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Uncertainties of QCD predictions for Higgs boson decay into bottom quarks at NNLO and beyond [*]

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The importance of detailed studies of theoretical QCD predictions for the decay width of the Standard Model Higgs boson into bottom quarks, in the case when $\mathrm{M_H} \leq 2\mathrm{M_W}$, is emphasized. The effects of higher order perturbative QCD corrections up to order α_s^4-terms are considered. The resummation of the π^2 terms resulting from analytical continuation, typical of Minkowskian quantities, is undertaken for the $\Gamma(H \to \overline{b}b)$ decay width. The uncertainties in the calculation of this decay width are analyzed.

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1. Introduction

Production cross-sections and decay widths of the Standard Electroweak Model Higgs boson are nowadays among the most extensively analyzed theoretical quantities (for recent reviews, see, e.g., [1], [2]). Indeed, the main hope of scientific community is that this essential ingredient of the Standard Model may be discovered, if not at Fermilab Tevatron, then at the forthcoming LHC experiments at CERN. There is great interest in the "low-mass" region 114.5 GeV$\leq \mathrm{M_H} \leq 2\mathrm{M_W}$, because a "low-mass" Higgs boson is heavily favored by Standard Model analysis of the available precision data. The lower bound, 114.5 GeV, was obtained from the direct searches of Higgs boson at the LEP2 e^+e^--collider primarily through Higgs boson decay into a $\overline{b}b$ pair. The decay mode $H \to \overline{b}b$, which is dominant for the "low-mass" region, is important for Higgs boson searches in certain associated (semi-inclusive) Higgs boson production processes at the Tevatron and the LHC. It is also the main decay mode for diffractively produced Higgs boson searches in CMS-TOTEM and, possibly, FP420 experiments.

It should be stressed, that the uncertainties in $\Gamma(H \to \overline{b}b)$, analytically calculated in QCD using the $\overline{\mathrm{MS}}$-scheme at the α_s^4-level [3], dominate the theoretical uncertainty for the branching ratio of $H \to \gamma\gamma$ decay, which is considered to be the most important process in searches for a "low mass" Higgs boson by CMS and ATLAS collaborations at the LHC. Moreover, since at present the QCD corrections to the Higgs boson production cross-sections at Tevatron and LHC are known at the next-to-next-to-leading order (NNLO) and, partly, even beyond (see, e.g., [7]), it becomes important to understand how to estimate the theoretical error-bars of QCD-predictions for both the production cross-sections and Higgs-boson decay widths, which are also calculated in QCD beyond the next-to-leading order (NLO) level. Here we will focus on the analysis of the concrete uncertainties of the QCD predictions for $\Gamma_{\mathrm{H\bar{b}b}} = \Gamma(H \to \overline{b}b)$, including those which come from the on-shell mass parameterizations of this quantity (previous related discussions see in [8]-[13]) and from the resummations of the π^2 terms, typical of the Minkowskian region (see [14]- [18]).

2. QCD expressions for $\Gamma_{\mathrm{H\bar{b}b}}$ in the $\overline{\mathrm{MS}}$-scheme

2.1 Basic results in terms of running b-quark mass

The QCD prediction for $\Gamma_{\mathrm{H\bar{b}b}}$ in the $\overline{\mathrm{MS}}$-scheme is of the form

$$\Gamma_{\mathrm{H\bar{b}b}} = \Gamma_0^b \frac{\overline{\mathrm{m}}_\mathrm{b}^2(\mathrm{M_H})}{\mathrm{m_b^2}} \left[1 + \sum_{i\geq 1} \Delta\Gamma_\mathrm{i}\, a_s^i(\mathrm{M_H})\right]. \tag{2.1}$$

Here $\Gamma_0^b = 3\sqrt{2}/8\pi \mathrm{G_F M_H m_b^2}$, $\overline{\mathrm{m}}_\mathrm{b}$ and $\mathrm{M_H}$ are the pole b-quark and Higgs boson masses, $a_s(\mathrm{M_H}) = \alpha_\mathrm{s}(\mathrm{M_H})/\pi$ and $\overline{\mathrm{m}}_\mathrm{b}(\mathrm{M_H})$ are the QCD running parameters, defined in the $\overline{\mathrm{MS}}$-scheme. The coefficients $\Delta\Gamma_\mathrm{i}$ can be expressed through the sum of the following contributions: the positive contributions $\mathrm{d_i^E}$, calculated directly in the Euclidean region, and the ones proportional to π^2- factors, which are typical for the Minkowski time-like region.

The corresponding expressions were derived at the α_s^4-level in Ref. [19] and have the following form:

$$\Delta\Gamma_1 = \mathrm{d}_1^{\mathrm{E}} = \frac{17}{3}; \tag{2.2}$$

$$\Delta\Gamma_2 = \mathrm{d}_2^{\mathrm{E}} - \gamma_0(\beta_0+2\gamma_0)\pi^2/3; \tag{2.3}$$

$$\Delta\Gamma_3 = \mathrm{d}_3^{\mathrm{E}} - \left[\mathrm{d}_1^{\mathrm{E}}(\beta_0+\gamma_0)(\beta_0+2\gamma_0)+\beta_1\gamma_0+2\gamma_1(\beta_0+2\gamma_0)\right]\pi^2/3; \tag{2.4}$$

$$\begin{aligned}\Delta\Gamma_4 = {} & \mathrm{d}_4^{\mathrm{E}} - \left[\mathrm{d}_2^{\mathrm{E}}(\beta_0+\gamma_0)(3\beta_0+2\gamma_0)+\mathrm{d}_1^{\mathrm{E}}\beta_1(5\beta_0+6\gamma_0)/2\right. \\ & + \left. 4\mathrm{d}_1^{\mathrm{E}}\gamma_1(\beta_0+\gamma_0)+\beta_2\gamma_0+2\gamma_1(\beta_1+\gamma_1)+\gamma_2(3\beta_0+4\gamma_0)\right]\pi^2/3 \\ & + \gamma_0(\beta_0+\gamma_0)(\beta_0+2\gamma_0)(3\beta_0+2\gamma_0)\pi^4/30, \end{aligned} \tag{2.5}$$

where the $\mathrm{n_f}$-dependence of $\mathrm{d}_i^{\mathrm{E}}$ ($2 \leq i \leq 4$) was evaluated in [20], [19] and [3] (the detailed results are presented in [13]). The coefficients β_i and γ_i are the perturbative coefficients of the QCD renormalization group (RG) β-function and mass anomalous dimension function γ_m of the MS-like schemes. The QCD β-function will be considered at the 5-loop level:

$$\frac{da_s}{d\ln\mu^2} = \beta(a_s) = -\beta_0 a_s^2 - \beta_1 a_s^3 - \beta_2 a_s^4 - \beta_3 a_s^5 - \beta_4 a_s^6 + O(a_s^7). \tag{2.6}$$

