In a previous paper, some deviations were found in the $\mathcal{O}(p^6)$ low-energy constants that contribute to the $\pi\pi$-scattering lengths. This work completes the study of all the relevant couplings ($r_1, ... r_6, r_{S_2}$). We also perform a reanalysis of the hadronical inputs used for the estimation (resonance masses, widths...), checking the impact of the input uncertainties on the determinations of the chiral couplings and the scattering lengths $a_f$. A good agreement is found with respect to former works, though our detailed analysis produces a more solid estimate of these couplings and slightly larger errors. The effect in the final values of the $a_f$ is negligible after combining them with the other uncertainties, being the previous scattering length determinations sound and reliable. Nevertheless, the uncertainties derived here for the $\mathcal{O}(p^6)$ contributions to the scattering lengths point out the limitation on further improvements unless the precision of the $\mathcal{O}(p^6)$ low-energy couplings is properly increased.
1. Introduction

This talk presents the culmination \cite{1} of a former work \cite{2}, where some of the $\mathcal{O}(p^6)$ Chiral Perturbation Theory low energy constants ($r_2, ..., r_6$) that describe the $\pi\pi$-scattering were calculated. Some shifts were found with respect to former estimates \cite{3}, inducing slight modifications on the corresponding predictions for the scattering lengths $a_J$ and effective ranges $b_J$ \cite{3, 4}. These constants provide the partial wave amplitudes for isospin $I$ and angular momentum $J$ near threshold $T_J(s) = \frac{k^{2J}}{J!(2\pi)^J} \left( a_J + b_J k^2 + \ldots \right)$, with $k = \sqrt{s/4 - m^2}$ the pion three-momentum in the dipion rest-frame \cite{3}. However, our previous article \cite{2} lacked of revised predictions for the $r_1$ low-energy constant (LEC), which also enters into the $\pi^+(p_1)\pi^- (p_2) \to \pi^0 (p_3)\pi^0 (p_4)$ amplitude at $\mathcal{O}(p^6)$ \cite{3}:

$$A(s,t,u) = \frac{m^4_s}{F^6} \left( r_2 - 2r_F \right) + \frac{m^2_s t^2}{F^6} + \frac{m^2_s (t-u)^2}{F^6} r_4 + \frac{s(t-u)^2}{F^6} r_5 + \frac{m^6_s}{F^6} \left( r_1 + 2r_F \right).$$

Likewise, the dispersive method considered by Colangelo et al. \cite{4} required the $\mathcal{O}(p^6)$ LEC $r_{S_2}$ instead of $r_5$ and $r_6$. Here we complete the study of these last LECs and perform a full reanalysis of the different hadronic inputs and their uncertainties.

2. Resonance estimates of $\mathcal{O}(p^6)$ LECs

- **Set A:**

  This is the group of estimates commonly employed in nowadays calculations \cite{3, 5}. The $\chi$PT couplings are assumed to be determined by the resonance exchanges provided by the phenomenological lagrangian

  $$\mathcal{L} = \frac{F^2}{4} \left( u_\mu u^\mu + \chi_+ \right) + \frac{1}{2} \langle \nabla_\mu S \nabla_\nu S \rangle - \frac{1}{2} M_5^2 \langle SS \rangle + c_d \langle Su_\mu u^\mu \rangle + c_m \langle S \chi_+ \rangle$$

  $$- \frac{i}{4} \langle \hat{\chi} \pi^\nu \hat{\chi} \rangle + \frac{1}{2} M_5^2 \langle \hat{\chi} \mu \rangle - \frac{i}{2} \langle \hat{\chi} \mu \rangle + f_\chi \langle \hat{\chi} [u^\mu, \chi_-] \rangle, \tag{2.1}$$

  where $\langle \ldots \rangle$ stands for trace in flavour space, $S$ and $\hat{\chi}$ account respectively for the scalar and vector multiplets. The tensor $u^\mu$ contains the chiral pseudo-Goldstone and $\chi_\pm$ is, in addition, proportional to the light quark masses. Their precise definitions can be found in Refs. \cite{3, 5, 6}. From the comparison of the $\rho \to \pi\pi$ and $K^* \to K\pi$ decays and other processes, Ref. \cite{3} obtained the set of parameters

  $$M_\rho = 770 \text{ MeV}, \quad g_\rho = 0.09, \quad f_\chi = -0.03, \quad M_5 = 983 \text{ MeV}, \quad c_m = 42 \text{ MeV}, \quad c_d = 32 \text{ MeV}. \tag{2.2}$$

