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Mathematical model of magnetically interacting rigid bodies

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Dynamics of two bodies, which interact by magnetic forces, is considered. The model of interaction is built on quasi-stationary approach for electromagnetic field, and symmetric tops with different moments of inertia of the bodies are considered. The general form of the interaction energy is discovered for the case of coincidence of mass and magnetic symmetries. Since the energy of interaction depends only on the relative position of the bodies, then the consideration is too much simplified in the c.m. system, notwithstanding that force is non-central. The task requires the development of the classic Hamilton formalism for the systems of magnetically interactive bodies, including the systems of magnets and/or superconductive magnets (mixed systems). Hamilton motion equations are obtained on the basis of Poisson structure in the dynamic variables area. Such an approach allows the equations to be represented in the galilei-invariant vector form in contrast to default definition in Euler's angles. Conservation laws following from the system symmetry are considered. This variant of Hamilton formalism easily spreads in the case of arbitrary number of magnetically interactive symmetric tops. All equations with Poisson brackets are tested with symbolic features of the Maple system. For the numeral modelling of magnetic rigid bodies dynamics Maple and MATLAB packages are used. The obtained mathematical model allows the possibility of orbital motion in the system of magnetically interactive bodies to be investigated.

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1. Introduction

This article describes the new results of investigating the dynamics of the magnetic interaction of rigid bodies and continues the cycle of papers devoted to the investigation of contact-free equilibrium of rigid bodies in magnetostatics. These papers show that the magnetic interaction for a wide class of magnetic bodies such as permanent magnets, inductance coils (superconductive and with direct current) and their different combinations ("mixed" type systems) can be described through potential energy of their interaction received from the Lagrangian formalism of electromechanical analogy. It was also shown, that there are such magnetic configurations of rigid bodies, including superconductive elements, that the potential energy has the minimum. Such systems with stable magnetic equilibrium are called "Magnetic Potential Well"[1].

To investigate not only quasistatic models but also dynamic stable configurations, as well as to consider a larger number of tasks (confinement, scattering, orbital motion) an adequate mathematical apparatus for investigating the dynamics of such systems is required. Such mathematical apparatus is the Hamilton formalism, presented below.

It should be noted that the Poisson structures were used when trying to classically describe the magnetic interaction of spins [2,3]. These papers do not consider spatial motion of bodies, i.e. spatial variables are absent. The energy of interaction given in these papers cannot be used even for describing classic magnetic dipoles.

A more realistic description of magnetic interaction of two magnets was given by V.V. Kozoriz based on the Lagrangian formalism [4]. But the mathematical apparatus he uses does not give any description of "mixed type" systems. Moreover, it is well known that the generalized coordinates used (Euler's angles) cannot correctly map all orientations of a rigid body, which becomes apparent in the peculiarities of the coefficients of the differential motion equations.

We developed a formalism which results in the coordinate-free, i.e. vector form of motion equations for the system of magnetically interactive bodies (including "mixed type" systems).

2. Hamiltonian formalism for two magnetic-interacting bodies

As it is shown in [5,6] Poisson structures are a suitable basis for Hamilton description of rigid body dynamics. The properties of Poisson brackets are fully determined with the structural functions $\{x^i,x^k\}$ and the structural tensor J^{ik}.

$$\{F,G\} = \sum_{i,k}\{x^i,x^k\}\frac{\partial F}{\partial x^i}\frac{\partial G}{\partial x^k} = \sum_{i,k} J^{ik}\frac{\partial F}{\partial x^i}\frac{\partial G}{\partial x^k}; \quad J^{ik} = -J^{ki} = \{x^i,x^k\}, \tag{1}$$

where $x^i, i = 1..dim(M)$ is a coordinate (generally speaking, local) system on the Poisson manifold

For a classical phase space with *global* coordinates $q^i, p^i, i = 1..n$ the structural tensor has the form of (block) matrix, which results the known expression for a classical Poisson bracket.

$$J_{(q,p)} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \tag{2}$$

We suppose that we consider only symmetric tops, i.e. the cases when two main moments of inertia are equal, $I_1 = I_2 = I_\perp$.

Such a restriction decreases the theoretic and practical value of the proposed model to a very small degree only, but the benefits are so great that this case is worthy of being considered independently.

Components of the moment with respect to a body are normally used, since the inertia tensor is not time-dependent in the system related to the body. However, when a symmetric top is concerned it is more suitable to use the components of the moment $\vec{m}$ and the axis of symmetry $\vec{v}$ with respect to a stationary coordinate system. It is particularly important for the description of several rotating bodies interacting with each other, as the invariant (vector) description of dynamic variables can be possible in this case.