The expressions for β_0 and β_1 are well-known and are scheme-independent. The coefficients β_2 and β_3 were analytically evaluated in [22] and [23] and confirmed by independent calculations at the level of 3-loops [24] and 4-loops [25]. The 5-loop coefficient β_4 is still unknown, and will be estimated using the Padé approximation procedure, developed in [26]. The mass anomalous dimension function is defined as

$$\frac{d\ln\overline{\mathrm{m}}_b}{d\ln\mu^2} = \gamma_m(a_s) = -\gamma_0 a_s - \gamma_1 a_s^2 - \gamma_2 a_s^3 - \gamma_3 a_s^4 - \gamma_4 a_s^5 + O(a_s^6). \tag{2.7}$$

The 4-loop correction $\gamma_3 a_s^4$ was independently calculated in [27] and [28]. At 5-loops γ_4 may be modelled using the Padé approximation procedure of Ref.[26] mentioned above. Note that the consideration of the explicit $\mathrm{n_f}$ dependence of the 5-loop coefficients of Eq.(2.6) and Eq.(2.7) presented in Ref.[13] strongly suggests that the available Padé estimate of the γ_4-coefficient contains more uncertainties than the Padé estimate of the coefficient β_4, given in Ref.[26]. This conclusion is supported in part by the fact that the Padé estimated value of the $\mathrm{n_f^3}$-part of γ_4 [26] is over 3 times smaller, than the result of the explicit analytical calculation of Ref. [29]. It should be stressed, however, that the uncertainties of the estimated 5-loop contributions to the QCD β-function and mass anomalous dimension function γ_m are not so important in the definition of the running of the b-quark mass from the pole mass $\mathrm{m_b}$ to the pole mass of Higgs boson $\mathrm{M_H}$. This effect of running is described by the solution of the following RG equation:

$$\overline{\mathrm{m}}_\mathrm{b}^2(\mathrm{M_H}) = \overline{\mathrm{m}}_\mathrm{b}^2(\mathrm{m_b})\exp\left[-2\int_{a_s(\mathrm{m_b})}^{a_s(\mathrm{M_H})}\frac{\gamma_m(x)}{\beta(x)}dx\right] = \overline{\mathrm{m}}_\mathrm{b}^2(\mathrm{m_b})\left(\frac{a_s(\mathrm{M_H})}{a_s(\mathrm{m_b})}\right)^{2\gamma_0/\beta_0}\left(\frac{AD(a_s(\mathrm{M_H}))}{AD(a_s(\mathrm{m_b}))}\right)^2 \tag{2.8}$$

where $AD(a_s)$ is a polynomial of 4-th order in the QCD expansion parameter $a_s = \alpha_s/\pi$ (see [13]). We will complete this section by presenting the numerical values of Eq.(2.2)-Eq.(2.5) in the case of $\mathrm{n_f} = 5$ active quark flavours and comment on the significance of other QED and QCD contributions to $\Gamma_{\rm H\bar{b}b}$, first considered in Refs.[8],[30], [31], [32]. In the Higgs boson masses region of interest, the expression for Eq.(2.1) may be expressed as

$$\begin{aligned}\Gamma_{\rm H\bar{b}b} &= \Gamma_0^b\frac{\overline{\rm m}_{\rm b}^2({\rm M_H})}{\rm m_b^2}\left[1+\sum_{\rm i\geq 1}\Delta\Gamma_{\rm i}\,a_s^{\rm i}({\rm M_H})\right] \qquad (2.9)\\ &= \frac{3\sqrt{2}}{8\pi}{\rm G_F M_H}\overline{\rm m}_{\rm b}^2({\rm M_H})\left[1+5.667a_s({\rm M_H})+29.15a_s({\rm M_H})^2+41.76a_s({\rm M_H})^3-825.7a_s({\rm M_H})^4\right]\end{aligned}$$

Substituting the value $a_s({\rm M_H})\approx 0.0366$ (which corresponds to $\alpha_s({\rm M_H}=120~{\rm GeV})\approx 0.115$) into Eq.(2.9), and decomposing the coefficients in the Minkowskian series into Euclidean contributions and Minkowskian-type π^2-effects, one can get from the work of Ref.[3] the following numbers

$$\begin{aligned}\Gamma_{\rm H\bar{b}b} &= \Gamma_0^b\frac{\overline{\rm m}_{\rm b}^2({\rm M_H})}{\rm m_b^2}\left[1+0.207+0.039+0.0020-0.0015\right] \qquad (2.10)\\ &= \Gamma_0^b\frac{\overline{\rm m}_{\rm b}^2({\rm M_H})}{\rm m_b^2}\left[1+0.207+(0.056-0.017)+(0.017-0.015)+(0.0063-0.0078)\right]\end{aligned}$$

where the negative numbers in the round brackets come from the effects of analytical continuation. Having a careful look at Eq. (2.10) we may conclude that in the Euclidean region the perturbative series is well-behaved and the π^2-contributions typical of the Minkowskian region are also decreasing from order to order. However, in view of the strong interplay between these two effects in the third and fourth terms, the latter ones are becoming numerically equivalent. This feature spoils the convergence of the perturbation series in the Euclidean region. Therefore, **to improve the precision** of the perturbative prediction in the Minkowskian region it seems natural to **sum up these π^2- terms** using the ideas, developed in the 80s (see, e.g., [33], [34], [35] and [36]). Due to the works discussed, e.g., in [37], these ideas now have a more solid theoretical background. We will describe some applications of these resummation procedures to $\Gamma(H\to\overline{b}b)$ later on. Here we stress that the truncated perturbative expansions of Eq.(2.9) have some additional uncertainties. These include $\rm M_H$ and t-quark mass dependent QCD [31], [32] and QED [30] contributions:

$$\Delta\Gamma_{\rm H\bar{b}b} = \frac{3\sqrt{2}}{8\pi}{\rm G_F M_H}\overline{\rm m}_{\rm b}^2({\rm M_H})\left[\Delta_{\rm t}+\Delta^{\rm QED}\right] \qquad (2.11)$$

where $\Delta_{\rm t}$ and $\Delta^{\rm QED}$ is defined following Refs. [32], [30] as

$$\begin{aligned}\Delta_{\rm t} &= \overline{a}_s^2\left((3.111-0.667L_t)+\frac{\overline{\rm m}_{\rm b}^2}{\rm M_H^2}(-10+4L_t+\frac{4}{3}ln(\overline{\rm m}_{\rm b}^2/{\rm M_H^2}))\right) \qquad (2.12)\\ &+ \overline{a}_s^3\left(50.474-8.167L_t-1.278L_t^2\right)+\overline{a}_s^2\frac{\rm M_H^2}{\rm m_t^2}\left(0.241-0.070L_t\right)\\ &+ X_t\left(1-4.913\overline{a}_s+\overline{a}_s^2(-72.117-20.945L_t)\right)\end{aligned}$$

