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Feynman Diagrams, Differential Reduction, and Hypergeometric Functions

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We will present some (formal) arguments that any Feynman diagram can be understood as a particular case of a sum of Horn-type multivariable hypergeometric functions. The advantages and disadvantages of this type of approach to the evaluation of Feynman diagrams is discussed.

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1. Introduction

Any dimensionally-regularized [1] multiloop Feynman diagram with propagators $1/(p^2-m^2)$ can be written in the form of a finite sum of multiple Mellin-Barnes integrals [2, 3] obtained via a Feynman-parameter or "α" representation: [4]

$$F\left(a_{js},b_{km},c_i,d_j,\vec{x}\right) = \int_{\gamma+i\mathbb{R}} dz_1 \ldots dz_r \frac{\Pi_{j=1}^{p}\Gamma\left(\sum_{s=1}^{r} a_{js}z_s + c_j\right)}{\Pi_{k=1}^{q}\Gamma\left(\sum_{m=1}^{r} b_{km}z_m + d_k\right)} x_1^{-z_1} \ldots x_r^{-z_r} \,, \tag{1.1}$$

where $a_{js}, b_{km} \in \mathbb{Q}, c_i, d_j \in \mathbb{C},$ and γ is chosen such that the integral exists. In this integral, the x_a are rational functions of kinematic invariants (masses and momenta) and the matrices a_{js}, b_{kr} and vectors c_i, d_j depend linearly on the dimension n of space-time.[1]. In dimensional regularization, n is treated as an arbitrary parameter, so that $n = I - 2\varepsilon$ for a positive integer I.

Formally,[2] this integral can be expressed in terms of a sum of residues of the integrated expression

$$F\left(a_{js},b_{km},\vec{c},\vec{d},\vec{\alpha},\vec{x}\right) = \sum_{\vec{\alpha}} B_{\vec{\alpha}} \vec{x}^{\,\vec{\alpha}} \Phi(\vec{\gamma};\vec{\sigma};\vec{x}) \,, \tag{1.2}$$

where the components of the vector $\vec{\alpha}$ are defined in terms of the matrix components a_{js}, b_{km} and vectors $\vec{c}, \vec{d}$, the coefficients $B_{\vec{\alpha}}$ are ratios of Γ-functions with arguments depending on $\vec{\alpha}$, and the functions $\Phi(\vec{\gamma};\vec{\sigma};\vec{x})$ have the form

$$\Phi(\vec{\gamma};\vec{\sigma};\vec{x}) = \sum_{m_1,m_2,\cdots,m_r=0}^{\infty} \left(\frac{\Pi_{j=1}^{K}\Gamma\left(\sum_{a=1}^{r}\mu_{ja}m_a + \gamma_j\right)}{\Pi_{k=1}^{L}\Gamma\left(\sum_{b=1}^{r}\nu_{kb}m_b + \sigma_k\right)} \right) x_1^{m_1} \cdots x_r^{m_r} \,, \tag{1.3}$$

with $\mu_{ab}, \nu_{ab} \in \mathbb{Q},\ \gamma_j, \sigma_k \in \mathbb{C}$.

Let $\vec{e}_j = (0,\cdots,0,1,0,\cdots,0)$ denote the unit vector with unity in its j^{th} entry, and define $\vec{x}^{\vec{m}} = x_1^{m_1}\cdots x_r^{m_r}$ for any integer multi-index $\vec{m} = (m_1,\cdots,m_r)$. In accordance with the definition of Refs. [15, 16], the (formal) multiple series $\sum_{\vec{m}=0}^{\infty} C(\vec{m})\vec{x}^{\vec{m}}$, is called *hypergeometric* if, for each $i = 1,\cdots,r$,

[1] Starting from the α-representation for Feynman diagrams with (irreducible) numerators [4], the Mellin-Barnes representation can be written explicitly in terms of Symanzik polynomials.[2] In Refs. [5, 6], it was shown that this α-representation can be understood as a linear combination of scalar Feynman diagrams of the original type with shifted powers of propagators and space-time dimension multiplying a tensor factor depending on external momenta. Through fifty years of the evaluation of Feynman diagrams, many auxiliary programs have been created for the generation of Symanzik polynomials. In particular, the matrix representation for these polynomials derived by Nakanishi (see Eqs. (3.23)$-$(3.34) in Ref. [7]) has been realized on FORM by Oleg Tarasov in Ref. [8], by Oleg Veretin in [9], and recently in Ref. [10].