If a system consists of such two bodies and the potential energy depends only on the mutual position of the bodies, we normally proceed to the c.m. system.

Therefore, the Poisson structure for a system of 2 magnetically interactive bodies has the generatrices $\vec{r},\vec{p}$, $\vec{v}^{'},\vec{m}^{'}$, $\vec{v}^{''},\vec{m}^{''}$, where $\vec{r},\vec{p}$ are orbital coordinates and impulses; $\vec{v}^{'},\vec{m}^{'}$ and $\vec{v}^{''},\vec{m}^{''}$ are axes of symmetry and moments of impulses of the 1st and 2nd body, respectively.

Each of the 3 groups describes independent degrees of freedom, therefore the structural tensor is of a block form of the following kind

$$J = \begin{bmatrix} J_{(r,p)} & 0 & 0 \\ 0 & J_{(v',m')} & 0 \\ 0 & 0 & J_{(v'',m'')} \end{bmatrix} \tag{3}$$

and the kinetic energy of the system is as follows

$$T(p^2,\vec{m}^{'2},\vec{m}^{''2}) = \frac{1}{2m}p^2 + \frac{\alpha^{'}}{2}\vec{m}^{'2} + \frac{\alpha^{''}}{2}\vec{m}^{''2} \tag{4}$$

where $\alpha^{'} = \frac{1}{I_{\perp}^{'}}$, $\alpha^{''} = \frac{1}{I_{\perp}^{''}}$, $m = \frac{m_1 m_2}{m_1+m_2}$ is the reduced mass of the system.

The following dynamic variables are Casimir functions for this Poisson structure:

$$\vec{v}^{\,'2} = 1,\ \vec{v}^{\,''2} = 1; (\vec{v}^{\,'},\vec{m}^{'}) = M_3^{'} = const_1,\ (\vec{v}^{\,''},\vec{m}^{''}) = M_3^{''} = const_2 \tag{5}$$

Proposition 1. *As it will be shown below the potential energy of the type $U(r,c^{'},c^{''},c^{'''})$ describes interaction for a rather wide class of magnetic bodies, where $r = |\vec{r}|; \vec{e}_r = \vec{r}/r; c^{'} = (\vec{e}_r,\vec{v}^{'}); c^{''} = (\vec{e}_r,\vec{v}^{''}); c^{'''} = (\vec{v}^{'},\vec{v}^{''})$.*

There are many physical models for which it is known that the potential energy adequately describes interaction and has a reduced form in the axisymmetric case: permanent magnets - in classical courses; superconductive elements - in the monograph on electromechanics by White-Woodson (within quasistationary approximation for electromaghetic field); systems consisting of superconductive elements and constant magnets - in my Ph.D. thesis (also within quasistationary approximation for electromagnetic field).

Proposition 2. *In the case of a permanent magnet having axisymmatric form, when the scalar magnetic potential outside the body can be written as $\psi = \psi(r,z) = \psi(r,rc^{'})$, the potential energy of its interaction both with a magnetic dipole and a "dumbbell" has the form $U(r,c^{'},c^{''},c^{'''})$. Here*

z is a dipole coordinate in the coordinate system, the axis $\vec{z}$ of which coincides with the axis of the magnetic symmetry of the body; r is the distance to the magnetic dipole from the datum point located on the axis $\vec{z}$.

Specifically, we can consider a case with two «magnetic charges» separated with a fixed distance. Such a system, we will call it a dumbbell, modulates the field of a long thin cylinder. The potential energy for this system will have the form:

$$U(r,c',c'',c''') = \frac{\mu_0 \kappa' \kappa''}{4\pi} \sum_{\varepsilon',\varepsilon''=\pm 1} \frac{\varepsilon' \varepsilon''}{\sqrt{r^2 + l'^2 + l''^2 + 2r(\varepsilon'' l'' c'' - \varepsilon' l' c') - 2\varepsilon' \varepsilon'' l' l'' c'''}}, \tag{6}$$

where $\kappa = \mu / l$, à l, μ, κ are the length, magnetic moment of the dumbbell and the corresponding «magnetic charge».

It is this system that we used to check the capability of the orbital motion in the system of two magnets. The potential energy of the magnetic interaction of 2 bodies has the same form for the cases: two "dumbbells"; two superconductive loops of a ring form; magnetic dipole - superconductive loop of a ring form.