  Taking this inputs and the phenomenological lagrangian (2.1), Ref. \cite{3} provided

  $$r_1^A = -0.6 \times 10^{-4}, \quad r_2^A = 1.3 \times 10^{-4}, \quad r_3^A = -1.7 \times 10^{-4}, \quad r_4^A = -1.0 \times 10^{-4}, \quad r_5^A = 1.1 \times 10^{-4}, \quad r_6^A = 0.3 \times 10^{-4}, \quad r_{S_2}^A = -0.3 \times 10^{-4}. \tag{2.3}$$

- **Set B:**

  However, some scalar meson contributions were found to be missing in previous estimates of the $\mathcal{O}(p^6)$ LECs \cite{3, 4, 5}. The couplings $r_2, ..., r_6$ were fully calculated at large $N_C$ \cite{2}, being expressed in terms of the ratios

  $$\Gamma_R \equiv \Gamma_R \left[ 1 + \alpha_r \frac{m^2_s}{M_R} + m^4_s \frac{M_R}{M^2_R} + \mathcal{O}(m^6_s) \right], \quad \frac{\Gamma_R}{M^3_R} = \frac{\Gamma_R}{M^3_R} \left[ 1 + \beta_r \frac{m^2_s}{M_R} + \mathcal{O}(m^4_s) \right]. \tag{2.4}$$

\[\text{Page 2}\]
3. Phenomenology of the resonance parameters

3.1 Mass splitting up to $O(m^2)$

In the large-$N_C$ limit, the mass splitting of the resonance multiplets can be described at leading order by one single operator $m^R$ [7]

$$
\frac{\mathcal{M}^2}{2} = -\frac{\mathcal{M}^R}{2} + e^R_{m}(RR\chi^+),
$$

(3.1)

which leads at large $N_C$ to the mass eigenstates

$$
M_{I=1} = \mathcal{M}^R - 4e^R_{m}\pi^2 + O(m^4_p) = M_{I=0}^{\text{had}_{d\bar{d}}},
$$

$$
M_{I=\frac{3}{2}} = \mathcal{M}^R - 4e^R_{m}\kappa^2 + O(m^4_p),
$$

$$
M_{I=0}^{(s)} = \mathcal{M}^R - 4e^R_{m}(2\kappa^2 - m^2) + O(m^4_p).
$$

(3.2)

The combined study of the $\rho(770)$, $K^*(892)$ and $\phi(1020)$ masses leads to the values [1]

$$
\mathcal{M}_V = 764.3 \pm 1.1 \text{MeV}, \quad e^V_{m} = -0.228 \pm 0.015.
$$

(3.3)

In the case of the scalars, the lightest $I=1$ resonance is identified with the $a_0(980)$: $M_{I=1} = 984.7 \pm 1.2 \text{MeV}$ [8]. In order to avoid the problem of the mixing of iso-singlet scalars, the analysis is performed with the $I=1/2$ state. The broad $\kappa(800)$ seems to be a possible candidate although the first clear $I=1/2$ scalar resonance signal is provided by the $K^*_0(1430)$ [8]. Hence, we take the conservative estimate $M_{I=1/2} = 1050 \pm 400$ MeV, which ranges from the $\kappa$ up to the $K^*_0(1430)$ mass. This leads then to the values

$$
\mathcal{M}_S = 980 \pm 40 \text{MeV}, \quad e^S_{m} = -0.1 \pm 0.9.
$$

(3.4)