[2] This is true under the condition that there is a common domain of convergence. For particular values of the kinematic variables and powers of the propagators, the Mellin-Barnes representation may contain terms like $\Gamma(a-s)\times\Gamma(a+s)$ or $\Gamma^2(a-s)\times\Gamma^2(b+s)$, where a and b are parameters and s is an integration variable. Two algorithms for the practical construction of the ε-expansion of Mellin-Barnes integrals with such singularities have been described in Refs. [11], These are now implemented in several packages [12]. It will be quite interesting to apply modern mathematical algorithms, as was done in Ref. [13], to the problem of singularities in the Mellin-Barnes representation as well. In the framework of our approach, such singularities can be regularized by introducing an additional analytical regularization for each propagator (see Ref. [14]) or by introducing masses to regulate IR singularities. General theorems on the properties of Feynman diagrams in dimensional regularization guarantee that a smooth limit of the auxiliary regularization should exist.

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the ratio $C(\vec{m}+\vec{e}_i)/C(\vec{m})$ is a rational function in the index of summation $(m_1,\cdots,m_r)$. Ore and Sato [17] found (see also Ref. [16]) that the coefficients of such a series have the general form

$$C(\vec{m}) = \Pi_{i=1}^r \lambda_i^{m_i} R(\vec{m}) \left(\frac{\Pi_{j=1}^N \Gamma(\mu_j(\vec{m})+\gamma_j)}{\Pi_{k=1}^M \Gamma(\nu_k(\vec{m})+\delta_k)} \right) , \tag{1.4}$$

where $N,M \geq 0$, $\lambda_j, \delta_j, \gamma_j \in \mathbb{C}$ are arbitrary complex numbers, $\mu_j, \nu_k : \mathbb{Z}^r \to \mathbb{Z}$ are arbitrary integer-valued linear maps, and R is an arbitrary rational function. A series of this form is called a *Horn-type* hypergeometric series.

We deduce from Eqs. (1.3) and (1.4) that, assuming there is a region of variables where each multiple series of Eq. (1.2) is convergent, any Feynman diagram can be understood as a special case of a Horn-type hypergeometric function with the functions $R(\vec{m})$ equal to unity.[3] One interesting property of Horn-type hypergeometric functions is the existence of a set of differential contiguous relations between functions with shifted arguments. We consider such relations in the following section.

2. Contiguous Relations via Linear Differential Operators

Let us consider a formal series of type (1.3). The sequences $\vec{\gamma} = (\gamma_1,\cdots,\gamma_K)$ and $\vec{\sigma} = (\sigma_1,\cdots,\sigma_L)$ are called *upper* and *lower* parameters of the hypergeometric function, respectively. Two functions of type (1.3) with sets of parameters shifted by a unit, $\Phi(\vec{\gamma}+\vec{e}_c;\vec{\sigma};\vec{x})$ and $\Phi(\vec{\gamma};\vec{\sigma};\vec{x})$, are related by a linear differential operator:

$$\begin{aligned}
\Phi(\vec{\gamma}+\vec{e}_c;\vec{\sigma};\vec{x}) &= \sum_{m_1,m_2,\cdots,m_r=0}^{\infty} \left(\frac{(\sum_{a=1}^r \mu_{ca} m_a + \gamma_c) \Pi_{j=1}^K \Gamma\left(\sum_{c=1}^r \mu_{jc} m_c + \gamma_j\right)}{\Pi_{k=1}^L \Gamma\left(\sum_{b=1}^r \nu_{kb} m_b + \sigma_k\right)} \right) x_1^{m_1} \cdots x_r^{m_r} \\
&= \sum_{m_1,m_2,\cdots,m_r=0}^{\infty} \left(\frac{\left(\sum_{a=1}^r \mu_{ca} x_a \frac{\partial}{\partial x_a} + \gamma_c\right) \Pi_{j=1}^K \Gamma\left(\sum_{c=1}^r \mu_{jc} m_c + \gamma_j\right)}{\Pi_{k=1}^L \Gamma\left(\sum_{b=1}^r \nu_{kb} m_b + \sigma_k\right)} \right) x_1^{m_1} \cdots x_r^{m_r} \\
&= \left(\sum_{a=1}^r \mu_{ca} x_a \frac{\partial}{\partial x_a} + \gamma_c \right) \Phi(\vec{\gamma};\vec{\sigma};\vec{x}) \equiv U^+_{[\gamma_c \to \gamma_c+1]} \Phi(\vec{\gamma},\vec{\sigma},\vec{x}) \, .
\end{aligned} \tag{2.1}$$

Similar relations also exist for the lower parameters:

$$\begin{aligned}
\Phi(\vec{\gamma};\vec{\sigma}-\vec{e}_c;\vec{x}) &= \sum_{m_1,m_2,\cdots,m_r=0}^{\infty} \sum_{b=1}^r (\nu_{cb} m_b + \sigma_c) \left(\frac{\Pi_{j=1}^K \Gamma\left(\sum_{a=1}^N \mu_{ja} m_a + \gamma_j\right)}{\Pi_{k=1}^L \Gamma\left(\sum_{b=1}^r \nu_{kb} m_b + \sigma_k\right)} \right) x_1^{m_1} \cdots x_r^{m_r} \\
&= \left(\sum_{b=1}^r \nu_{cb} x_b \frac{\partial}{\partial x_b} + \sigma_c \right) \Phi(\vec{\gamma};\vec{\sigma};\vec{x}) \equiv L^-_{[\sigma_c \to \sigma_c-1]} \Phi(\vec{\gamma};\vec{\sigma};\vec{x}) \, .
\end{aligned} \tag{2.2}$$

The linear differential operators $U^+_{\gamma_c \to \gamma_c+1}$, $L^-_{\sigma_c \to \sigma_c-1}$ are called *step-up* and *step-down* operators for the upper and lower index, respectively. If additional step-down and step-up operators $U^-_{\gamma_c}$, $L^+_{\sigma_c}$ satisfying

$$U^-_{[\gamma_c+1 \to \gamma_c]} U^+_{[\gamma_c \to \gamma_c+1]} \Phi(\vec{\gamma},\vec{\sigma},\vec{x}) = L^+_{[\sigma_c-1 \to \sigma_c]} L^-_{[\sigma_c \to \sigma_c-1]} \Phi(\vec{\gamma},\vec{\sigma},\vec{x}) = \Phi(\vec{\gamma},\vec{\sigma},\vec{x}) \, ,$$

[3] Using the technique presented in Ref. [18], this statement can also be shown to be valid for the phase-space integral.

(*i.e.*, the inverses of $U^+_{\gamma_c}$, $L^-_{\sigma_c}$), are constructed, we can combine these operators to shift the parameters of the hypergeometric function by any integer. This process of applying $U^{\pm}_{\gamma_c}, L^{\pm}_{\sigma_c}$ to shift the parameters by an integer is called **differential reduction** of a hypergeometric function (1.3).

Algebraic relations between functions $\Phi(\vec{\gamma},\vec{\sigma};\vec{x})$ with parameters shifted by integers are called **contiguous relations**. The development of systematic techniques for obtaining a complete set of contiguous relations has a long story. It was started by Gauss, who found the differential reduction for the ${}_2F_1$ hypergeometric function in 1823. [19] Numerous papers have since been published [20] on this problem. An algorithmic solution was found by Takayama in Ref. [21], and those methods have been extended in a later [4] series of publications [23].