$L_t = ln({\rm M_H^2}/{\rm m_t^2})$, $X_t = {\rm G_F m_t^2}/(8\pi^2\sqrt{2})$, $\rm m_t$ is the t-quark pole mass, $\overline{\rm m}_{\rm b}=\overline{\rm m}_{\rm b}({\rm M_H})$

$$\Delta^{\rm QED} = \left(0.472-3.336\frac{\overline{\rm m}_{\rm b}^2}{\rm M_H^2}\right)a-1.455a^2+1.301aa_s \qquad (2.13)$$

Using $a = \alpha(\mathrm{M_H})/\pi$=0.0027 ($\alpha(\mathrm{M_H})^{-1} \approx 129$), $\mathrm{m_t} = 175$ GeV, $\mathrm{M_H} = 120$ GeV, $\overline{\mathrm{m}}_\mathrm{b} = 2.8$ GeV, $\mathrm{G_F} = 1.1667 \times 10^{-5}$ GeV^{-2} we get

$$\Delta_\mathrm{t} = \Big[4.84 \times 10^{-3} - 1.7 \times 10^{-5} \tag{2.14}$$

$$+ \; 2.27 \times 10^{-3} + 1.85 \times 10^{-4} \tag{2.15}$$

$$+ \; 3.2 \times 10^{-3} - 5.75 \times 10^{-4} - 2.42 \times 10^{-4}\Big] \tag{2.16}$$

$$\Delta^{\mathrm{QED}} = \Big[1.1 \times 10^{-3} - 4.5 \times 10^{-6} - 9 \times 19^{-6} - 1.2 \times 10^{-4}\Big] \tag{2.17}$$

Comparing the numbers presented in Eq.(2.10) and Eq.(2.15)-Eq.(2.17), we conclude that **it seems more natural** to take into account order α_s^4-terms in Eq.(2.10) **only after** the **possible discovery** of the Standard Model Higgs boson . Indeed, one can see, that even for the light Higgs boson the numerical values of the order α_s^4-contributions to Eq.(2.10) are comparable with the leading $\mathrm{M_H}$- and $\mathrm{m_t}$- dependent terms in Eqs. (2.14)-(2.16) and with the leading QED correction in Eq.(2.17). These terms can be neglected at the current level of the experimental precision of "Higgs-hunting".

2.2 The relations between different definitions of b-quark mass.

In the discussions above we used two definitions of the b-quark mass, namely the pole mass $\mathrm{m_b}$ and the running mass $\overline{\mathrm{m}}_\mathrm{b}$. At the maximal order we are interested in, i.e. at the next-to-next-to-next-to-leading order (N^{3}LO), these two definitions are related in the following way

$$\frac{\overline{\mathrm{m}}_\mathrm{b}^2(\mathrm{m_b})}{\mathrm{m_b^2}} = 1 - \frac{8}{3}a_s(\mathrm{m_b}) - 18.556a_s(\mathrm{m_b})^2 - 175.76a_s(\mathrm{m_b})^3 - 1892a_s(\mathrm{m_b})^4 \tag{2.18}$$

where the first three coefficients come from the calculations of Refs. [38], [39], while the numerical estimate of the α_s^4-one is the updated variant of the estimate of Ref. [19]. It takes into account the explicit expression for the $O(a_s^3)$ term in Eq.(2.18) in the effective charges procedure applied in Ref.[19] and developed developed previously in Ref. [40]. The pronounced feature of Eq.(2.18) is the rapid growth of the coefficients in this relation. This property agrees with the expectations for the fast increase of the coefficients of this perturbation series, revealed in the process of applications of the QCD renormalon approach in Refs.[41], [42] (for a discussion see [43]).

There are different points of view concerning the application of various definitions of b-quark mass in phenomenological studies.

1. The most popular one is that in view of the factorial growth of the coefficients evident in Eq.(2.18) it is better to avoid application of the pole mass $\mathrm{m_b}$ and to use instead the $\overline{\mathrm{MS}}$-scheme running b-quark mass $\overline{\mathrm{m}}_\mathrm{b}(\mu)$ normalized at the scale $\mu = \overline{\mathrm{m}}_\mathrm{b}$ (see e.g. [44], [45]).

2. It is also possible to consider the invariant b-quark mass, which is related to the running mass, normalized at the scale $\mu = \mathrm{m_b}$ as

$$\hat{\mathrm{m}}_\mathrm{b} = \overline{\mathrm{m}}_\mathrm{b}(\mathrm{m_b})\left[a_s(\mathrm{m_b})^{\frac{\gamma_0}{\beta_0}}\mathrm{AD}(a_s(\mathrm{m_b}))\right]^{-1} \tag{2.19}$$

where AD is defined in Eq.(2.8). The concept of the invariant mass is rather useful in treating π^2-contributions to $\Gamma(H \to b\overline{b})$ (see [14]- [18]).

3. The pole *b*-quark mass is frequently used in the MOM-scheme [46]. Within this prescription threshold effects of heavy quarks may be understood rather easily [47], [48], [49]. Moreover, the concept of the pole *b*-quark mass is commonly applied in considerations of deep-inelastic scattering processes [50] , and **what is even more importantly for the LHC**, in **global fits of parton distributions** [51]. Note, however, that quite recently the three-loop transformation of the MOM-scheme to the $\overline{\rm MS}$-scheme was analysed in Ref.[52] where the equivalence of these two approaches outside the threshold region was demonstrated.

