity of chiral perturbation theory. In addition, the recent 2+1 flavor, partially quenched chiral perturbation theory (PQChPT) calculation of Aubin, Laiho, Li and Lin [5] provides the theoretical formulae necessary to extract low energy constants from our results and to explore the validity of this approach.

As we will see, $SU(3) \times SU(3)$ chiral perturbation theory describes our results poorly. This same conclusion was reached in a study of meson masses, decay constants and neutral kaon mixing using these same 2+1 flavor ensembles [6]. However, in the present case the use of $SU(3) \times SU(3)$ chiral perturbation theory is required by our strategy for calculating two-pion matrix elements, making the final results of the calculation presented here highly uncertain.

1. Description of the calculation

This calculation is based on the $24^3 \times 64$ RBC/UKQCD 2+1 flavor ensembles. These have an inverse lattice spacing of 1.73 GeV and a linear extent of 2.7 fm. Two ensembles are examined: one with a light quark mass $m_l = 0.005$ corresponding to a pion mass of 331 MeV and the second with $m_l = .01$ and a pion mass of 415 MeV. Both use 0.04 for the strange mass, approximately 15% larger than that of the physical strange quark. The fifth-dimension extent of the domain fermion lattice is $L_s = 16$ and the resulting residual mass 0.00315(2).

The weak matrix elements are computed using a Coulomb gauge-fixed wall source for the quarks making up the pion located at time slice $t = 5$ while the quarks in the K meson are created by a similar wall source located at $t = 59$. Interference from quarks which travel through the time boundary at $t = 63 - 0$ is suppressed by "doubling" the lattice, achieved by determining the quark propagators twice, first using periodic and second using anti-periodic boundary conditions in the time. The average of these propagators will have a source at $t = 5$ or 59 and no image at $t = 69$ or $t = -6$, respectively. As shown in Fig. 1, even with this large separation of 64 between the two sources, the 3-point function is quite accurately determined with some error reduction coming from the average over the long plateau.
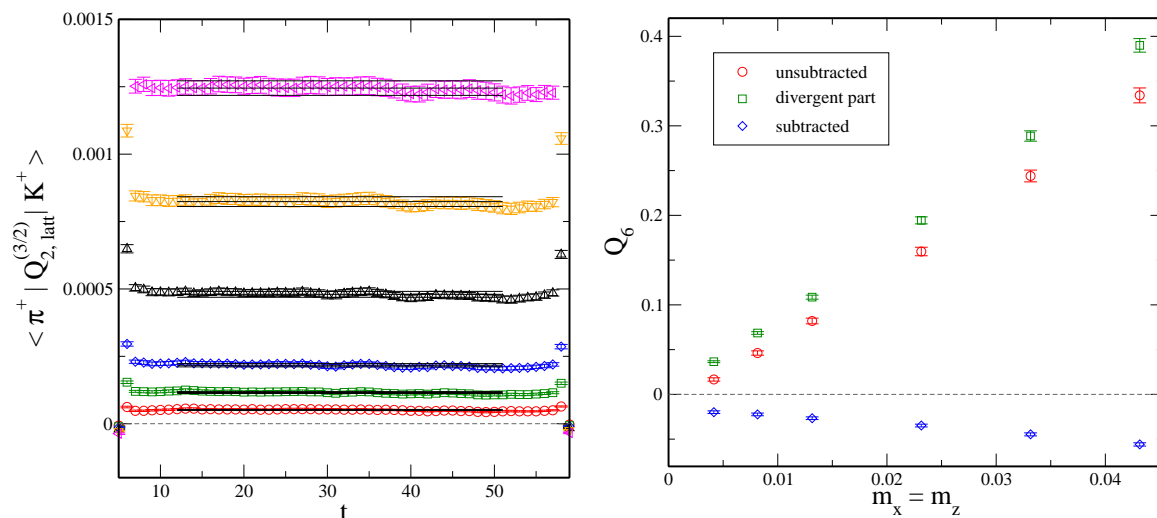

Figure 1: The left panel shows the dependence of $\pi - Q_2 - K$ amplitude on the location t of the Q_2 operator at which the four quark propagators coming from the sources at $t = 5$ and 59 are combined. From the top the curves describe the degenerate light quark masses 0.001, 0.005, 0.01, 0.02, 0.03, 0.04. The right panel shows the matrix elements of Q_6 (circles), the subtraction term (squares) and their difference (diamonds).