3.2 The splitting of the vector resonance decay width up to $O(m^2)$

The vector decay width into two light pseudo-scalars, $V \to \phi_1\phi_2$, shows the general structure

$$
\Gamma_{V\to\phi_1\phi_2} = C_{V12} \times \frac{M^2_{V\to\phi_1\phi_2}}{48\pi F_{\pi}\lambda^2_{\pi\pi}} \left[ 1 + \frac{m^2_{\phi_1} + m^2_{\phi_2}}{2M^2_{V\to\phi_1\phi_2}} + O(m^4_p) \right]^2,
$$

(3.5)
with the phase-space factor \( \rho_{\pi \pi 12} = M_{\pi}^{-2} \sqrt{(M_{\pi}^2 - (m_1 + m_2)^2)(M_{\pi}^2 - (m_1 - m_2)^2)} \). The \( F_i \) are the physical decay constants for the \( \phi \) pseudo-Goldstones \( (F_{\pi} \simeq 92.4 \text{ MeV} \text{ and } F_{K} \simeq 113 \text{ MeV}) \) and they appear due to the large\( -N_C \) wave function renormalization of the light pseudo-scalars \([2,9]\). \( M_{\pi} \) and \( m_i \) correspond, respectively, to the physical vector and pseudo-scalar mass. Depending on the channel, one has the Clebsch-Gordan \( C_{\phi \pi \pi} = 1 \), \( C_{K K \pi} = 3/4 \) and \( C_{\phi K K} = 1 \). The \( V \phi \phi \) coupling and the mass scaling \( M_{\pi}^n \) depend on the considered lagrangian realization, either Proca fourvector \((n=5)\) or Antisymmetric tensor formalism \((n=3)\) \([6,5,11]\). The combination of the experimental \( K \) and \( \rho \) widths yields \([1]\):

\[
\begin{align*}
\lambda_{\pi \pi} &= g_{\pi} = 0.0846 \pm 0.0008, & \epsilon_{\pi} &= 0.01 \pm 0.09, & \text{(Proca [5, 11])}, \\
\lambda_{\pi \pi} &= G_{\pi} = 63.9 \pm 0.6 \text{MeV}, & \epsilon_{\pi} &= 0.82 \pm 0.10, & \text{(Antisym. [6, 11])}.
\end{align*}
\]

### 3.3 The decay width for the scalar resonance

In the case of the scalar mesons the current knowledge nowadays is still very poor. We had then to rely on the phenomenological lagrangian \((2.1)\) for the description of the \( a_0(980) \rightarrow \pi \eta \) width \([1,6]\), and on the theoretical scalar form-factor constraint \(4c_d c_m = F^2\) \([13]\):

\[
c_d = 26 \pm 7 \text{MeV}, \quad c_m = 80 \pm 21 \text{MeV}, \quad (3.7)
\]

where their large errors stems essentially from the wide range we considered for the \( a_0(980) \) partial width, \( \Gamma_{a_0 \rightarrow \pi \eta} = 75 \pm 25 \text{ MeV} \) \([1]\).

### 3.4 Chiral corrections to \( F_\pi \)

At large\( -N_C \), the wave-function renormalization of the \( \pi \) field is related to the decay constant in the way \( F_\pi = F Z_{\pi}^{-1/2} \) \([9,10]\). The \( m_\pi^2 \) corrections to \( F_\pi \) can be parametrized in the form

\[
F_\pi = F \left[ 1 + \delta F_2 \frac{m_\pi^2}{M_S^2} + \delta F_4 \frac{m_\pi^4}{M_S^4} + \mathcal{O}(m_\pi^6) \right]. \quad (3.8)
\]

The scalar lagrangian \((2.1)\), the mass splitting \((3.2)\) and the former inputs produce the predictions

\[
\begin{align*}
\delta F_2 &= \frac{4c_d c_m}{F^2} = 1, & \delta F_4 &= \frac{8c_d c_m}{F^2} \left( \frac{3c_d c_m}{F^2} - \frac{4c_m^2}{F^2} \right) + \frac{16c_d c_m e^2}{F^2} = -5 \pm 5. \quad (3.9)
\end{align*}
\]