4. Keeping in mind the advantages of both running and pole definitions of the *b*-quark mass one may analyze the effects of the RG resummation of $\alpha_s^n \ln^m(q^2/m_b^2)$-terms ($m< n$) by comparing theoretical predictions for concrete physical quantities, which depend on $\overline{\rm m}_{\rm b}$ and $\rm m_b$.

Note, that for Eq.(2.18), Eq.(2.19) and other perturbative series, discussed in this work, we coordinate their truncated expressions with the truncation of the inverse logarithmic expressions for a_s. At the $\rm N^{(k-1)}$LO of perturbation theory ($1 \leq {\rm k} \leq 5$) they are defined as

$$a_s(\mu^2)_{\rm LO} = \frac{1}{\beta_0 {\rm Log}_1} \quad a_s(\mu^2)_{\rm NLO} = \frac{1}{\beta_0 {\rm Log}_2}\left[1 - \frac{\beta_1 \ln({\rm Log}_2)}{\beta_0^2 {\rm Log}_2^2}\right] \tag{2.20}$$

$$a_s(\mu^2)_{\rm N^2LO} = a_s(\mu^2)_{\rm NLO} + \Delta a_s(\mu^2)_{\rm N^2LO} \quad a_s(\mu^2)_{\rm N^3LO} = a_s(\mu^2)_{\rm N^2LO} + \Delta a_s(\mu^2)_{\rm N^3LO} \tag{2.21}$$

$$a_s(\mu^2)_{\rm N^4LO} = a_s(\mu^2)_{\rm N^3LO} + \Delta a_s(\mu^2)_{\rm N^4LO} \tag{2.22}$$

where ${\rm Log_k} = \ln(\mu^2/\Lambda_{\rm k}^2)$, $\Lambda_{\rm k}$ are the values of the QCD scale parameter $\Lambda_{\overline{\rm MS}}^{(\rm n_f)}$, extracted from the experimental data for concrete physical quantities taking into account $\rm N^{(k-1)}$LO perturbative QCD corrections, which depend on the number of active quark flavours, $\rm n_f$. The definitions of $\Delta a_s(\mu^2)_{\rm N^{(k-1)}LO}$ $3 \leq {\rm k} \leq 5$ in Eq.(2.21) and Eq.(2.22) contain high-order scheme-dependent coefficients β_{k-1} of the QCD β-function of Eq.(2.6) and $\Lambda_{\overline{\rm MS}}^{(\rm n_f)}$ as well (for details see Ref. [13]).

2.3 Explicit expression for $\Gamma_{\rm H\bar{b}b}$ in terms of pole mass

Consider the $\overline{\rm MS}$-scheme perturbative series for $\Gamma_{\rm H\bar{b}b}$ from Eq.(2.1). To transform it to the case when instead of the running mass $\overline{\rm m}_{\rm b}({\rm M_H})$ the pole mass $\rm m_b$ is used, one should make the following steps:

- express $\overline{\rm m}_{\rm b}({\rm M_H})$ in terms of $\overline{\rm m}_{\rm b}({\rm m_b})$ and $\alpha_{\rm s}({\rm m_b})$ by solving the RG equation for the running mass, defined in Eq.(2.8);
- apply Eq.(2.18), which relates the square of the running mass $\overline{\rm m}_{\rm b}({\rm m_b})$ to the square of the pole mass $\rm m_b$ via a perturbative expansion in powers $a_s({\rm m_b}) = \alpha_{\rm s}({\rm m_b})/\pi$;
- reexpress powers of $a_s({\rm m_b})$, which appear at the first and and second steps, in terms of powers of $a_s({\rm M_H})$ using the RG equation of Eq.(2.6) for the QCD coupling constant.

The explicit solutions of equations mentioned above, namely the solutions of Eq.(2.8) and Eq.(2.6), were written down in Ref.[53] and extended to higher order level in Ref.[19]. Their application result in the appearance in the expressions for $\Gamma_{\rm H\bar{b}b}$

$$\Gamma_{\rm H\bar{b}b} = \Gamma_0^b\left[1 + \Delta\Gamma_1^{\rm b} a_s({\rm M_H}) + \Delta\Gamma_2^{\rm b} a_s({\rm M_H})^2 + \Delta\Gamma_3^{\rm b} a_s({\rm M_H})^3 + \Delta\Gamma_4^{\rm b} a_s({\rm M_H})^4\right] \tag{2.23}$$

of the RG controllable $L = \ln(\mathrm{M_H^2/m_b^2})$-terms , which enter into the coefficients $\Delta\Gamma_i^b$ in the following way

$$\Delta\Gamma_1^b = 3 - 2L; \quad (2.24)$$

$$\Delta\Gamma_2^b = -4.5202 - 18.139L + 0.08333L^2; \quad (2.25)$$

$$\Delta\Gamma_3^b = -316.88 - 133.42L - 1.1551L^2 + 0.0509L^3; \quad (2.26)$$

$$\Delta\Gamma_4^b = -4366.2 - 1094.6L - 55.867L^2 - 1.8065L^3 + 0.0477L^4 \;. \quad (2.27)$$

The numerical values of the calculated logarithmic terms are not small. However, in the case of $\mathrm{i} \geq 2$ they tend to cancel each other in the final results for $\Delta\Gamma_i^b$.

The related variant of Eq.(2.1), where the RG-controllable terms are summed up, may be written down as

$$\Gamma_{\mathrm{H\bar{b}b}} = \Gamma_0^b \left(\frac{a_s(\mathrm{M_H})}{a_s(\mathrm{m_b})}\right)^{(24/23)} \frac{AD(a_s(\mathrm{M_H}))^2}{AD(a_s(\mathrm{m_b}))^2}\left[1 + \sum_{\mathrm{i}\geq 1}\Delta\Gamma_{\mathrm{i}}\, a_s^{\mathrm{i}}(\mathrm{M_H})\right] \quad (2.28)$$
$$\times \left(1 - \frac{8}{3}a_s(\mathrm{m_b}) - 18.556\, a_s(\mathrm{m_b})^2 - 175.76\, a_s(\mathrm{m_b})^3 - 1892\, a_s(\mathrm{m_b})^4\right),$$

where

$$AD(a_s)^2 = 1 + 2.351\, a_s + 4.383\, a_s^2 + 3.873\, a_s^3 - 15.15\, a_s^4 \quad (2.29)$$

We will factorize the term Γ_0^b out of the expressions for Eq.(2.23) and Eq.(2.28) as well. These representations are rather convenient for comparing different parameterizations of $\Gamma_{\mathrm{H\bar{b}b}}$ and of the ratio $\mathrm{R(M_H)} = \Gamma_{\mathrm{H\bar{b}b}}/\Gamma_0^b$.

