Past horizons in Robinson-Trautman spacetimes

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We exhibit properties of past quasi-local horizons in vacuum Robinson-Trautman spacetimes. Trapped surfaces have to intersect the surface $r = 2m$, which cannot be null unless $g$ is the Schwarzschild metric. The only Robinson-Trautman metric admitting a past nonexpanding horizon is the Schwarzschild metric and the C-metric.
1. Introduction

The Robinson-Trautman (RT) metrics [1]

\[ g = 2du(Hdu + dr) - 2\frac{r^2}{P^2}d\xi d\bar{\xi} \]  \hspace{1cm} (1.1)

were proposed in order to describe gravitational radiation from bounded sources. Here \( u, r, \xi, \bar{\xi} \)
are coordinates and \( P = P(u, \xi, \bar{\xi}) \) is an unknown function. For RT metrics the vacuum Einstein
equations reduce to the definition of \( H \) in terms of \( P \)

\[ H = P^2(\ln P)_{,\xi\bar{\xi}} - r(\ln P)_{,u} - \frac{m}{r} \]  \hspace{1cm} (1.2)

and the RT equation for the function \( P \)

\[ K_{,\xi\bar{\xi}} - 3m(P^{-2})_{,u} = 0 . \]  \hspace{1cm} (1.3)

Here \( m = \text{const} \) and

\[ K = 2P^2(\ln P)_{,\xi\bar{\xi}} \]

is the Gauss curvature of the surfaces \( u = \text{const}, \ r = 1 \).

Global structure, trapped surfaces and asymptotic behaviour of RT spacetimes were successfully studied by
Penrose [2], Foster and Newman [3], Lukacs, Perjes, Porter and Sebestyen [4], Schmidt [5], Rendall [6], Tod [7], Singleton [8], Chruściel [9, 10], Chruściel and Singleton [11],
Chow and Lun [12] and others. In this communication we summarize our results [13] on trapped
surfaces and quasi-local horizons in RT spacetimes. These geometrical objects play an important
role in modern theory of black (or white) holes (see e.g. [14] and references therein).

A nontrivial solution of the RT equation is given by

\[ P = 1 + \frac{1}{2\xi\bar{\xi}} . \]

It defines the Schwarzschild metric in the Eddington-Finkelstein coordinates

\[ g = du((1 - \frac{2m}{r})du + 2dr) - r^2 \frac{2d\xi d\bar{\xi}}{(1 + \frac{1}{2}\bar{\xi}\xi)^2} , \]

\( \xi \) being the complex stereographic coordinate of the 2-dimensional sphere \( S_2 \). These coordinates
cover the shaded half of the Penrose conformal diagram (Fig.1).

Let us consider RT metrics with sections \( u = \text{const}, \ r = \text{const} \) diffeomorphic to \( S_2 \). Then

\[ \hat{P} = \frac{P}{1 + \frac{1}{2}\xi\bar{\xi}} \]

is a smooth and positive function on \( S_2 \). By virtue of the RT equation we can set

\[ i \int_{S_2} P^{-2}d\xi \wedge d\bar{\xi} = 4\pi . \]  \hspace{1cm} (1.4)

Then the surface area of the sections is \( 4\pi r^2 \). For \( m > 0 \) these metrics can be developed from an
initial surface \( u = u_0 \) to cover a manifold shown in Fig.2 [10].

When \( u \to \infty \) they tend to the Schwarzschild metric. Thus, the future event horizon is given
by \( u = \infty, \ r = 2m \) and is similar to that in the Schwarzschild spacetime.
2. Past trapped surfaces and horizons

The question arises whether cross-sections of $r = 2m$ with $u = \text{const} < \infty$ form the past event horizon as in the Schwarzschild spacetime. In general, the answer is 'no' since solving condition $H = 0$ together with the RT equation shows that (see [13] for details)

- Surface $r = 2m$ is null for $u < \infty \iff g$ is Schwarzschild.

Let $\mathcal{S}$ be a 2-dimensional spacelike surface given by

$$u = \text{const}, \quad r = R(\xi, \bar{\xi}). \quad (2.1)$$

The ingoing and outgoing null vectors normal to $\mathcal{S}$ read

$$k = \partial_r, \quad l = \partial_u - \left( H - \frac{p^2}{r^2} |R,\xi|^2 \right) \partial_r + \frac{p^2}{r^2} (R,\xi) \partial_\xi + (R,\bar{\xi}) \partial_{\bar{\xi}}.$$  

Expansion of $k$ and $l$ on $\mathcal{S}$ are, respectively,

$$\theta_{(k)} = -R^{-1}$$

and

$$\theta_{(l)} = \frac{1}{R} \left( -p^2 (\ln R),\xi \xi + \frac{K}{2} - \frac{m}{R} \right).$$
Hence, the surface $S$ is trapped iff

$$-P^2(\ln R) \xi \bar{\xi} + \frac{K}{2} - \frac{m}{R} = 0 . \tag{2.2}$$

According to Tod [7] equation (2.2) admits unique (for each $u$) solution which defines an outermost marginally trapped surface $S$.

Integrating equation (2.2) over $S$ and using the Gauss-Bonnet theorem yields

$$\int_{S_2} \left( \frac{2m}{R} - 1 \right) d\sigma = 0 ,$$

where $d\sigma = iP^{-2}d\xi \wedge d\bar{\xi}$ is the surface 2-form. Hence, we obtain the following property

- The trapped surface $S$ intersects the surface $r = 2m$.

By varying $u$ in (2.1) and (2.2) one can define a hypersurface $H$ foliated by the marginally trapped surfaces $S$. Chow and Lun [12] showed that $H$ is a non-timelike surface (dynamical horizon). From the point of view of a theory of black holes it is important to know whether $H$ can be null (nonexpanding horizon). Note that if $H$ is null then the expansion-free null vector $l$ is tangent to $H$. It is also shear-free due to the Raychaudhuri equation. Independently of topological assumptions on intersections of $H$ with $u=\text{const}$ we obtain the following result (see [13] for a proof)

- The only vacuum Robinson-Trautman metrics admitting a past nonexpanding horizon is the Schwarzschild solution and the C-metric.

In the case of the C-metric coordinates $u$ and $\xi$ can be chosen in such a way that

$$K = K(x + u) , \quad P^2 = \frac{12m}{K_x} ,$$

where $x = \text{Re}\xi$. The function $K$ undergoes the equation

$$6mK_x = -\frac{1}{3}K^3 + bK + c$$

and the nonexpanding horizon $H$ is given by

$$r = 6m(K + a)^{-1} .$$

Here $a$, $b$ and $c$ are constants constrained by the condition

$$\frac{a^3}{3} - ab + c = 0 .$$

Note that in this case $H$ does not admit regular spherical sections.
Acknowledgments

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References

[7] Tod P. 1989, Analogues of the past horizon in Robinson-Trautman space-times, Class. Quantum Grav. 8, 1159