

Electromagnetic effects in $K_{\ell 3}$ decays

Helmut NEUFELD*[†]

Faculty of Physics, University of Vienna, Boltzmannngasse 5, A-1090 Wien, Austria

E-mail: Helmut.Neufeld@univie.ac.at

I discuss a recent calculation of the radiative corrections for all $K_{\ell 3}$ decay modes. I review a new update of the determination of the isospin breaking parameter $\delta_{\text{SU}(2)}^{K\pi}$ using current information on the electromagnetic contributions to the pseudoscalar masses. I compare the theoretical result with an “experimental” determination of this parameter from $K_{\ell 3}$ data.

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1. $K_{\ell 3}$ decays

In the standard model, transitions of quark flavours in weak interactions are described by a unitary matrix, the famous Cabibbo-Kobayshi-Maskawa (CKM) mixing matrix. The $K_{\ell 3}$ decays

$$K^0(p_K) \rightarrow \pi^-(p_\pi)\ell^+(p_\ell)\nu_\ell(p_\nu), \quad K^+(p_K) \rightarrow \pi^0(p_\pi)\ell^+(p_\ell)\nu_\ell(p_\nu) \quad (1.1)$$

(and their charge conjugate modes) are an important source of information on the the CKM matrix element V_{us} which is needed for a test of the unitarity relation $|V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 = 1$. These decays also provide competitive probes of lepton universality, the ratios of light quark masses and the dependence of the associated form factors on the momentum transfer to the lepton pair. With the present high-statistics experiments (see [1] and references therein), the Cabibbo angle V_{us} can be measured with a precision at the subpercent level. In order to fully exploit these amazing experimental achievements, it is mandatory to have full theoretical control of these decays at the percent level or better. This requires in particular a proper treatment of electromagnetic corrections and other isospin breaking effects at all stages of the analysis.

Including electromagnetic corrections, the fully inclusive decay rate for all four $K_{\ell 3}$ modes ($K = K^\pm, K^0, \bar{K}^0; \ell = e, \mu$) can be written as

$$\Gamma(K_{\ell 3}[\gamma]) = \frac{G_F^2 |V_{us}|^2 M_K^5 C_K^2}{128 \pi^3} S_{\text{EW}} |f_+^{K^0 \pi^-}(0)|^2 I_{K\ell}^{(0)}(\lambda_i) \left(1 + \delta_{\text{EM}}^{K\ell} + \delta_{\text{SU}(2)}^{K\pi}\right). \quad (1.2)$$

In this formula, G_F is the Fermi decay constant as extracted from muon decays, the Clebsch-Gordan coefficient C_K differs for neutral and charged kaons ($C_K = 1$ for $K_{\ell 3}^0$ and $C_K = 1/\sqrt{2}$ for $K_{\ell 3}^\pm$), while $I_{K\ell}^{(0)}$ is a phase-space integral depending on slope and curvature of the form factors $f_\pm^{K\pi}(t)$. As usual, the vector form factor of the K^0 decay at zero momentum transfer has been pulled out. The strong isospin breaking correction is defined as $\delta_{\text{SU}(2)}^{K\pi} = (f_+^{K\pi}(0)/f_+^{K^0 \pi^-}(0))^2 - 1$. The electroweak short-distance enhancement factor [2, 3]

$$S_{\text{EW}} = 1 + \frac{2\alpha}{\pi} \left(1 - \frac{\alpha_s}{4\pi}\right) \log \frac{M_Z}{M_\rho} + \mathcal{O}\left(\frac{\alpha\alpha_s}{\pi^2}\right) = 1.0223 \pm 0.0005 \quad (1.3)$$

is universal for all semileptonic decays. The long-distance electromagnetic corrections $\delta_{\text{EM}}^{K\ell} = \delta_{\text{EM}}^{K\ell}(\mathcal{D}_3) + \delta_{\text{EM}}^{K\ell}(\mathcal{D}_{4-3})$ receive contributions from three-particle and four-particle final states.