Let us recall that any hypergeometric function can be considered to be the solution of a proper system of partial differential equations (PDEs). In particular, for a Horn-type hypergeometric function, the system of PDEs can be derived from the coefficients of the series. In this case, the ratio of two coefficients can be presented as a ratio of two polynomials,

$$\frac{C(\vec{m}+e_j)}{C(\vec{m})} = \frac{P_j(\vec{m})}{Q_j(\vec{m})} \, , \tag{2.3}$$

so that the Horn-type hypergeometric function satisfies the following system of equations:

$$0 = D_j(\vec{\gamma},\vec{\sigma},\vec{x})\Phi(\vec{\gamma},\vec{\sigma},\vec{x}) = \left[Q_j\left(\sum_{k=1}^r x_k \frac{\partial}{\partial x_k} \right) \frac{1}{x_j} - P_j\left(\sum_{k=1}^r x_k \frac{\partial}{\partial x_k} \right) \right] \Phi(\vec{\gamma},\vec{\sigma},\vec{x}) \, , \quad j = 1,\cdots,r. \tag{2.4}$$

Let $\mathfrak{R}$ be the left ideal of the ring $\mathfrak{D}$ of differential operators generated by the system of differential equations for a hypergeometric function (2.4), $D_j(\vec{\gamma},\vec{\sigma},\vec{x}), j = 1,\cdot\cdots,r$. The first step in Takayama's algorithm is the construction of a Gröbner basis $\mathfrak{G} = \{G_i(\vec{\gamma},\vec{\sigma},\vec{x}), i = 1,\cdots,q\}$ of $\mathfrak{R}$. Then the step-up operator corresponding to $U^+_{\gamma_c}$ and step-down operator corresponding to $L^-_{\sigma_c}$ are solutions to the linear equations

$$\sum_{i=1}^q C_i G_i(\vec{\gamma},\vec{\sigma},\vec{x}) + U^-_{[\gamma_c+1\to\gamma_c]} U^+_{[\gamma_c\to\gamma_c+1]} = 1 \, , \tag{2.5}$$

$$\sum_{i=1}^q E_i G_i(\vec{\gamma},\vec{\sigma},\vec{x}) + L^+_{[\sigma_c-1\to\sigma_c]} L^-_{[\sigma_c\to\sigma_c-1]} = 1 \, , \tag{2.6}$$

where C_i, E_i are arbitrary functions. This system has a solution if the left ideal generated by $\mathfrak{G} \cup \left\{ U^+_{\gamma_c} \right\}$ is equal to $\mathfrak{D}$ (see details in Ref. [21]).

In this way, the Horn-type structure provides an opportunity to reduce hypergeometric functions to a set of basis functions with parameters differing from the original values by integer shifts:

$$P_0(\vec{x})\Phi(\vec{\gamma}+\vec{k};\vec{\sigma}+\vec{l};\vec{x}) = \sum_{m_1,\cdots,m_p=0}^{\sum|k_i|+\sum|l_i|} P_{m_1,\cdots,m_r}(\vec{x}) \left(\frac{\partial}{\partial x_1} \right)^{m_1} \cdots \left(\frac{\partial}{\partial x_r} \right)^{m_r} \Phi(\vec{\gamma};\vec{\sigma};\vec{x}) \, , \tag{2.7}$$

where $P_0(\vec{x})$ and $P_{m_1,\cdots,m_p}(\vec{x})$ are polynomials with respect to $\vec{\gamma},\vec{\sigma}$ and $\vec{x}$ and $\vec{k},\vec{l}$ are lists of integers.

[4] The problem also can be solved via an Ore algebra [22] approach to the relevant system of linear differential and difference (shift) operators.