A particularly delicate part of the calculation is associated with the mixing of the (8,1) operators with the lower dimensional operators $\overline{s}d$ and $\overline{s}\gamma^5 d$. While these quadratically divergent contributions, (*i.e.* $\sim 1/a^2$) vanish when the weak operator carries no four-momentum, these operators do contribute when chiral perturbation theory is used. However, just as in our earlier quenched work, with proper care this subtraction, determined by the ChPT expression, can be done to better than 10% accuracy. The right panel of figure 1 shows the two amplitudes which must be subtracted and their well-resolved difference.

2. Chiral extrapolation

We first discuss the $\Delta I = 3/2$ LEC α_{27} which can be determined from the chiral limit of the matrix element $\langle \pi | Q^{(27,1)} | K \rangle$. This operator makes up the $\Delta I = 3/2$ part of Q_1, Q_2, Q_9 and Q_{10} and with $K - \overline{K}$ external states determines B_K. Following the ChPT studies in Ref [6], we limit the range of input masses in our chiral fits to those whose average is 0.01 or less in lattice units. We were unable to obtain a sensible, NLO chiral fit to this quantity. While it was possible to describe our data by the NLO chiral formula, the large chiral log with coefficient of -34/3 was sufficiently inconsistent with our data and the other LEC's sufficiently unconstrained, that the resulting fits gave a leading order term contributing 3% to the total with the next leading order terms providing 97%. (When defined in the same fashion the coefficient of the NLO logarithm in m_π^2 is 2/3.)

In order to sidestep this difficulty, we next fit the ratio $\langle \pi | Q^{(27,1)} | K \rangle / (f_K f_\pi m_K^2 m_\pi^2)$, similar to the amplitude giving B_K, at NLO. Taking this ratio reduces the statistical errors in our computed quantities and reduces the coefficient of the chiral logarithm to -6. The NLO fit to this ratio is more satisfactory and is shown in Fig. 2. While the left-hand panel shows a reasonable chiral fit to this ratio, the right-hand panel shows the contribution of the various terms in the chiral expansion and reveals that the NLO correction is of the same size as the LO term. In order to test the robustness of our use of this *ad hoc* ratio, we also divided by a second factor of $f_K f_\pi$ and found similar results for the corresponding NLO ChPT fit. The low energy constants obtained in the fit to the first ratio, for each of the $\Delta I = 3/2$ operators $Q^{(27,1)}$, Q_7 and Q_8 are shown in Table 1. The statistical error obtained by a standard jackknife analysis is shown in the left bracket. Our estimate of a systematic error is given in the right bracket.
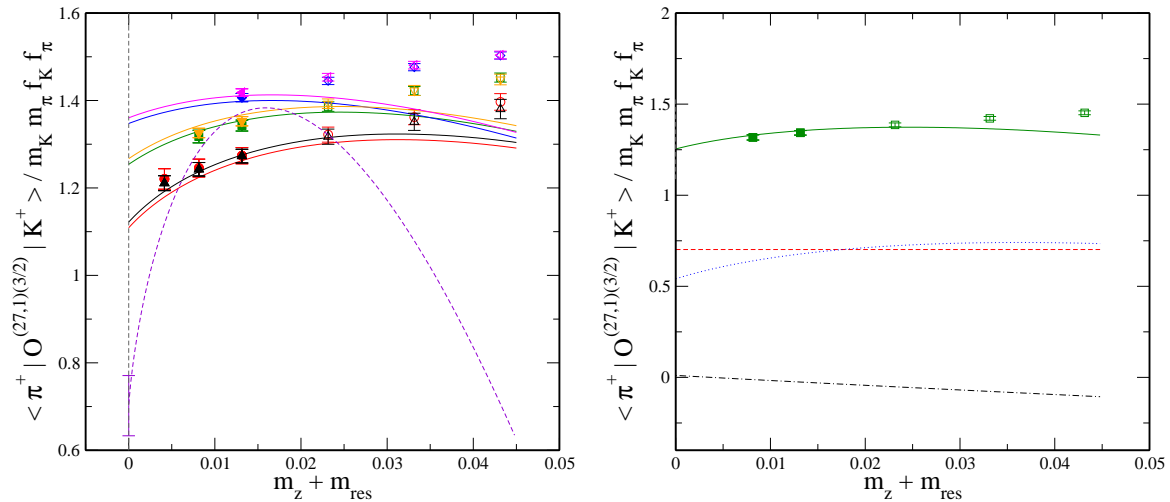
Figure 2: The left panel shows the NLO ChPT fit to the ratio $\langle \pi | Q^{(27,1)} | K \rangle / (f_K f_\pi m_K^2 m_\pi^2)$. The three bands of points from bottom to top correspond to light quark masses of 0.001, 0.005 and 0.01. In each band of points the lower have a light sea quark of mass 0.005 and upper, 0.01. The solid lines are the chiral fits and the dotted line a unitary extrapolation (based on those fits) to the chiral limit. In the right-hand panel contributions of the individual terms are broken out and the uncomfortably large size of the NLO terms shown. The solid curve passing through the data points is the sum of all terms. The next two moving downward are the leading order term and the NLO logarithms. The bottom curve is the NLO analytic terms.