2. Long-distance electromagnetic corrections

The appropriate theoretical tool to analyze electromagnetic corrections is provided by chiral perturbation theory with virtual photons [4–6] and leptons [7]. Using this theoretical framework, the calculation of electromagnetic contributions to $\mathcal{O}(e^2 p^2)$ in $K_{\ell 3}$ decays was presented in [8]. Based on this analysis, full numerical results on the K_{e3} decay modes were given in [9] adopting a specific prescription for treating real photon emission and a specific factorization scheme for soft photons. This approach resulted in the partial inclusion of higher order terms in the chiral expansion.

In a recent publication [10] we have extended our numerical analysis of electromagnetic corrections also to a complete study of $K_{\mu 3}$ decays. At the same time, we have updated our previous

results for the K_{e3} modes using the new estimates [11, 12] of the electromagnetic low-energy constants which affect the structure-dependent electromagnetic contributions. Rather than using the soft-photon factorization procedure of [8], we were working to fixed chiral order $e^2 p^2$. In contrast to our previous studies, we were also employing a fully inclusive prescription for real photon emission being more appropriate for comparison with experimental results.

Table 1 summarizes our numerical results for the long-distance radiative corrections. Two characteristic features can be understood by qualitative arguments. Firstly, the electromagnetic corrections for the neutral K decays are expected to be positive and sizable on account of the final state Coulomb interaction between ℓ^+ and π^- producing a correction factor of $\pi\alpha/v_{\ell^+\pi^-}^{\text{rel}} \sim 2\%$ over most of the Dalitz plot. While the exact correction and the relative size of $K_{\mu 3}^0$ and K_{e3}^0 depend on other effects such as the emission of real photons, the qualitative expectation based on the Coulomb interaction is confirmed by the detailed calculation. Secondly, the large hierarchy $\delta_{\text{EM}}^{K\mu}(\mathcal{D}_{4-3}) \ll \delta_{\text{EM}}^{Ke}(\mathcal{D}_{4-3})$ admits a simple interpretation in terms of bremsstrahlung off the charged lepton in the final state. The probability of emitting soft photons is a function of the lepton velocity v_ℓ which becomes logarithmically singular as $v_\ell \rightarrow 1$, thus enhancing the electron emission. For typical values of v_ℓ in \mathcal{D}_{4-3} , the semiclassical emission probability implies $\delta_{\text{EM}}^{Ke}(\mathcal{D}_{4-3})/\delta_{\text{EM}}^{K\mu}(\mathcal{D}_{4-3}) \sim 20 \rightarrow 40$.

	$\delta_{\text{EM}}^{K\ell}(\mathcal{D}_3)(\%)$	$\delta_{\text{EM}}^{K\ell}(\mathcal{D}_{4-3})(\%)$	$\delta_{\text{EM}}^{K\ell}(\%)$
K_{e3}^0	0.50	0.49	0.99 ± 0.22
K_{e3}^\pm	-0.35	0.45	0.10 ± 0.25
$K_{\mu 3}^0$	1.38	0.02	1.40 ± 0.22
$K_{\mu 3}^\pm$	0.007	0.009	0.016 ± 0.25

Table 1: Electromagnetic corrections to the $K_{\ell 3}$ decay rates given in [10], calculated to fixed order in chiral perturbation theory.

The theoretical uncertainties assigned to the $\delta_{\text{EM}}^{K\ell}$ in Table 1 arise from two sources: the input parameters (low-energy constants and form factor parameters) used in the calculation and unknown higher order terms in the chiral expansion (the latter would require a complete analysis at order $e^2 p^4$). The experimental errors of λ_+ , λ_+'' and λ_0 induce a fractional uncertainty in $\delta_{\text{EM}}^{K\ell}$ well below the percent level. This source of uncertainty can safely be neglected. The structure dependent contributions to the electromagnetic corrections are determined by the electromagnetic low-energy couplings X_1 and X_6^{phys} . Taking the central values from [12] and assigning very conservative 100% fractional uncertainties to these parameters, we find an absolute uncertainty of $\pm 0.11\%$ in $\delta_{\text{EM}}^{K^0\ell}$ and of $\pm 0.16\%$ in $\delta_{\text{EM}}^{K^\pm\ell}$.