### 3.5 Next-to-next-to-leading order chiral corrections to \( M_V \), \( \Gamma_V \) and \( \Gamma_S \)

The next-to-next-to-leading order corrections (NNLO) to the vector mass are also needed in order to extract the LEC \( r_2 \) \([1]\). At large \( N_C \), the quark mass corrections are given at NNLO by

\[M_{I=1}^2 = M_{V}^2 - 4c_m^2 m_\pi^2 - 4c_m^4 m_\pi^4 / M_{V}^2.\]

Demanding that the NNLO terms never overcome the NLO corrections in the vector multiplet sets the range \(|e_{\pi}^V| \leq \frac{M_V^2}{2m_\pi^2 - m_\pi^2} |e_{\pi}^V| \simeq 0.3 \) \([1]\).

The determination of \( r_2 \) also needs the NNLO chiral corrections \( \tilde{c}_R \) to the resonance widths

\[
\begin{align*}
\Gamma_{\rho \rightarrow \pi \pi} &= \frac{M_\rho^3}{4 \pi F_\pi^3} \lambda_{\rho \pi \pi}^2 \left[ 1 + \epsilon_\pi \frac{m_\pi^2}{M_{V}^2} + \bar{\epsilon}_\pi \frac{m_\pi^2}{M_{V}^2} + \mathcal{O}(m_\pi^6) \right]^2,
\quad \Gamma_{\sigma \rightarrow \pi \pi} = \frac{3M_\sigma^3}{16 \pi F_\pi^4} \lambda_{\sigma \pi \pi}^2 \left[ 1 + \epsilon_\pi \frac{m_\pi^2}{M_{S}^2} + \bar{\epsilon}_\pi \frac{m_\pi^2}{M_{S}^2} + \mathcal{O}(m_\pi^6) \right]^2. \quad (3.10)
\end{align*}
\]
The phenomenological lagrangian (2.1) \([6, 5]\) yields the predictions
\[
\bar{\varepsilon}_V = \varepsilon_V \left[ \frac{8c_m(c_d - c_m)M_V^2}{F^2} + 4\varepsilon_m^V \right] = \begin{cases} 
-0.03 \pm 0.18, & \text{(Proca)} \\
-1.6 \pm 0.9, & \text{(Antisym.)}
\end{cases}
\]
\[
\bar{\varepsilon}_S = \frac{16c_m^2(c_d - c_m)}{c_d F^2} + 8(c_m - c_d)\varepsilon_m^S = -7 \pm 12.
\]

(3.11)

4. Low-energy constant determination at \(\mathcal{O}(p^6)\)

Based on the partial-wave dispersion relations developed in Refs. \([2, 12]\), it is possible to extract the large–\(N_C\) values of \(r_2, \ldots, r_6\) from the \(I = 1\) vector and \(I = 0\) scalar \((\bar{u} u + \bar{d} d)\) width and mass ratios \(\Gamma_R/M_R^2\) and \(\Gamma_R/M_R^2\): the couplings \(r_5\) and \(r_6\) are determined by the \(\Gamma_R/M_R^2\) ratio in the chiral limit; \(r_3\) and \(r_4\) also require its first \(m_K^2\) correction \(\beta_R\); those and the NNLO \(m_K^2\) contribution to \(\Gamma_R/M_R^2\) are needed in order to obtain \(r_2\). All these LECs have been found to be dominated by the vector resonance exchanges. The \(\mathcal{O}(p^6)\) couplings \(r_1\) \([5]\) and \(r_{S_2}\) \([5]\) could not be computed through the partial-wave dispersion relations in \([2, 12]\). They were calculated directly from the phenomenological lagrangian (2.1):

\[
r_{1,\text{Proca}} = -\frac{16c_d c_m (8c_d^2 - 17c_d c_m + 12c_m^2)}{M_s^4} + \frac{32(c_d - c_m)^2 F^2}{M_s^4} \varepsilon_m^S \\
- \frac{16g_V^2 F^2}{M_s^4} \left[ 1 + \varepsilon_V + \frac{1}{4} \varepsilon_V^2 - \frac{8c_d c_m M_V^2}{F^2 M_s^2} \right],
\]