Hamiltonian of the system is given in the following expression

$$H = T(p^2, \vec{m}'^2, \vec{m}''^2) + U(r,c',c'',c''') \tag{7}$$

The motion equations for this Hamiltonian have the form

$$\begin{cases} \dot{\vec{r}} = \frac{1}{m}\vec{p}; \\ \dot{\vec{p}} = -\partial_r U \vec{e} - \frac{1}{r}(\partial_{c'} U P^e_\perp(\vec{\nu}\,') + \partial_{c''} U P^e_\perp(\vec{\nu}\,'')); \\ \dot{\vec{\nu}}\,' = \alpha'(\vec{m}' \times \vec{\nu}\,'); \\ \dot{\vec{m}}\,' = \partial_{c'} U(\vec{e} \times \vec{\nu}\,') - \partial_{c'''} U(\vec{\nu}\,' \times \vec{\nu}\,''); \\ \dot{\vec{\nu}}\,'' = \alpha''(\vec{m}'' \times \vec{\nu}\,''); \\ \dot{\vec{m}}\,'' = \partial_{c''} U(\vec{e} \times \vec{\nu}\,'') + \partial_{c'''} U(\vec{\nu}\,' \times \vec{\nu}\,''); \end{cases} \tag{8}$$

where the operator $P^e_\perp$ is the projector on the plane perpendicular to the vector $\vec{e}$, i.e. $P^e_\perp(\vec{\nu}') = \vec{\nu}' - c'\vec{e}$.

The components of the total momentum of the system are integrals of motion.

$$\vec{j} = \vec{l} + \vec{m}' + \vec{m}'' = const_3, \quad \vec{l} = \vec{x} \times \vec{p}; \tag{9}$$

This is the result of symmetry considerations, but, besides, this can be checked by direct calculation. Thus, the system of equations (8) should be supplemented with the relations (5) and (9).

These relations can be used to reduce the order of the system differential equations. In particular, it is very easy to exclude for example the dynamic variable $\vec{m}''$ using (9). Using relations (5) and (9) it is possible to exclude $4+3=7$ dynamic variables and have, in essence, a system of the 11th order.

3. Application of Maple and MatLab systems

Presently, the analytical possibilities of the computer algebra system (in our case it is the Maple system) allow the complete symbolic modelling of our task to be realized, namely: to deduce a structural tensor and check up the implementation of all properties, first of all, Jacobi identity; to conduct computation of the Poisson brackets between all dynamic variables interesting for us (see below); to deduce the Hamilton equations of motion; to check up the implementation of the conservation laws in our model.

As an example of proposed Hamilton formalism application we checked an interesting but little known prediction of V.V. Kozoriz [8] about the possibility of stable orbital motions in the systems of cramped magnetically interacted bodies.

There is the Bertrand's theorem in classical mechanics (sec.2,§2 [7]). It states that all restricted orbits in the central field are closed in two cases only: $U = ar^2, a \geq 0$ and $U = -k/r, k \geq 0$.

Based on the given calculations and results [4] the uncritical generalization of this theorem for the case of non-centric forces is not correct. For example, generalizations of this kind were used as a case against the «magnetic theory of matter»[8].

Orbital motion was modeled in the system of two dumbbells with the following parameters. For long cylindrical magnets d=0.0025 [m], h=0.02 [m] are the diameter and length; m=0.0003828816 [kg]. Other parameters are $\alpha = 3.87228183489 * 10^7$ $[kg^{-1}m^{-2}]$; $\mu = 0.15546875$ $[Am^2]$. Orbit radius and impulse were found from equality of centrifugal and magnetic force. Choosing R_{orb}=0.01[m] we get P_{orb}=0.0006491 $[Ns]$; T_{orb}=0.037062129 [s].

For increasing the efficiency of computational modelling the motion equations were coded in MatLab. Computational modelling showed stability of the orbital motion. The system accomplished 1000 turns for 11 minutes of modelling on a PC with the processor of Pentium M 2.0 GHz with RAM of 512 Mb. Thus there was no noticeable change of the orbit parameters. In accordance with predictions of V.V. Kozoriz the system of two magnetic dipoles does not demonstrate stable orbital motion.

During the computation process the constancy of Casimir functions and the total moment of momentum conservation laws of the system were checked.

4. Summary

The Hamilton formalism has been developed which results in a coordinate-free, i.e. vector form of motion equations for a system of magnetically interactive bodies provided that the axial symmetry of distribution of the mass of a body and its magnetic properties is the same.

In the Maple system of symbolic computation the procedure of calculating Poisson brackets has been programmed for a system of 2 magnetically interactive symmetric tops.

Using symbolic methods the following has been checked: the Jacobi identity for a structural tensor of the Poisson structure; Poisson brackets between all dynamic variables which are of interest for our problem; system motion equations; conservation of the components of the total moment of momentum.

The orbital motion of two magnets ("dumbbells" model) has been modeled numerically both in Maple and MatLab. During computational modeling the uniformity of Casimir functions and integrals of motions have been checked. The example demonstrates the stability of the orbital motion of magnets with certain relations between their parameters.

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