

Permutation group S_N and hadron spectroscopy

Dan Pirjol*

*Department of Particle Physics, National Institute for Physics and Nuclear Engineering, 077125
Bucharest, Romania
E-mail: pirjol@mac.com*

Carlos Schat

*CONICET and Departamento de Física, FCEyN, Universidad de Buenos Aires, Ciudad
Universitaria, Pab. 1, (1428) Buenos Aires, Argentina
Department of Physics and Astronomy, Ohio University, Athens, Ohio 45701, USA
E-mail: carlos.schat@gmail.com*

We discuss the application of the permutation group S_N to a few problems in hadron physics. In Ref. [11] a method was proposed for matching a quark model Hamiltonian onto the effective Hamiltonian of the $1/N_c$ expansion, which makes use of the transformation properties of the states and operators under S_N . This method is used in [13] to obtain information about the spin-flavor structure of the quark interaction Hamiltonian from the spectrum of the negative parity $L = 1$ excited baryons. Assuming the most general 2-body quark Hamiltonian, we derive two correlations among the masses and mixing angles of these states which should hold in any quark model. These correlations constrain the mixing angles, and can be used to test for the presence of 3-body quark interactions. We find that the pure gluon-exchange model is disfavored by data, independently of any assumptions about the hadronic wave functions.

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1. Introduction

Quark models provide a simple and intuitive picture of the physics of ground state baryons and their excitations [1, 2]. An alternative description is provided by the $1/N_c$ expansion, which is a systematic and model-independent approach to the study of baryon properties [3]. This program can be realized in terms of a quark operator expansion, which gives rise to a physical picture similar to the one of the phenomenological quark models, but is closer connected to QCD. In this context quark models gain additional significance.

The $1/N_c$ expansion has been applied both to the ground state and excited nucleons [4, 5, 6, 7]. In the system of negative parity $L = 1$ excited baryons this approach has yielded a number of interesting insights:

- The three towers [5, 8, 9] predicted by \mathcal{K} -symmetry for the $L = 1$ negative parity N^* baryons, labeled by $\mathcal{K} = 0, 1, 2$ with \mathcal{K} related to the isospin I and spin J of the N^* 's by $I + J \geq \mathcal{K} \geq |I - J|$.
- The vanishing of the strong decay width $\Gamma(N_{\frac{1}{2}}^* \rightarrow [N\pi]_S)$ for $N_{\frac{1}{2}}^*$ in the $\mathcal{K} = 0$ tower, which provides a natural explanation for the relative suppression of pion decays for the $N^*(1535)$ [5, 8, 9].
- The order $\mathcal{O}(N_c^0)$ mass splitting of the $SU(3)$ singlets $\Lambda(1405) - \Lambda(1520)$ in the $[70, 1^-]$ multiplet [7].

The $1/N_c$ expansion for the excited nucleons has been extended also to the first subleading order in $1/N_c$ [4, 5, 6, 7, 8, 10].

In a recent paper [11] we showed how to match an arbitrary quark model Hamiltonian onto the operators of the $1/N_c$ expansion, thus making the connection between these two physical pictures. This method makes use of the transformation of the states and operators under $S_N^{\text{sp-fl}}$, the permutation group of N objects acting on the spin-flavor degrees of the quarks. This is similar to the method discussed in Ref. [12] for $N_c = 3$ in terms of S_3^{orb} , the permutation group of 3 objects acting on the orbital degrees of freedom.

The main result of [11] can be summarized as follows: consider a two-body quark Hamiltonian $V_{qq} = \sum_{i < j} O_{ij} R_{ij}$, where O_{ij} acts on the spin-flavor quark degrees of freedom, and R_{ij} acts on the orbital degrees of freedom. Then the hadronic matrix elements of the quark Hamiltonian on a baryon state $|B\rangle$ contains only the projections O_α of O_{ij} onto irreducible representations of S_N , the permutation group of N objects acting on the spin-flavor degrees of freedom $\langle B | V_{qq} | B \rangle = \sum_\alpha C_\alpha \langle O_\alpha \rangle$. The coefficients C_α are related to reduced matrix elements of the orbital operators R_{ij} , and are given by overlap integrals of the quark model wave functions.