Q_i	$\alpha_{i,\mathrm{ren}}^{(1/2)}$	$\alpha_{i,\mathrm{ren}}^{(3/2)}$
1	$-6.6(15)(66) \times 10^{-5}$	$-2.48(24)(39) \times 10^{-6}$
2	$9.9(21)(99) \times 10^{-5}$	$-2.47(24)(39) \times 10^{-6}$
3	$-0.8(31)(21) \times 10^{-5}$	0.0
4	$1.62(44)(162) \times 10^{-4}$	0.0
5	$-1.52(29)(152) \times 10^{-4}$	0.0
6	$-4.1(7)(41) \times 10^{-4}$	0.0
7	$-1.11(17)(18) \times 10^{-5}$	$-5.53(85)(91) \times 10^{-6}$
8	$-4.92(72)(75) \times 10^{-5}$	$-2.46(37)(37) \times 10^{-5}$
9	$-9.8(20)(98) \times 10^{-5}$	$-3.72(37)(59) \times 10^{-6}$
10	$6.8(15)(68) \times 10^{-5}$	$-3.69(37)(59) \times 10^{-6}$

**Table 1:** The values for the low energy constants which describe the ten operators $Q_1, ..., Q_{10}$ at leading order in ChPT. These correspond to operators renormalized in the $\overline{\mathrm{MS}}$ scheme at the scale $\mu = 2.15$ GeV. The LEC's for $Q_1, ...Q_6$, Q_9 and Q_{10} are expressed in $(\mathrm{GeV})^4$ while those for Q_7 and Q_8 are given in $(\mathrm{GeV})^6$.

Given the failure of simple NLO PQChPT to directly describe the matrix element $\langle \pi | Q^{(27,1)} | K \rangle$, assessing the systematic error in a chiral extrapolation of this quantity is difficult. The systematic errors presented for the three $\Delta I = 3/2$ LECs in Table 1 were obtained by computing the difference between the results from the NLO chiral fits to the two ratios described above and then doubling the result in an attempt to account for the uncertainties in this *ad hoc* procedure and the uncomfortably large relative size of the NLO terms.

Next we describe the extraction of the $\Delta I = 1/2$ low energy constants. Here difficulties arose because of the large number of LECs which appear in the NLO expression for an (8,1) operator. Our twelve PQ data points with average input quark masses at or below the 0.01 upper limit proved insufficient to give a stable fit to the required eight LO and NLO LECs. As a result we could perform only the LO fit shown in left panel of Fig 3 for the case of Q_6. As can be seen, the fit nicely describes both the data included in the fit and that at larger masses as well. However, the omitted chiral logarithms can substantially alter the resulting LEC (the slope of the correct curve, including the logarithms, at $m = 0$). This difficulty is shown in the right panel of Fig 3 where the second, upward bending curve at $m = 0$ is obtained by adding an $m_\pi^4 \ln(m_\pi^2/\Lambda^2)$ term with the correct coefficient for the case of vanishing degenerate light quark masses but fixed strange quark mass. The constant Λ as well as the two parameters in the altered linear terms are matched to the LO linear fit in value, slope and curvature at our lightest dynamical quark mass ($m_l = 0.005$). As can be seen in the figure, this increases the slope at zero by a factor of two.