For an estimate of the uncertainties due to higher order chiral corrections not included in the analysis of [10], it is useful to decompose the EM corrections $\delta^{K\ell}$ in terms of parameters $\delta_{1,2,3,4}$,

$$\begin{aligned} \delta^{K^0e} &= \delta_1 + \delta_2 + \delta_3 + \delta_4, & \delta^{K^0\mu} &= \delta_1 + \delta_2 - \delta_3 - \delta_4, \\ \delta^{K^\pm e} &= \delta_1 - \delta_2 + \delta_3 - \delta_4, & \delta^{K^\pm\mu} &= \delta_1 - \delta_2 - \delta_3 + \delta_4, \end{aligned} \quad (2.1)$$

where δ_1 represents a correction common to all modes, δ_2 a correction anti-correlated in kaon isospin but blind to lepton flavor, and finally $\delta_{3,4}$ are lepton-universality breaking terms, correlated

and anti-correlated in kaon isospin, respectively. Our calculation at order $e^2 p^2$ gives $\delta_1 = 0.63\%$, $\delta_2 = 0.57\%$, $\delta_3 = -0.08\%$, $\delta_4 = -0.12\%$. On the basis of chiral power counting we expect the higher order corrections to scale with the factor $M_K^2/(4\pi F_\pi)^2 \sim 0.2$ which gives the main contributions to the errors shown in the last column of Table 1.

This error estimate is validated by comparison of the fixed chiral order results with the ones obtained within the ‘‘soft-photon factorization’’ approach of [8], which include a class of higher order chiral corrections (see Table 2). The only anomaly appears to be in the coefficient δ_3 , where one finds $\delta_3 : -0.08\% \rightarrow -0.16\%$ when going from fixed chiral order to the soft factorization scheme. This can be traced back to the cancellation between the negative contribution from \mathcal{D}_3 (-0.31%) and the positive contribution from \mathcal{D}_{4-3} (0.23%). Multiplying these individual pieces by 0.2 gives ~ 0.06 and ~ 0.05 , respectively, which is just the order of magnitude of the shift one sees ($-0.08 \rightarrow -0.16$).

	$\delta_{\text{EM}}^{K\ell}(\mathcal{D}_3)(\%)$	$\delta_{\text{EM}}^{K\ell}(\mathcal{D}_{4-3})(\%)$	$\delta_{\text{EM}}^{K\ell}(\%)$
K_{e3}^0	0.41	0.59	1.0
K_{e3}^\pm	-0.564	0.528	-0.04
$K_{\mu 3}^0$	1.57	0.04	1.61
$K_{\mu 3}^\pm$	-0.006	0.011	0.005

Table 2: Electromagnetic corrections to the $K_{\ell 3}$ decay rates taken from [10], using the ‘‘soft photon factorization’’ approach of [8] which includes a part of the higher order terms in the chiral expansion. Comparison with the results shown in the previous table validates the estimate of theoretical uncertainties.

The differential decay distribution can be written in the form

$$\frac{d\Gamma}{dydz} = \frac{G_F^2 |V_{us}|^2 M_K^5 C_K^2}{128 \pi^3} S_{\text{EW}} |f_+^{K\pi}(0)|^2 \left[\bar{\rho}^{(0)}(y, z) + \delta \bar{\rho}^{\text{EM}}(y, z) \right], \quad (2.2)$$

where the Lorentz invariants $y = 2p_K \cdot p_\ell / M_K^2 = 2E_\ell / M_K$ and $z = 2p_\pi \cdot p_K / M_K^2 = 2E_\pi / M_K$ are related to the energy of the charged lepton and the pion, respectively, measured in the rest frame of the kaon. We emphasize that (2.2) should be the basis to properly determine the momentum dependence of the QCD $K_{\ell 3}$ form factors appearing in $\bar{\rho}^{(0)}(y, z)$. For a further detailed discussion, we refer to [10]. Here we just show the density plots of the ratio $\bar{\rho}^{\text{EM}}/\bar{\rho}^{(0)}$ for K_{e3}^0 , K_{e3}^\pm , $K_{\mu 3}^0$ and $K_{\mu 3}^\pm$ in the Figures 1 - 4. The theoretical uncertainties discussed above induce an overall uncertainty on $\bar{\rho}^{\text{EM}}/\bar{\rho}^{(0)}$ of about $\pm 0.3\%$. We wish to stress that the correction to the Dalitz distribution can be locally large ($\sim 10\%$) and does not have a definite sign, implying cancellations in the integrated total electromagnetic corrections.