The explicit calculation in Ref. [11] confirms the N_c power counting rules of Ref. [4, 6], in particular the leading order $\mathcal{O}(N_c^0)$ contribution to the mass coming from the spin-orbit interaction $\vec{s} \cdot \vec{l}$, and confirms in a direct way the prediction of the breaking of the $SU(4)$ spin-flavor symmetry at leading order in N_c [4]. The calculation in Ref. [11] confirms that the nonrelativistic quark model with gluon mediated quark interactions displays the same breaking phenomenon.

Another important conclusion following from the S_N analysis is that operators depending on excited and core quarks are indeed required by a correct implementation of the $1/N_c$ expansion, in

Table 1: The most general two-body spin-flavor quark interactions and their projections onto irreducible representations of S_3 , the permutation group of three objects acting on the spin-flavor degrees of freedom. $C_2(F) = \frac{F^2-1}{2F}$ is the quadratic Casimir of the fundamental representation of $SU(F)$.

Operator	\mathcal{O}_{ij}	O_S	O_{MS}
Scalar	1	1	–
	$t_i^a t_j^a$	$T^2 - 3C_2(F)$	$T^2 - 3t_1 T_c - 3C_2(F)$
	$\vec{s}_i \cdot \vec{s}_j$	$\vec{S}^2 - \frac{9}{4}$	$\vec{S}^2 - 3\vec{s}_1 \cdot \vec{S}_c - \frac{9}{4}$
	$\vec{s}_i \cdot \vec{s}_j t_i^a t_j^a$	$G^2 - \frac{9}{4}C_2(F)$	$3g_1 G_c - G^2 + \frac{9}{4}C_2(F)$
Vector (symm)	$\vec{s}_i + \vec{s}_j$	$\vec{L} \cdot \vec{S}$	$3\vec{L} \cdot \vec{s}_1 - \vec{L} \cdot \vec{S}$
	$(\vec{s}_i + \vec{s}_j) t_i^a t_j^a$	$\frac{1}{2} L^i \{G^{ia}, T^a\} - C_2(F) L^i S^i$	$2\frac{1-F}{F} L^i S_c^i + L^i g_1^a T_c^a + L^i t_1^a G_c^{ia}$
Vector (anti)	$\vec{s}_i - \vec{s}_j$	–	$3\vec{L} \cdot \vec{s}_1 - \vec{L} \cdot \vec{S}$
	$(\vec{s}_i - \vec{s}_j) t_i^a t_j^a$	–	$L^i g_1^a T_c^a - L^i t_1^a G_c^{ia}$
Tensor (symm)	$\{s_i^a, s_j^b\}$	$L_2^{ij} \{S^i, S^j\}$	$3L_2^{ij} \{s_1^i, S_c^j\} - L_2^{ij} \{S^i, S^j\}$
	$\{s_i^a, s_j^b\} t_i^c t_j^c$	$L_2^{ij} \{G^{ia}, G^{ja}\}$	$L_2^{ij} g_1^{ia} G_c^{ja} - \frac{F-1}{4F} L_2^{ij} \{S^i, S^j\}$
Tensor (anti)	$[s_i^a, s_j^b]$	–	0
	$[s_i^a, s_j^b] t_i^c t_j^c$	–	0

contrast to the approach of Ref. [17] which does not include such operators, and does not predict a breaking of $SU(4)$ spin-flavor symmetry at leading order in N_c .

Any particular model of quark interactions, e.g the one-gluon exchange model (OGE) [1], or the Goldstone boson exchange model (GBE) [14], predicts a distinct hierarchy among the coefficients C_α of the $1/N_c$ expansion. This prediction can be used to discriminate among models by confronting it against the observed values of the coefficients.

In a recent paper [13] we used the S_N approach to study the predictions of the quark model with the most general 2-body quark interactions, and to obtain information about the spin-flavor structure of the quark interactions from the observed spectrum of the $L = 1$ negative parity baryons. This talk summarizes the main results of this paper.