The resulting LEC for the $\Delta I = 1/2$ operators are also shown in Table 1. The 100% systematic errors arise from our inability to constrain the place at which the simple linear behavior seen in our results is replaced by the proper, non-linear, logarithmic behavior required by ChPT. The factor of two increase in α_6 discussed above is typical for these amplitudes. Since a larger matching point than 0.005 is consistent with our data and leads to even larger effects, we do not believe that 100% is an obvious over estimate of this uncertainty.

Finally we attempt to combine the LECs determined above to determine the real and imaginary parts of the $I = 0$ and 2, $K \to \pi\pi$ decay amplitudes A_0 and A_2. Such a determination is made highly
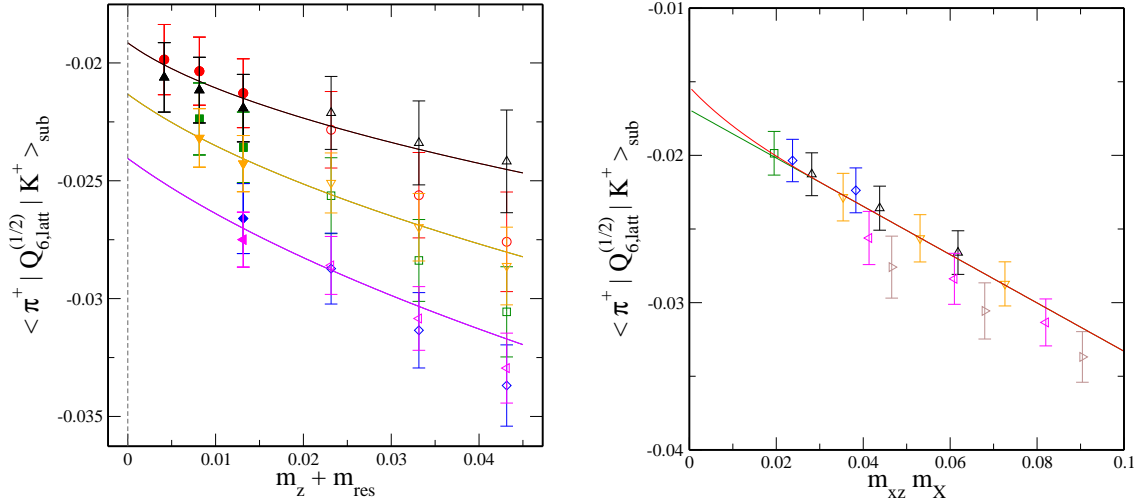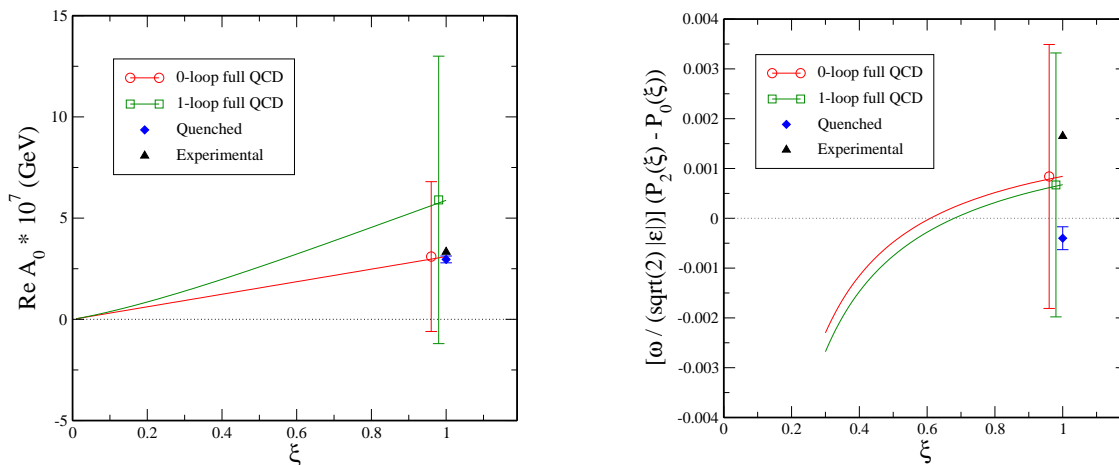
Figure 3: The left panel shows the LO ChPT fit to the matrix element $\langle\pi|Q_6|K\rangle$. The ordering of the data is similar to that in left panel of Fig. 2 except that the light valence quark mass now increases from top to bottom. The data is replotted in the right panel as a function of product of the partially quenched pion and kaon masses. The second, upward bending curve estimates the effects of the omitted chiral logarithms.