3. Quark mass ratios and $K_{\ell 3}$ decays

The theoretical expression for the isospin-breaking parameter of $K_{\ell 3}$ takes the form

$$\delta_{\text{SU}(2)}^{K^+\pi^0} = 2\sqrt{3} \left(\varepsilon^{(2)} + \varepsilon_S^{(4)} + \varepsilon_{\text{EM}}^{(4)} + \dots \right). \quad (3.1)$$

It is dominated by the the lowest-order $\pi^0 - \eta$ mixing angle $\varepsilon^{(2)}$ which is related to the quark mass ratio $R = (m_s - \hat{m})/(m_d - m_u)$ by $\varepsilon^{(2)} = \sqrt{3}/4R$ (\hat{m} denotes the mean-value of m_u and m_d).

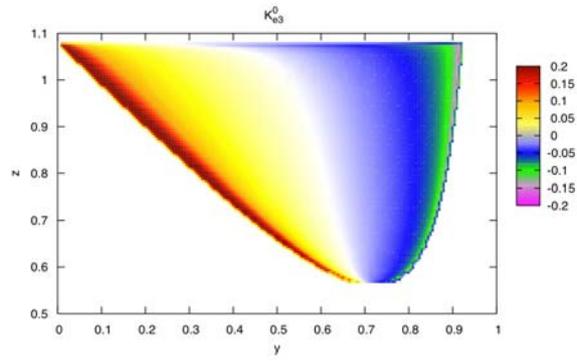


Figure 1: Ratio $\delta\bar{\rho}^{\text{EM}}(y,z)/\bar{\rho}^{(0)}(y,z)$ for K_{e3}^0

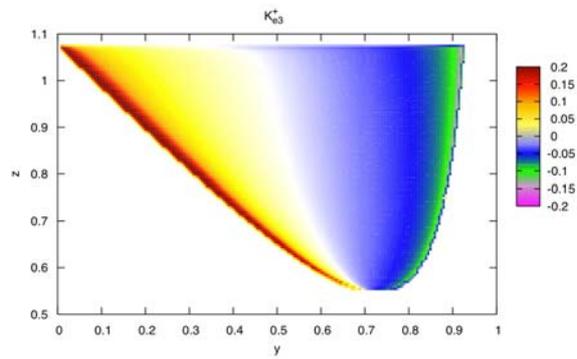


Figure 2: Ratio $\delta\bar{\rho}^{\text{EM}}(y,z)/\bar{\rho}^{(0)}(y,z)$ for K_{e3}^+

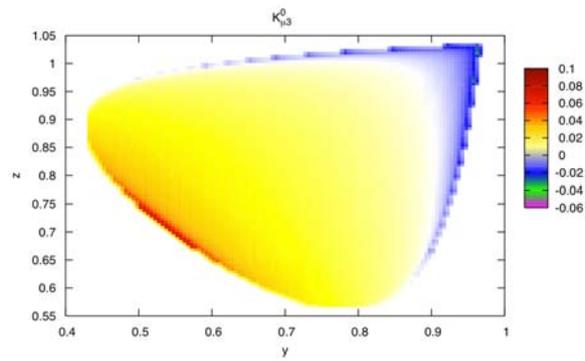


Figure 3: Ratio $\delta\bar{\rho}^{\text{EM}}(y,z)/\bar{\rho}^{(0)}(y,z)$ for $K_{\mu 3}^0$