2.4 Values of the QCD parameters used

To analyze the behavior of the truncated series in Eq.(2.23) and Eq. (2.28) and of the various approximations for the R-ratio it is necessary to fix definite values of the QCD parameters $\mathrm{m_b}$, $\Lambda_{\overline{\mathrm{MS}}}^{(\mathrm{n_f}=4)}$ and $\Lambda_{\overline{\mathrm{MS}}}^{(\mathrm{n_f}=5)}$. This is done in Table 1.

order	$\mathrm{m_b}$ GeV	$\Lambda_{\overline{\mathrm{MS}}}^{(\mathrm{n_f}=4)}$ MeV	$\Lambda_{\overline{\mathrm{MS}}}^{(\mathrm{n_f}=5)}$ MeV
LO	4.74	220	168
NLO	4.86	347	254
$\mathrm{N^2LO}$	5.02	331	242
$\mathrm{N^3LO}$	5.23	333	243
$\mathrm{N^4LO}$	5.45	333	241

Table 1: The values of the QCD parameters used.

The LO, $\mathrm{N^kLO}$ $(1 \leq \mathrm{k} \leq 3)$ results for the pole mass $\mathrm{m_b}$ are taken from Ref. [54], where they were extracted using the relation between the mass of the $\Upsilon(1S)$- resonance, $\mathrm{m_b}$ and the ground state energy of $\Upsilon(1S)$-system. The $\mathrm{N^4LO}$ estimate of $\mathrm{m_b}$ is our theoretical guess. The LO, NLO, $\mathrm{N^2LO}$ values for $\Lambda_{\overline{\mathrm{MS}}}^{(\mathrm{n_f}=4)}$, given in Table 1, come from the recent **parton distribution fits** of Ref. [55]. At the NLO and $\mathrm{N^2LO}$ level these numbers agree with the values of $\Lambda_{\overline{\mathrm{MS}}}^{(\mathrm{n_f}=4)}$, which were extracted from the first $\mathrm{N^3LO}$ QCD analysis of the experimental data, performed in Ref.[56]. The fitted data were the Tevatron experimental data points for the xF_3 structure function of the νN deep-inelastic scattering (DIS) process and were obtained by CCFR collaboration [57]. The $\mathrm{N^3LO}$ value of $\Lambda_{\overline{\mathrm{MS}}}^{(\mathrm{n_f}=4)}$ in Table 1 is one of the results of Ref.[56]. In view of the indications for a convergence of the fits performed at $\mathrm{N^4LO}$ revealed in Ref. [56] we will use the same value of $\Lambda_{\overline{\mathrm{MS}}}^{(\mathrm{n_f}=4)}$ as at the $\mathrm{N^3LO}$ level. To get the results for $\Lambda_{\overline{\mathrm{MS}}}^{(\mathrm{n_f}=5)}$, given in the last column of Table 1, we apply the NLO and $\mathrm{N^2LO}$ matching conditions of Ref.[58] (the latter ones were corrected a bit in Ref.[59]). At the $\mathrm{N^3LO}$ we use the expressions from Ref. [61]. At the $\mathrm{N^4LO}$ the analytical relation from Ref. [61] was applied. Note, that the calculations of Ref. [61] confirmed the validity of analogous expressions, obtained (partly numerically) in Ref.[62] and Ref.[63]. The related NLO-$\mathrm{N^4LO}$ results for $\alpha_s(\mathrm{M_Z})$ are contained in the interval (0.118-0.119). Thus they agree with the world average value of $\alpha_s(\mathrm{M_Z})$ (for a recent review see Ref.[64]).

2.5 The comparisons of the renormalization group improved and the truncated pole-mass parameterizations

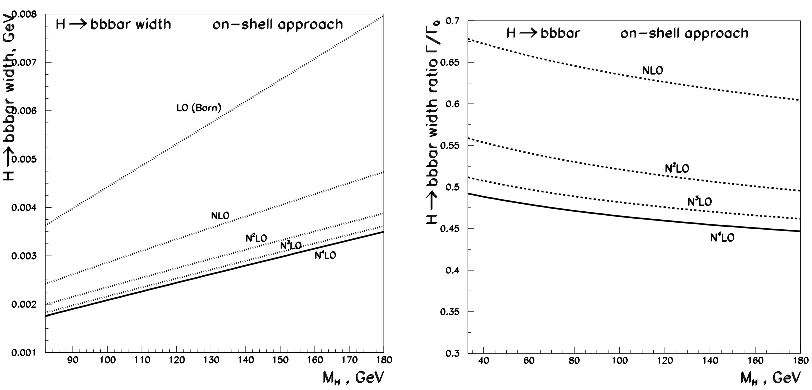

Figure 1: The quantities analyzed in the pole (or on-shell) mass approach.

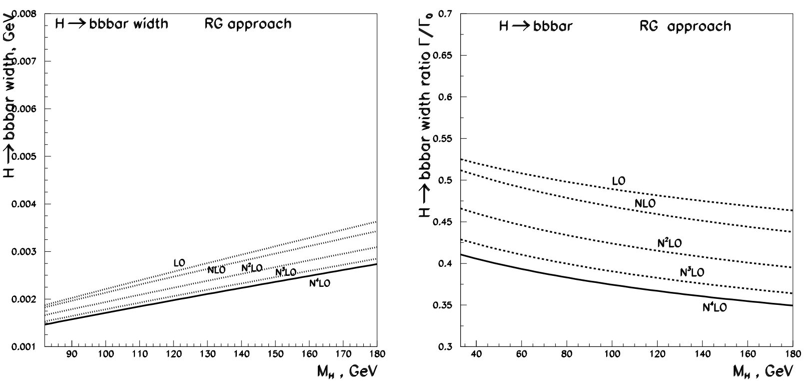

Figure 2: The quantities analyzed in the approach with explicit RG-resummation

Different perturbative approximations for two parameterizations of $\Gamma_{\mathrm{H\bar{b}b}}$ (see Eq.(2.23) and Eq.(2.28)) and of the related ratios $\mathrm{R(M_H)}$ are compared in the plots of Figure 1 and Figure 2. Looking carefully at these plots, we conclude that

1. the partial cancellation of the RG-controllable large contributions to Eqs.(2.25)-(2.27), which are proportional to $L = \ln(\mathrm{M_H^2}/\mathrm{m_b^2})$ and which appear in the coefficients of the pole-mass parameterization of $\Gamma_{\mathrm{H\bar{b}b}}$, leads to a reduction in the size of perturbative corrections. Taking them into account results in a decrease of the difference bewteen the behaviour of the curves of **Fig.1** and **Fig.2**. This feature demonstrates the numerical importance of step-by-step application of the RG-resummation approach.