Quantity	This analysis	Quenched [1]	Experiment
$\mathrm{Re}A_0$ (GeV)	$4.5(11)(53)\times 10^{-7}$	$2.96(17)\times 10^{-7}$	3.33×10^{-7}
$\mathrm{Re}A_2$ (GeV)	$8.57(99)(300)\times 10^{-9}$	$1.172(53)\times 10^{-8}$	1.50×10^{-8}
$\mathrm{Im}A_0$ (GeV)	$-6.5(18)(77)\times 10^{-11}$	$-2.35(40)\times 10^{-11}$	
$\mathrm{Im}A_2$ (GeV)	$-7.9(16)(39)\times 10^{-13}$	$-1.264(72)\times 10^{-12}$	
$1/\omega$	$50(13)(62)$	$25.3(1.8)$	22.2
$\mathrm{Re}(\varepsilon'/\varepsilon)$	$7.6(68)(256)\times 10^{-4}$	$-4.0(2.3)\times 10^{-4}$	1.65×10^{-3}

Table 2: Values for physical quantities describing $K \to \pi\pi$ decay. Here $\omega = \mathrm{Re}A_2/\mathrm{Re}A_0$. Note, only statistical errors are given for the quenched results.

uncertain by the failure of chiral perturbation theory to describe physics for quark masses in the region of the strange quark as discussed above and in Ref [6]. This leads to the large uncertainties in the necessary LECs in Table 1 and makes the extrapolation from those LECs to the physical $K \to \pi\pi$ amplitudes unreliable. Further, the needed 2+1 flavor NLO ChPT formulae are not available nor have we determined full the set of LEC's needed for such a NLO extrapolation. The resulting values for the physical quantities of interest are given in Table 2. The large systematic errors are determined by adding in quadrature the propagated systematic errors from the LO LECs and the difference between a simple LO extrapolation and a NLO extrapolation in which only the effects of the logarithms are included, using the chiral scale $\Lambda = 1$ GeV but with no NLO analytic terms. Fits with and without the logarithm terms are compared in Figure 4.

We have presented preliminary first results for $K \to \pi\pi$ decays determined by lattice calculation and chiral perturbation theory. These have been obtained in a theory with accurate chiral symmetry, with a proper treatment in full QCD of the light and strange quarks and a complete renormalization, including the effects of operator mixing, of the seven independent four-quark operators needed to describe this decay. The errors on the $\Delta I = 3/2$ LECs could be reduced by

Figure 4: The left panel shows the real part of A_0 as a function the factor ζ by which both m_π^2 and m_K^2 have been scaled to allow a connection to be made with the region where ChPT should become accurate. The lower curve is a LO ChPT extrapolation while the upper curve includes the NLO chiral logarithm. The right panel shows similar LO and NLO logarithmic extrapolations for the quantity equal to Re $(\varepsilon'/\varepsilon)$ when $\zeta = 1$. The triangles show the experimental values while the diamonds the quenched results of Ref [1].

extending our calculation to smaller quark masses while those for $\Delta I = 1/2$ could be decreased if more quark mass combinations were studied. Likewise some uncertainties related to the extrapolation to the physical $K \to \pi\pi$ amplitudes could be reduced if the full NLO ChPT formulae were available. However, these steps would not reduce the large, dominate uncertainties associated with the use of ChPT at the scale of the kaon as is required by this treatment the π-π final state.

We conclude that useful results for these important quantities require the direct study of π-π final states. The RBC and UKQCD collaborations are now actively pursing calculations with larger volumes and lattice spacings [7] and lighter pions, working toward this goal.

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