(4.1)

\[
r_{S_2} = \frac{8c_m(c_m - c_d) F^2}{M_s^4} - \frac{32c_d^2 c_m^2}{M_s^4} + \frac{16c_d c_m F^2}{M_s^4} \varepsilon_m^S.
\]

(4.2)

The expression for \(r_1\) in the Antisymmetric tensor formalism is similar to (4.1) but with the second line replaced by \(-\frac{16g_V^2 F^2}{M_s^4} \left[ 1 + \varepsilon_V + \frac{8c_d c_m M_V^2}{F^2 M_s^2} + 2\varepsilon_V^2 \right]\). All this leads to the values of the low-energy constants shown in Table 1. The first error derives from the phenomenological inputs and the second one stems from the uncertainty on the saturation scale \(\mu_s\) where \(r_i^\infty(\mu_s) = r_i^{N_C \to \infty}\) \([1]\).

<table>
<thead>
<tr>
<th>(10^4 \cdot r_i^1)</th>
<th>ND est. ([14])</th>
<th>set A ([3, 5])</th>
<th>set C (Proca)</th>
<th>set C (Antisym.)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(10^4 \cdot r_1^2)</td>
<td>±80</td>
<td>-0.6</td>
<td>-14 ± 17 ± 3</td>
<td>-20 ± 17 ± 3</td>
</tr>
<tr>
<td>(10^4 \cdot r_2^2)</td>
<td>±40</td>
<td>1.3</td>
<td>22 ± 16 ± 4</td>
<td>7 ± 10 ± 4</td>
</tr>
<tr>
<td>(10^4 \cdot r_3^2)</td>
<td>±20</td>
<td>-1.7</td>
<td>-3 ± 1 ± 3</td>
<td>-4 ± 1 ± 3</td>
</tr>
<tr>
<td>(10^4 \cdot r_4^2)</td>
<td>±3</td>
<td>-1.0</td>
<td>-0.22 ± 0.13 ± 0.05</td>
<td>0.13 ± 0.13 ± 0.05</td>
</tr>
<tr>
<td>(10^4 \cdot r_5^2)</td>
<td>±6</td>
<td>1.1</td>
<td>0.9 ± 0.1 ± 0.5</td>
<td>0.9 ± 0.1 ± 0.5</td>
</tr>
<tr>
<td>(10^4 \cdot r_6^2)</td>
<td>±2</td>
<td>0.3</td>
<td>0.25 ± 0.01 ± 0.05</td>
<td>0.25 ± 0.01 ± 0.05</td>
</tr>
<tr>
<td>(10^4 \cdot r_{S_2})</td>
<td>±1</td>
<td>-0.3</td>
<td>1 ± 4 ± 1</td>
<td>1 ± 4 ± 1</td>
</tr>
</tbody>
</table>

Table 1: Different predictions for the \(\mathcal{O}(p^6)\) LECs \(r_i^\infty(\mu)\) for \(\mu = 770\) MeV: The first column presents the order of magnitude estimate based on naive dimensional analysis \([14]\); In the set A column we show former estimates from Refs. \([3, 5]\); in the last two columns, one can find the values for the present reanalysis.
5. Scattering lengths

The $\chi$PT expression for the scattering lengths contains at $\mathcal{O}(p^6)$ a series of logarithmic terms together with analytical $\mathcal{O}(p^6)$ contributions [3]. Although the first ones are the most complicated to compute, their value is nevertheless rather sound and under control. On the other hand, the local terms are determined by the $\mathcal{O}(p^6)$ LECs $a_i$ and, although they can be easily computed, these couplings are badly known and their estimation is pretty cumbersome.