2. The cancellations between the contributions proportional to L in the coefficient $\Delta\Gamma_4^b$ result in the small value of the whole $\alpha_s(\mathrm{M_H})^4$ correction to the perturbative expression for $\Gamma_{\mathrm{H\bar{b}b}}$. This fact demonstrates the convergence of different theoretical approximants. From a **phenomenological** point of view this means that at present the α_s^4 corrections **may be neglected** in the related computer codes for calculating some branching ratios, e.g., for $H \to \gamma\gamma$, $H \to \bar{b}b$ and $H \to \tau^+\tau^-$ processes. This conclusion is consistent with our similar statement, made in the case of using running b-quark $\overline{\mathrm{m}}_\mathrm{b}$, but motivated by different theoretical arguments (see the end of **Sec. 2.1**).

3. The behavior of the RG-resummed expressions for $\Gamma_{\mathrm{H\bar{b}b}}$ and $\mathrm{R_{H\bar{b}b}}$ are more stable than in the case, when RG summation of the mass-dependent terms is not used (see **Fig.1**). This feature supports the application of the complete RG-improved parameterization of Eq.(2.28), which is closely related to the one, defined through the running b-quark mass (see Eq.(2.9)).

4. We observed the existence of a difference $\Delta\Gamma_{\mathrm{H\bar{b}b}}$ between the truncated pole-mass approach and the RG-improved parameterization of $\Gamma_{\mathrm{H\bar{b}b}}$. A pleasant feature is that this difference becomes smaller and smaller in each successive order of perturbation theory considered. Indeed, for the phenomenologically interesting value of Higgs boson mass $\mathrm{M_H} = 120$ GeV we find that at the α_s^2-level $\Delta\Gamma_{\mathrm{H\bar{b}b}} \approx 0.7$ MeV, while for the α_s^3-curves it becomes smaller, namely $\Delta\Gamma_{\mathrm{H\bar{b}b}} \approx 0.3$ MeV. At the α_s^3-level of the RG-improved $\overline{\mathrm{MS}}$-scheme series one has $\Gamma_{\mathrm{H\bar{b}b}} \approx 1.85$ MeV for $\mathrm{M_H} = 120$ GeV. For this scale the value of $\Gamma_{\mathrm{H\bar{b}b}}$ with the explicit dependence from the pole-mass is 16 % higher, than its RG-improved estimate.

We hope, that further studies may clarify whether it is possible to formulate more well-defined procedures for determining the theoretical errors for this important characteristic of the Higgs boson and for the predictions of other perturbative QCD series as well. Indeed, even the $\overline{\mathrm{MS}}$-scheme expression for $\Gamma_{\mathrm{H\bar{b}b}}$ at the α_s^3-level reveal additional theoretical uncertainties, which are related not only to the the estimates of theoretical errors for the $\overline{\mathrm{MS}}$-scheme parameters $\overline{\mathrm{m}}_\mathrm{b}(\mathrm{m_b})$ and $\overline{\alpha}_s(M_Z)$ (for a discussion see e.g. Ref.[65]). These not previously specified uncertainties result from application of different approaches to the treatment of the typical Minkowskian π^2-contributions in the perturbative expressions for physical quantities. In the case of $\Gamma_{\mathrm{H\bar{b}b}}$ the results from application of these different procedures will be compared below.

3. Resummations of π^2-terms

3.1 The definitions of the resummed approximants

As was demonstrated in Ref.[3], the kinematic π^2-terms in the coefficients of perturbative series for $\Gamma_{\mathrm{H\bar{b}b}}$ become comparable with the Euclidean contributions starting from the $\mathrm{N^3LO}$ α_s^3

corrections (see Eq.(2.10)). In general, π^2-terms are resulting from analytical continuation to the Minsowskian region of theoretical expressions for physical quantities, defined through the the two-point functions of quark or gluon currents, which are calculated in the Euclidean region. These terms are starting to manifest themselves from the N^{2}LO. In Ref. [14] the idea of resummation of kinematic π^2-effects of Ref.[35] and Ref. [36] was generalized to the case when the related RG-equation has the non-zero anomalous dimension. In that work the case of the N^{2}LO approximation for $\Gamma_{H\bar{b}b}$ was analyzed. However, since the negative π^2-contributions to the N^{2}LO correction to $\Gamma_{H\bar{b}b}$ turn out to be smaller than the value of corresponding Euclidean term (see Eq.(2.10)), the possible development of the π^2 resummation procedure was overlooked by the authors of the works of Ref. [14] and Ref. [20], aimed at the study of scheme-dependence of α_s^2 approximations for this Higgs boson characteristic. The appearance of Contour Improved Perturbation Theory (CIPT) and its application in the semi-hadronic decay channel of the τ-lepton [66], [67] pushed ahead the real interest in the development of π^2-resummation approaches both in theoretical and phenomenological investigations.

In order to resum the kinematic π^2-contributions to perturbative predictions for $\Gamma_{H\bar{b}b}$ the authors of Ref. [68] supplemented the CIPT method with the procedure of "Naive Non-Abelianization" (NNA) [68], commonly used in the renormalon calculus approach (a detailed discussion of this method can be found in Ref.[69]). As the result, the following approximation for $\Gamma_{H\bar{b}b}$ was obtained [15]:

$$\Gamma^{BKM}_{H\bar{b}b} = \Gamma_0^b \frac{\hat{m}_b^2}{m_b^2}\left[(a_s(M_H))^{\nu_0}A_0 + \sum_{n\geq 1}(a_s(M_H))^{\nu_0} d_n^E A_n(a_s(M_H))\right] \tag{3.1}$$

where the A_n-functions are defined as:

$$A_n = \frac{1}{\beta_0\delta_n\pi}\left[1+\beta_0^2\pi^2 a_s{}^2\right]^{-\delta_n/2}(a_s)^{n-1}\sin\left(\delta_n \arctan(\beta_0\pi a_s)\right), \tag{3.2}$$