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3. Differential Reduction in Practice

In real physical problems, the variables are generally not linearly independent: some of them can be equal to one another $x_i = x_j, i \neq j$. In this case, the Horn-type hypergeometric function generally is not expressible in terms of Horn hypergeometric functions with fewer variables. Another important physical case is when variables belong to the surfaces where coefficient $P_0(\vec{x})$ of Eq. (2.7) vanishes (in the one-variable case it corresponds to $z = 1$). In all of these cases, the differential reduction cannot be directly applied. But if the l.h.s. of Eq. (2.7) is defined for that limit, the smooth limit of the r.h.s. of Eq. (2.7) will exist too.

The problem is then to find this smooth limit. Here, "physics" plays a role. For the evaluation of physical processes, exact results in terms of hypergeometric function are not necessary, but only the coefficients of a Laurent expansion around an integer value of the space-time dimension. From that point of view, to guarantee that smooth limit exists, it is enough to prove that a smooth limit exists for all coefficients of the all-order ε expansion.

Recently, physicists have proven several theorems on the all-order ε expansion of hypergeometric functions about integer and/or rational values of parameters [24, 25, 26, 27]. A remarkable property of this construction is that for special values of parameters, the coefficients are expressible in terms of multiple polylogarithms [28]. For functions of this type, the limiting procedure is well understood. Unfortunately, existing theorems are not adequate to cover all values of parameters for hypergeometric functions generated by Feynman diagrams [27, 29].

4. Discussion and Conclusions

We have presented formal arguments that any Feynman diagram can be treated as a finite sum of Horn-type hypergeometric functions. This applies, in particular, for off-shell diagrams with different values of masses in internal lines. In the physically interesting cases when some of the arguments are equal to one another, or belong to some singularity surface, a limiting procedure should be constructed. To find this limit, and strongly prove that the proper limit exists, the all-order ε expansion of hypergeometric function around rational values of parameters can be used.

The Horn-type hypergeometric functions possess useful properties: the system of differential equation they satisfy is enough (i) for reduction of original function to a restricted set of basis functions (the number of basis functions follows directly from the system of equations); (ii) for the construction of the all-order ε expansion for the basis hypergeometric functions in form of the Lappo-Danilevky solution. The first part of this algorithm has been discussed in [30, 31]; the second part in [26, 27]. The validity of our approach has been confirmed (in particular cases) by full agreement with the evaluation of the first coefficients of the ε expansion constructed in [32], by theorems about the all-order ε expansion of hypergeometric functions proven in [25] with the help of another technique, and by comparisons of the results of the differential reduction of some Feynman diagrams with the results of reductions obtained via computer programs [33].

The above-mentioned properties of the hypergeometric representation, in particular Takayama's reduction algorithm, demonstrate that the hypergeometric representation is a universal tool for the evaluation of Feynman diagrams. Obtaining a decomposition of the integration region into regions

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where each individual term is well-defined is one of the main problems in the hypergeometric approach (beyond the one-loop and one-fold cases).

Series representations (in four dimensions) have been used in Ref. [34] to obtain a system of partial differential equations for N-point one-loop diagrams. In Ref. [35], the hypergeometric representation has been derived for an N-point one-loop diagram with arbitrary powers of the propagators. The possibility of using a hypergeometric representation for the reduction of Feynman diagrams was considered in Ref. [30]. The idea of using the Gröbner basis technique for the reduction of Feynman diagrams has been proposed by Tarasov [36] and received further extension in Ref. [37]. For practical applications of the "limiting" procedure, the first few coefficients of the ε expansion are necessary. The ε expansion is implemented in several packages [38] for a restricted class of hypergeometric functions. Some results for the finite harmonic sums are available in [39].

Another important class of developments includes techniques such as integration-by-parts [40], generalized recurrence relations [6], and the differential equation approach [41], which make it possible to work directly with parameters of the Feynman diagram (the l.h.s. of Eq. (1.2)) without splitting the diagram into a linear combination of Horn-type hypergeometric functions. Such direct analyses of Feynman diagrams as hypergeometric functions are based on the properties of dimensional regularization [1] , which treats all types of singularities (IR and UV) simultaneously.

Some partial results of our research have been used in Ref. [42]. A more detailed discussion will be presented in a forthcoming publication [43].

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