2. The most general two-body quark Hamiltonian

The most general 2-body quark interaction Hamiltonian in the constituent quark model can be written in generic form as $V_{qq} = \sum_{i < j} V_{qq}(ij)$ with

$$V_{qq}(ij) = \sum_k f_{0,k}(r_{ij}) O_{S,k}(ij) + f_{1,k}^a(r_{ij}) O_{V,k}^a(ij) + f_{2,k}^{ab}(r_{ij}) O_{T,k}^{ab}(ij), \quad (2.1)$$

where O_S, O_V^a, O_T^{ab} act on spin-flavor, and $f_k(r_{ij})$ are functions of $r_{ij} = |\mathbf{r}_i - \mathbf{r}_j|$. Their detailed form is unimportant for our considerations. $a, b = 1, 2, 3$ denote spatial indices.

We list in Table 1 a complete set of spin-flavor 2-body operators with all possible Lorentz structures allowed by the orbital angular momentum $L = 1$. Columns 3 and 4 of Table 1 list the projections of the spin-flavor operators O_S, O_V^a, O_T^{ab} onto the irreducible representations of the S_3 permutation group, computed as explained in Ref. [11]. The representation content depends on the symmetry of O_{ij} under the permutation $[ij]$: the symmetric operators O_{ij} are decomposed as $S + MS$, and antisymmetric O_{ij} as $MS + A$.

The symmetric S projection depends only on quantities acting on the entire hadron S^i, T^a, G^{ia} , while the mixed-symmetric MS operators depend on operators acting on the core and excited quarks. We express them in a form commonly used in the application of the $1/N_c$ expansion [6], according to which their matrix elements are understood to be evaluated on the spin-flavor state $|\Phi(SI)\rangle$ constructed as a tensor product of an excited quark with a symmetric core with spin-flavor $S_c = I_c$. The antisymmetric operators contain also an A projection; its orbital matrix element vanishes for $N_c = 3$ because of T-invariance [11, 12], such that these operators do not contribute, and are not shown in Table 1.

The orbital matrix elements yield factors of $L^i, L_2^{ij} = \frac{1}{2}\{L^i, L^j\} - \frac{1}{3}\delta^{ij}L(L+1)$, which are the only possible structures which can carry the spatial index.

From Table 1 one finds that the most general form of the mass operator in the presence of 2-body quark interactions is a linear combination of 10 operators

$$\begin{aligned} O_1 &= T^2, \quad O_2 = \vec{S}_c^2, \quad O_3 = \vec{s}_1 \cdot \vec{S}_c, \quad O_4 = \vec{L} \cdot \vec{S}_c, \quad O_5 = \vec{L} \cdot \vec{s}_1, \quad O_6 = L^i t_1^a G_c^{ia}, \\ O_7 &= L^i g_1^{ia} T_c^a, \quad O_8 = L_2^{ij} \{S_c^i, S_c^j\}, \quad O_9 = L_2^{ij} s_1^i S_c^j, \quad O_{10} = L_2^{ij} g_1^{ia} G_c^{ja}. \end{aligned} \quad (2.2)$$

This gives the most general form of the hadronic mass operator of the negative parity $L = 1$ states allowing only 2-body quark operators.

3. Correlations

The $L = 1$ quark model states include the following SU(3) multiplets: two spin-1/2 octets $8_{\frac{1}{2}}, 8'_{\frac{1}{2}}$, two spin-3/2 octets $8_{\frac{3}{2}}, 8'_{\frac{3}{2}}$, one spin-5/2 octet $8'_{\frac{5}{2}}$, two decuplets $10_{\frac{1}{2}}, 10_{\frac{3}{2}}$ and two singlets $1_{\frac{1}{2}}, 1_{\frac{3}{2}}$. States with the same quantum numbers mix, and we define the relevant mixing angles in the nonstrange sector as

$$\begin{cases} N(1535) = \cos \theta_{N_1} N_{1/2} + \sin \theta_{N_1} N'_{1/2} \\ N(1650) = -\sin \theta_{N_1} N_{1/2} + \cos \theta_{N_1} N'_{1/2} \end{cases}, \quad \begin{cases} N(1520) = \cos \theta_{N_3} N_{3/2} + \sin \theta_{N_3} N'_{3/2} \\ N(1700) = -\sin \theta_{N_3} N_{3/2} + \cos \theta_{N_3} N'_{3/2} \end{cases} \quad (3.1)$$

It turns out that the 11 coefficients C_{0-10} contribute to the mass operator of the negative parity N^* states only in 9 independent combinations: $C_0, C_1 - C_3/2, C_2 + C_3, C_4, C_5, C_6, C_7, C_8 + C_{10}/4, C_9 - 2C_{10}/3$. This implies the existence of two universal relations among the masses of the 9 multiplets plus the two mixing angles, which must hold in any quark model containing only 2-body quark interactions.