Here $\delta_n = n+\nu_0-1$ and $\nu_0 = 2\gamma_0/\beta_0$ depends on the first coefficient of the QCD β-function $\beta_0 = (11-2/3n_f)/4$, introduced in Eq.(2.6) and $a_s(M_H) = 1/(\beta_0\ln(M_H^2/\Lambda^2))$ It should be stressed that the NNA approach is dealing with the leading terms in expansions of perturbative coefficients for Euclidean quantities in powers of number of flavor n_f and it provides the basis of "large β_0-approximation". Within this approximation it is assumed that the terms, proportional to β_0^p (where p is "large", namely $1\leq p\leq\infty$) give a qualitatively good approximation for the structure of the Euclidean perturbative contributions to physical quantities under study. Indeed, as was shown in Ref. [70] this approach gives correct both for sign for order of magnitude estimates of the perturbative coefficients d_i^E to various physical quantities, including d_4^E- contribution to $\Gamma_{H\bar{b}b}$, defined in Eq.(2.5). The explicit calculations of Ref.[3] demonstrate, that at n_f=5 the real value of d_4^E-coefficient is higher, than its NNA estimate from Ref.[70] by the factor 5 approximately. [1].

Within this "large β_0-approximation" is seems more consistent to approximate the perturbative QCD expansion parameter $a_s(M_H) = \alpha_s(M_H)/\pi$ by its LO-expression of Eq.(2.20). Fixing now $n=0$ in Eq.(3.2) and expanding A_0 to the first order in a_s, the authors of Ref.[15] got

$$A_0 = \frac{1}{b_0 L_{M_H}{}^{b_0}}\frac{\sin(b_0\ \arctan(\pi/L_{M_H}))}{(1+\pi^2/L_{M_H}{}^2)^{b_0/2}}, \tag{3.3}$$

[1] Unfortunately, NNA results from Table I of the important work of Ref. [3] contain misprints.

where $\mathrm{b_0} = \nu_0 - 1$, $\mathrm{L_{M_H}} = \ln(\mathrm{M_H^2}/\Lambda^2)$. This concrete expression was first derived in Ref. [14].

Other ways of resumming π^2-contributions to $\Gamma_{\mathrm{H\bar{b}b}}$ were considered within the framework of Fractional Analytical Perturbation Theory (FAPT) [16] and its variant (for a review of FAPT see Ref. [17] where "flavor-corrected global FAPT" was also proposed). It should be stressed that the cornerstone of FAPT is the Analytical Perturbation Theory approach, which was developed in the studies, initiated by the work [71].

Applying the "large β_0-expansion", which is equivalent to the choice of the LO expression for a_s, one can get the FAPT analog of Eq.(3.1). It can be written down as

$$\Gamma^{(1;\mathrm{FAPT})}_{\mathrm{H\bar{b}b}} = \Gamma_0^{\mathrm{b}} \frac{\hat{\mathrm{m}}_{\mathrm{b}}^2}{\mathrm{m}_{\mathrm{b}}^2} \left[\mathfrak{A}^{(1)}_{\nu_0}(\mathrm{M_H}) + \sum_{\mathrm{n}\geq 1}^{4} \mathrm{d}_{\mathrm{n}}^{\mathrm{E}} \frac{\mathfrak{A}^{(1)}_{\mathrm{n}+\nu_0}(\mathrm{M_H})}{\pi^{\mathrm{n}}} \right] . \tag{3.4}$$

This approximant almost coincides with the expression of Eq.(3.1). This conclusion was made in Ref. [16] taking into account the following definitions of the functions in Eq.(3.4), namely

$$\mathfrak{A}^{(1)}_{\nu}(\mathrm{M_H}) = \frac{\sin[(\nu-1)\arccos(\mathrm{L_{M_H}}/\sqrt{\pi^2+\mathrm{L^2_{M_H}}})]}{\pi(\nu-1)(\pi^2+\mathrm{L^2_{M_H}})^{\nu-1/2}} \tag{3.5}$$

$$\mathfrak{A}^{(1)}_{n+\nu}(\mathrm{M_H}) = \frac{\Gamma(\nu)}{\Gamma(n+\nu)} \left(-\frac{\mathrm{d}}{\mathrm{d\,L_{M_H}}} \right) \mathfrak{A}^{(1)}_{\nu}(\mathrm{M_H}) \tag{3.6}$$

The FAPT N^{3}LO expression of Ref.(3.4) was given in Refs.[16], [17]. It has the following form:

$$\Gamma^{(3;\mathrm{FAPT})}_{\mathrm{H\bar{b}b}} = \Gamma_0^{\mathrm{b}} \frac{\hat{\mathrm{m}}_{\mathrm{b}}^2}{\mathrm{m}_{\mathrm{b}}^2} \left[\mathfrak{B}^{(3)}_{\nu_0}(\mathrm{M_H}) + \sum_{\mathrm{n}\geq 1}^{3} \mathrm{d}_{\mathrm{n}}^{\mathrm{E}} \frac{\mathfrak{B}^{(3)}_{\mathrm{n}+\nu_0}(\mathrm{M_H})}{\pi^{\mathrm{n}}} \right] \tag{3.7}$$

Within the framework of FAPT the functions $\mathfrak{B}^{(3)}_{\mathrm{n}+\nu_0}(\mathrm{M_H})$ $(0 \leq \mathrm{n} \leq 3)$ absorb the evolution of the $\overline{\mathrm{MS}}$-scheme running b-quark mass and π^2-contributions in Eq.(2.1), which are proportional to high-order coefficients of RG-functions $\beta(a_s)$ and $\gamma_m(a_s)$ in Eqs.(2.1)-(2.5).

In the "flavour-corrected (f-c) global FAPT", proposed in Ref.[17], the FAPT expression for $\Gamma_{\mathrm{H\bar{b}b}}$ is expanded into the functions $\mathfrak{B}^{(1)}_{1;\mathrm{d_n}}(\mathrm{s})$, which absorb all $\mathrm{n_f}$ dependence from the Euclidean coefficients $\mathrm{d}_{\mathrm{n}}^{\mathrm{E}}$ in Eqs.(2.3)-(2.5).The explicit expression for this approximant reads [18]

$$\Gamma^{(3;\mathrm{FAPT,f-c})}_{\mathrm{H\bar{b}b}} = \Gamma_0^{\mathrm{b}} \frac{\hat{\mathrm{m}}_{\mathrm{b}}^2}{\mathrm{m}_{\mathrm{b}}^2} \left[\mathfrak{B}^{(3)}_{\nu_0}(\mathrm{M_H}) + \sum_{\mathrm{n}\geq 1}^{3} \frac{\mathfrak{B}^{(3)}_{\mathrm{n}+\nu_0;\mathrm{d_n^E}}(\mathrm{M_H})}{\pi^{\mathrm{n}}} \right] \tag{3.8}$$

The more detailed study of the characteristic feature of formalizm [17], [18] is now in progress [72].