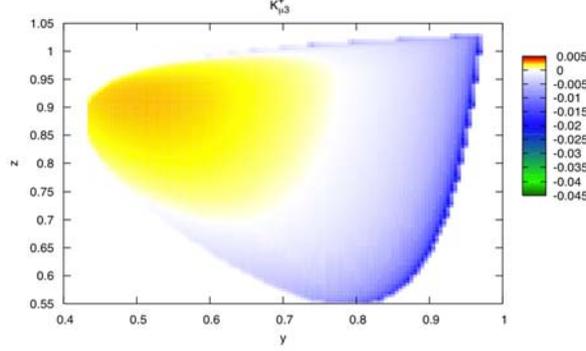


Figure 4: Ratio $\delta \bar{\rho}^{\text{EM}}(y, z) / \bar{\rho}^{(0)}(y, z)$ for $K_{\mu 3}^+$

The next-to-leading order corrections $\varepsilon_S^{(4)}$ (order p^4) and $\varepsilon_{\text{EM}}^{(4)}$ (order $e^2 p^2$) were computed in [13] and [5], respectively. The explicit expressions for these quantities can be found in [8]. The dots refer to next-to-next-to-leading order contributions (arising at order p^6), the latest results can be found in [14].

The standard strategy up to now has been to use all known information on the ratios of light quarks to predict $\delta_{\text{SU}(2)}$ using Eq. (3.1). However, the present precision of the decay rates and the radiative corrections permits also an “experimental” determination of $\delta_{\text{SU}(2)}$, which can be used as a constraint on the quark mass ratio R via the formula (3.1).

The traditional strategy starts with the observation [15] that the double ratio of quark masses $Q^2 = (m_s^2 - \hat{m}^2) / (m_d^2 - m_u^2) = R(m_s / \hat{m} + 1) / 2$ can be expressed in terms of meson masses and a purely electromagnetic EM contribution:

$$Q^2 = \frac{\Delta_{K\pi} M_K^2 (1 + \mathcal{O}(m_q^2))}{M_\pi^2 [\Delta_{K^0 K^+} + \Delta_{\pi^+ \pi^0} - (\Delta_{K^0 K^+} + \Delta_{\pi^+ \pi^0})_{\text{EM}}]}, \quad \Delta_{PQ} = M_P^2 - M_Q^2. \quad (3.2)$$

The electromagnetic term $(\Delta_{K^0 K^+} + \Delta_{\pi^+ \pi^0})_{\text{EM}}$ vanishes to lowest order $e^2 p^0$, a calculation at order $e^2 p^2$ gives [4, 5]

$$\begin{aligned} (\Delta_{K^0 K^+} + \Delta_{\pi^+ \pi^0})_{\text{EM}} = e^2 M_K^2 & \left[\frac{1}{4\pi^2} \left(3 \ln \frac{M_K^2}{\mu^2} - 4 + 2 \ln \frac{M_K^2}{\mu^2} \right) \right. \\ & \left. + \frac{4}{3} (K_5 + K_6)^r(\mu) - 8(K_{10} + K_{11})^r(\mu) + 16ZL_5^r(\mu) \right] + \mathcal{O}(e^2 M_\pi^2). \end{aligned} \quad (3.3)$$

The current estimates [11] of the electromagnetic low-energy constants appearing in this expression imply a large deviation from Dashen’s limit, $(\Delta_{K^0 K^+} + \Delta_{\pi^+ \pi^0})_{\text{EM}} = -1.5 \Delta_{\pi^+ \pi^0}$ [11], which gives $Q = 20.7 \pm 1.2$ [16]. Such a small value of Q (compared to $Q = 22.7 \pm 0.8$ given by Leutwyler in 1996 [17]) was also supported by other studies ($Q = 22.0 \pm 0.6$ in [18] and $Q \simeq 20$ in [19]). It should be noted, however, that the rather large value $Q = 23.2$ was obtained from an analysis of $\eta \rightarrow 3\pi$ at two loops [20]. On the other hand, the determinations of the second input parameter $m_s / \hat{m} \sim 24$ remained rather stable over the last years. Combining $Q = 20.7 \pm 1.2$ with $m_s / \hat{m} = 24.7 \pm 1.1$, we find $R = 33.5 \pm 4.3$ and finally $\delta_{\text{SU}(2)} = 0.058(8)$ [16]. It is amazing to see that this value agrees exactly with $\delta_{\text{SU}(2)\text{exp.}} = 0.058(8)$ obtained from $K_{\ell 3}$ data by the FLAVIANet Kaon Working Group [1].

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