The first universal relation involves only the nonstrange hadrons, and requires only isospin symmetry. It can be expressed as a correlation among the two mixing angles θ_{N_1} and θ_{N_3} (see Fig. 1 left)

$$\begin{aligned} & \frac{1}{2}(N(1535) + N(1650)) + \frac{1}{2}(N(1535) - N(1650))(3 \cos 2\theta_{N_1} + \sin 2\theta_{N_1}) \\ & - \frac{7}{5}(N(1520) + N(1700)) + (N(1520) - N(1700)) \left[-\frac{3}{5} \cos 2\theta_{N_3} + \sqrt{\frac{5}{2}} \sin 2\theta_{N_3} \right] \\ & = -2\Delta_{1/2} + 2\Delta_{3/2} - \frac{9}{5}N_{5/2}. \end{aligned} \quad (3.2)$$

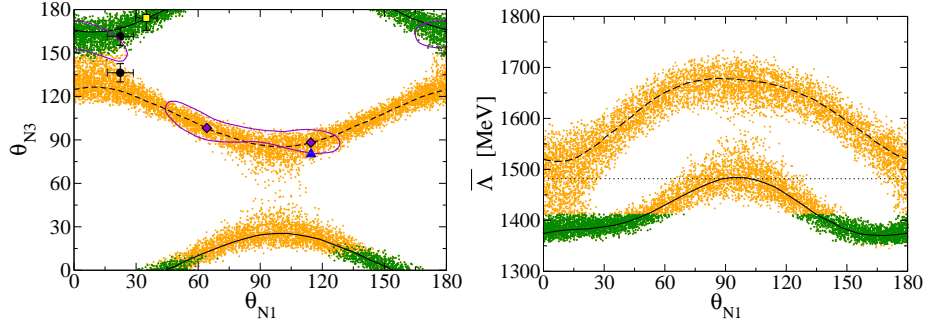


Figure 1: Left: correlation in the $(\theta_{N1}, \theta_{N3})$ plane in the quark model with the most general 2-body quark interactions. Right: prediction for the spin-weighted $\bar{\Lambda}$ mass in the SU(3) limit as a function of the θ_{N1} mixing angle, corresponding to the two solutions for θ_{N3} . The green points correspond to $\bar{\Lambda} = \bar{\Lambda}_{\text{exp}} - (100 \pm 30)$ MeV, with $\bar{\Lambda}_{\text{exp}} = 1481.7 \pm 1.5$ MeV.

This correlation holds also model independently in the $1/N_c$ expansion, up to corrections of order $1/N_c^2$, since for non-strange states the mass operator to order $O(1/N_c)$ [6, 7] is generated by the operators in Eq. (2.2). An example of an operator which violates this correlation is $L^i g^{ja} \{S_c^j, G_c^{ia}\}$, which can be introduced by 3-body quark forces.

On the same plot we show also the values of the mixing angles obtained in several analyses of the $N^* \rightarrow N\pi$ strong decays and N^* hadron masses. The two black dots correspond to the mixing angles $(\theta_{N1}, \theta_{N3}) = (22.3^\circ, 136.4^\circ)$ and $(22.3^\circ, 161.6^\circ)$ obtained from a study of the strong decays in Ref. [15]. The second point is favored by a $1/N_c$ analysis of photoproduction amplitudes Ref. [16]. The yellow square corresponds to the values used in Ref. [6, 7] $(\theta_{N1}, \theta_{N3}) = (35.0^\circ, 174.2^\circ)$, and the triangle gives the angles corresponding to the solution 1' in the large N_c analysis of Ref. [8] $(\theta_{N1}, \theta_{N3}) = (114.6^\circ, 80.2^\circ)$. All these determinations (except the triangle) are compatible with the ranges $\theta_{N1} = 0^\circ - 35^\circ, \theta_{N3} = 135^\circ - 180^\circ$. They are also in good agreement with the correlation Eq. (3.2), and provide no evidence for the presence of 3-body quark interactions.