In Ref. [4], Colangelo et al. combined the NNLO chiral perturbation theory computation of the scattering lengths [3] with a phenomenological dispersive representation. This allowed them to produce one of the most precise determinations of the scattering lengths. They were expressed in terms of some dispersive integrals, the pion quadratic scalar radius $\langle r^2 \rangle_\pi$, the $\mathcal{O}(p^4)$ coupling $\ell_3$ and a set of $\mathcal{O}(p^6)$ LECs $(r_1, r_2, r_3, r_4, r_S)$. Following the work of Ref. [4], we extracted the part of their scattering lengths that depended on the inputs $a_i(\mu)$ [1, 4]:

\[
\begin{align*}
\left(\frac{m_\pi^2}{F_\pi F_\pi}\right) C_0 |_{r_1} &= \frac{m_\pi^6}{32\pi F_\pi^6} \left[ 5r'_1 + 12r'_2 + 28r'_4 - 28r'_4 - 14r_S \right], \\
\left(\frac{m_\pi^2}{16\pi F_\pi^6}\right) C_2 |_{r_i} &= \frac{m_\pi^6}{16\pi F_\pi^6} \left[ r'_1 - 4r_3 + 4r_4 + 2r_S \right].
\end{align*}
\]

The largest contributions to the $a_0^0$ and $a_0^2$ errors are found to be produced in similar terms by $r_1, r_2, r_3$ and $r_S$, being the impact of $r_4$ negligible.

| $(\times 10^{-3})$ | Total: Ref. [4] | $a_j^i |_{r_i}$ [4] | $a_j^i |_{r_i}$ Set C (Proca) | $a_j^i |_{r_i}$ Set C (Antisym.) |
|-------------------|-----------------|-----------------|-----------------|-----------------|
| $a_0^0$           | 220 ± 5         | 0.0 ± 1.0       | 1.0 ± 1.5 ± 1.0 | −1.6 ± 1.5 ± 1.0 |
| $10a_0^2$         | −444 ± 10       | 0.4 ± 2.0       | 0 ± 4 ± 2       | 0 ± 4 ± 2       |

Table 2: The first and second columns show, respectively, the total scattering lengths and the $r_i$ contribution to them in the dispersive method from Colangelo et al. [4], where the authors used the $r_i$ in Eq. (2.3), $F_\pi = 92.4$ MeV and $m_\pi = 139.57$ MeV. The last two columns show the reanalyzed quantities $a_j^i |_{r_i}$ (set C) for the Proca and antisymmetric tensor formalisms for the usual scale $\mu = 770$ MeV. There, the first error derives from the inputs and the second one from the saturation scale uncertainty.

6. Summary and conclusion

The combination of the Proca and antisymmetric results yields for our prediction of the LECs the final numbers (for $\mu = 770$ MeV),

\[
\begin{align*}
\left(\times 10^{-4}\right) & & \\
r'_1 &= (−17 ± 20) & & \left(\times 10^{-4}\right), \\
r'_2 &= (17 ± 21) & & \left(\times 10^{-4}\right), \\
r'_4 &= (0.0 ± 0.3) & & \left(\times 10^{-4}\right), \\
r'_5 &= (0.9 ± 0.5) & & \left(\times 10^{-4}\right), \\
r'_6 &= (0.25 ± 0.05) & & \left(\times 10^{-4}\right), \\
r'_S &= (1 ± 4) & & \left(\times 10^{-4}\right).
\end{align*}
\]

The $r'_i$ contributions to the scattering lengths with the Colangelo et al.’s method [4] can be summarized in the predictions

\[
10^3 a_0^0 |_{r_i} = 0 ± 3, \quad 10^4 a_0^2 |_{r_i} = 0 ± 5.
\]
Following the analysis of global uncertainties of Ref. [4] leads to the updated values $a_0^0 = 0.220 \pm 0.005$ and $10a_0^2 = -0.444 \pm 0.011$. These values leave essentially unchanged the previous determinations $a_0^0 = 0.220 \pm 0.005$ and $10a_0^2 = -0.444 \pm 0.010$ [4].

This calculation shows that the determinations of the scattering lengths through the dispersive method and $r_i$ resonance saturation estimates are rather solid [4]. Our detailed analysis shows that the error stemming from the $r_i$ does not modify the final numbers quoted in Ref. [4]. Nonetheless, unless the the precision in the $O(p^6)$ low-energy constants is conveniently increased, it will be difficult to carry on further relevant improvements in the scattering length determinations.

References