3.2 The comparison of the results of π^2-resummations.

We now consider the result of applying these different procedures for resummations of kinematic π^2-effects in the perturbative coefficients for $\Gamma_{\mathrm{H\bar{b}b}}$. We will follow the results obtained in Ref.[18] and discuss the comparison of the behavior of various parameterizations for this quantity, which were defined in the previous section, with the α_s^2- and α_s^3- truncated $\overline{\mathrm{MS}}$-scheme expression of Eq.(2.1).

In the process of these discussions we use plots, similar to those, presented in Ref.[18], but modified at our request by changing the values of the QCD parameters to the ones presented in Table 1. The results of these additional studies were kindly communicated [72] to us in the form of the figures presented below.

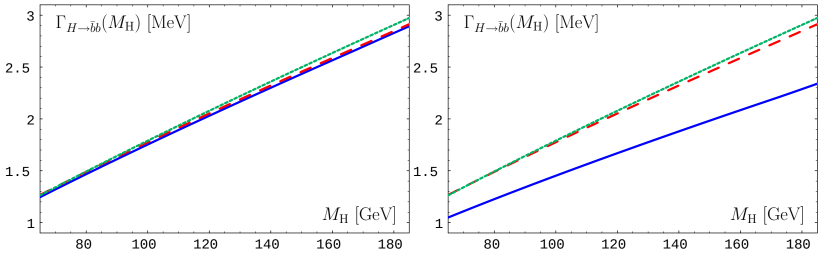

Figure 3: The comparison of N^{2}LO approximations for $\Gamma_{\mathrm{H}\bar{\mathrm{b}}\mathrm{b}}$. On both panels the dashed line is the truncated α_s^2-approximation of the $\overline{\mathrm{MS}}$-scheme result. The dotted line is the 1-loop FAPT expression of Eq.(3.4), which is identical to the CIPT expression of Eq.(3.1). The solid line on the left figure displays the 2-loop FAPT analog of Eq.(3.7), while on the right figure it corresponds to 2-loop variant of the "flavour-corrected global FAPT" approximant of Eq.(3.8).

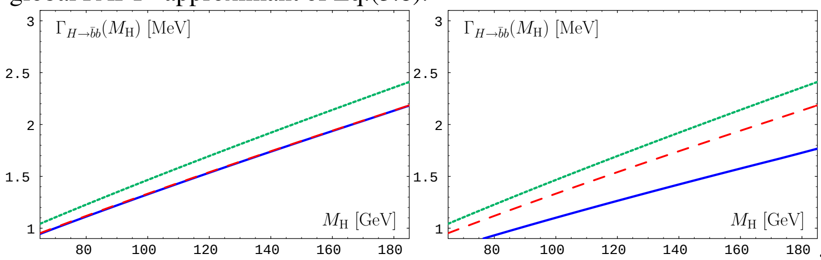

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Figure 4: The N^{3}LO version of Fig.3. The dashed line is the truncated α_s^3 $\overline{\mathrm{MS}}$-scheme expression.

A careful look to the curves from Fig.3 and Fig.4 leads us to the following observations:

1. at the N^{2}LO-level the resummation of the π^2-terms **do not** lead to detectable effects. The related curves almost coincide with the α_s^2 $\overline{\mathrm{MS}}$-scheme approximation for $\Gamma_{\mathrm{H}\bar{\mathrm{b}}\mathrm{b}}$, obtained in Ref. [20];

2. at the N^{3}LO level the effects of resummation of the kinematic π^2-contributions are visible. Indeed, for $\mathrm{M_H} = 120$ GeV the value of the resummed approximant is over 0.2 MeV lower, than the value of the truncated α_s^3 $\overline{\mathrm{MS}}$-scheme expression for $\Gamma_{\mathrm{H}\bar{\mathrm{b}}\mathrm{b}}$, obtained in Ref.[21]. The behavior of the curves of the left plot from Fig.3 indicate that for a Higgs boson with this mass, the value of the α_s^3-approximation for its decay width to $\bar{b}\,b$ quarks is $\Gamma_{\mathrm{H}\bar{\mathrm{b}}\mathrm{b}} \approx 1.7$ MeV, while the resummation of the π^2-effects decreases this by over 11%;

3. an obvious message, which comes from the consideration of the right-hand plots of Fig.3 and Fig.4 is that the application of "the flavor-corrected global FAPT" with full analytization of $\mathrm{n_f}$ dependence in $\Gamma_{\mathrm{H}\bar{\mathrm{b}}\mathrm{b}}$ leads to smaller values of the decay width for Higgs boson, than in the case of the truncated $\overline{\mathrm{MS}}$-scheme approach, and 2-loop and 3-loop FAPT approach [16]. An attempt to explain the phenomenological reason for this reduction is now in progress [72]. It may be also of interest to compare FAPT applications with the existing applications to $\Gamma_{\mathrm{H}\bar{\mathrm{b}}\mathrm{b}}$ of the CIPT based resummation procedure from Ref. [73].

Conclusions.

In the discussions presented above we found that even for the fixed values of the QCD parameters used in Table 1, which do not take into account the existing theoretical and experimental uncertainties in the values of $\alpha_s(\mathrm{M_Z})$ (and thus $\Lambda_{\overline{\mathrm{MS}}}^{(\mathrm{n_f}=5)}$), and of the running and pole b-quark masses, different parameterizations of the decay width of the $H \to \bar{b}b$ process deviate from the $\overline{\mathrm{MS}}$-scheme prediction. In our view, this feature demonstrates the existence of additional theoretical QCD uncertainties, which are not usually considered in phenomenological studies. We still do not know what is the role of similar QCD uncertainties in the theoretical predictions for other characteristics of Higgs boson production and its other decays (see however, the recent studies of resummation of terms proportional to π^2 in the N^{2}LO perturbative QCD predictions for the Higgs -boson production cross-section and for the hadronic decay rate $\Gamma(\mathrm{H} \to \mathrm{gg})$ [74]). It will also be interesting to understand whether these uncertainties survive in the branching ratios of the Standard Model Higgs boson for different values of its mass.

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