The second universal relation expresses the spin-weighted SU(3) singlet mass $\bar{\Lambda} = \frac{1}{6}(2\Lambda_{1/2} + 4\Lambda_{3/2})$ in terms of the nonstrange hadronic parameters

$$\begin{aligned} \bar{\Lambda} = & \frac{1}{6}(N(1535) + N(1650)) + \frac{17}{15}(N(1520) + N(1700)) - \frac{3}{5}N_{5/2}(1675) - \Delta_{1/2}(1620) \\ & - \frac{1}{6}(N(1535) - N(1650))(\cos 2\theta_{N1} + \sin 2\theta_{N1}) + (N(1520) - N(1700))\left(\frac{13}{15}\cos 2\theta_{N3} - \frac{1}{3}\sqrt{\frac{5}{2}}\sin 2\theta_{N3}\right). \end{aligned} \quad (3.3)$$

The rhs of Eq. (3.3) is plotted as a function of θ_{N1} in the right panel of Fig. 1, where it can be compared against the experimental value $\bar{\Lambda} = 1481.7 \pm 1.5$ MeV. Allowing for SU(3) breaking effects ~ 100 MeV, this constraint is also compatible with the range for θ_{N1} obtained above from direct determinations of the mixing angles.

Combining the Eqs. (3.2) and (3.3) gives a determination of the mixing angles from hadron masses alone, in contrast to their usual determination from $N^* \rightarrow N\pi$ decays. The green area in Fig. 1 shows the allowed region for $(\theta_{N1}, \theta_{N3})$ compatible with a positive SU(3) breaking correction in $\bar{\Lambda}$ of 100 ± 30 MeV. One notes a good agreement between this determination of the mixing angles and that from $N^* \rightarrow N\pi$ strong decays.

4. Spin-flavor structure of the quark interactions

We derive next constraints on the spin-flavor structure of the quark interaction, which can discriminate between models of effective quark interactions. There are two popular models used in the literature. The first model is the one-gluon exchange model (OGE) [1] which includes operators in Table 1 without isospin dependence. Expressed in terms of the operator basis O_{1-10} this gives the constraints

$$OGE : \quad C_1 = C_6 = C_7 = C_{10} = 0. \quad (4.1)$$

An alternative to the OGE model is the Goldstone boson exchange model (GBE) [14]. In this model quark forces are mediated by Goldstone boson exchange, and the quark Hamiltonian contains all the operators in Table 1 which contain the flavor dependent factor $t_i^a t_j^a$. The coefficients of the hadronic Hamiltonian C_i satisfy the constraints ($F = 3$ is the number of light quark flavors)

$$GBE : \quad C_1 = \frac{F}{4} C_3, \quad C_5 = C_9 = 0. \quad (4.2)$$

We would like to determine the coefficients C_i , and compare their values with the predictions of the two models Eqs. (4.1), (4.2). As mentioned, only 9 combinations of the 11 coefficients can be determined from the available data: $C_0, C_1 - C_3/2, C_2 + C_3, C_4, C_5, C_6, C_7, C_8 + C_{10}/4, C_9 - 2C_{10}/3$. In particular, as the coefficients of the spin-orbit interaction terms C_{4-7} can be determined, we propose to use their values to discriminate between different models of quark interaction.

The values of C_{4-7} can be compared with the hierarchy expected in each model. In the OGE model the flavor-dependent operators have zero coefficients $C_{6,7} \sim 0 \ll |C_{4,5}|$, while in the GBE model the spin-orbit interaction of the excited quark vanishes $C_5 \sim 0 \ll |C_{4,6}|$.

The coefficient $C_5 = 75.7 \pm 2.7$ MeV is fixed by the $\Lambda_{3/2} - \Lambda_{1/2}$ splitting [7]. This indicates the presence of the operators $s_i \pm s_j$ in the quark Hamiltonian, which is compatible with the OGE model. A suppression of the coefficients $C_{6,7}$ would be further evidence for the OGE model. We show in Fig. 2 the coefficients of the spin-orbit operators $C_{6,7}$ as functions of θ_{N1} . Within errors small values for C_7 are still allowed, however no suppression is observed for C_6 . This indicates the presence of the operators $(s_i \pm s_j)t_i^a t_j^a$ in the quark Hamiltonian. These results show that the quark Hamiltonian is a mix of the OGE and GBE interactions.

In the pure OGE model Eq. (4.1) the 7 nonvanishing coefficients C_i can be determined from the 7 nonstrange N^*, Δ^* masses (assuming only isospin symmetry but no specific form of the wave functions). This fixes the mixing angles, and the $\Lambda_{3/2} - \Lambda_{1/2}$ splitting, up to a 2-fold ambiguity. The allowed region for mixing angles is shown as the violet region in Fig. 1 left, and the central values as diamonds $(\theta_{N1}, \theta_{N3}) = (64.2^\circ, 98.2^\circ), (114.5^\circ, 88.2^\circ)$. Note that they are different from the angles obtained in the Isgur-Karl model $(31.7^\circ, 173.6^\circ)$ in Refs. [2, 18, 19].

The violet region near $\theta_{N1} \sim 0$ is consistent with the determinations from strong decays and from the SU(3) universal relation Eq. (3.3), but is ruled out by the prediction for the Λ splitting, in agreement with the non-zero value of C_6 that can be read off from Fig. 2. This implies that the pure OGE model is disfavored ¹.

¹Note that this argument neglects possible long-distance contributions to the Λ splitting, due to the proximity of the $\Lambda(1405)$ to the KN threshold. Such threshold effects are not described by the quark Hamiltonian Eq. (2.1), and their presence could invalidate the prediction of the Λ splitting in the OGE model.

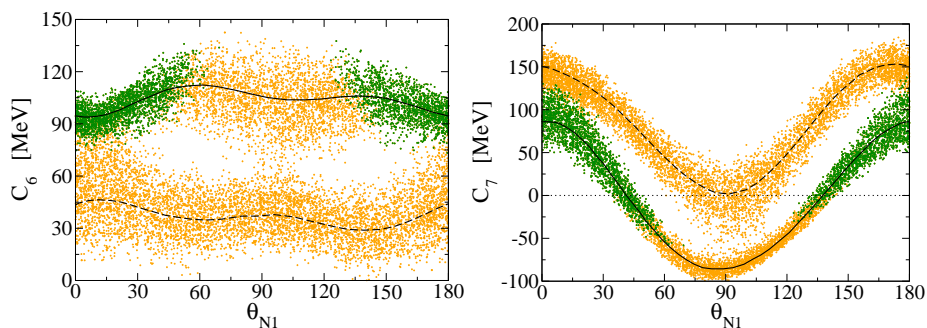


Figure 2: The coefficients of the spin-orbit operators $C_{6,7}$ as functions of the mixing angle θ_{N1} , in the quark model with the most general 2-body interactions. The green area is obtained by imposing the $\bar{\Lambda}$ constraint.

5. Conclusions

We discussed a few applications of the permutation group S_N to the study of baryonic properties in the quark model. The applications are based on a simple result: the spin-flavor contents of the mass operator is directly related to the projections of the spin-flavor part of the quark interaction onto irreducible representations of S_N . Using this result, any quark Hamiltonian can be matched onto the effective Hamiltonian of the $1/N_c$ expansion.

Following Ref. [13], we discussed the predictions of the most general 2-body quark Hamiltonian for the spin-flavor structure of the negative parity $L = 1$ excited baryons, without making any assumptions about the orbital hadronic wave functions. We derive two universal correlations among masses and mixing angles, which constrain the mixing angles, and can test for the presence of 3-body quark interactions. In addition, we derive constraints on the spin-flavor structure of the quark forces from the observed spectrum, and conclude that the gluon-exchange model is disfavored by data, independently on any assumptions about the hadronic wave functions.

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