PROCEEDINGS OF SCIENCE

Penrose Limits, Singularities and DLCQ

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Space-time singularities of gravitational fields are a sign of a break down in the classical geometric description of gravitational physics. We look at some of the approaches that modern string theory and quantum gravity indicate for understanding the physics of regions with extreme gravitational fields. These notes principally deal with an approach that combines string theory with the Penrose limit to obtain a weakly coupled non-abelian Yang-Mills theory as an alternative description of the space-time near a singularity.

Black Holes in General Relativity and String Theory
24-30 August, 2008
Veli Lošinj, Croatia

*Speaker.

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1. Singularities and what to do with them

That singularities arise generically in solutions to the Einstein equations is the substance of the Hawking-Penrose singularity theorems [1]. The most general singular space-times have been studied very little due to the complicated nature of the non-linear Einstein equations. The most well studied examples arise for space-times with large isometry groups and amongst these are black hole and cosmological space-times.

Singularities can be categorised in many different ways using measures of strength, dimension, orientation and topology.

The orientation of a singularity may be;

- Space-like - the most well-known being the big bang of Friedmann Robertson Walker cosmology and that of the Schwarzschild Black Hole.
- Time-like - here the classic example is the Reissner-Nördstrom black-hole.
- Null - conjectured to arise under generic perturbations of inner horizons. Analytic examples of null singularities are the metrics of singular plane waves.

There are various approaches to studying the physics of singularities.

- Singularities may be resolved in a geometric sense, for instance by replacing the region close to and including the singularity by a smooth geometry. This would be a classical resolution where an additional source type term is added to the stress-energy tensor in the region close to the singularity in such a way as to deform the geometry away from it's singular form. An example of this construction in the context of plane waves is given in [2]. Often such additional terms in the stress-energy tensor violate energy conditions.
- Singularities are truly singular, however it may turn out that fields in the background of the singularity are nevertheless smooth, in the sense that they may be continued through the singularity. This can happen for scalar fields near singularities that are not too strong [5].
- There is some other quantum gravity related resolution: Fuzzballs that hide horizons and consequently also singularities [3]; Loop Quantum Gravity has a minimum distance element and thus there is encoded in this theory an upper bound on the space-time curvature [4].
- Gravitation and space-time enter a non-geometrical phase: a gas of black holes; a new phase of quantum geometry; or an alternative Yang-Mills description of the physics.

There is much that one could say about these different approaches, of which one could also make a more complete catalogue and exposition. We will here concentrate on null singularities and their non-geometric resolution via non-perturbative physics in String Theory described by a Yang-Mills theory. The references in the bibliography are largely to recent research papers and some historical references. The interested reader should refer to the research papers, on which this exposition is largely based, for more complete references.

2. Singularities - Probing with the Penrose Limit

A technique which has been extremely successful in recent years for the study of string theory in non-trivial space-time metrics is that of the Penrose limit. This corresponds to zooming in to a path in the space-time by blowing up that part of the metric close to a chosen null geodesic. This often leads to a non-trivial limit space-time that is much simpler than the original space-time and thus more tractable from a mathematical point of view while at the same time retaining important features of the original space-time and in the case of interetst the singularity. We will now explain how this works in some more detail (for further details see [7] and references therein).

Choose an affinely parametrised null geodesic $\gamma(u)$, the tangent vector to this geodesic,

$$E_{+}^{\mu} = \dot{\gamma}^{\mu} \tag{2.1}$$

is parallel transported along the geodesic and we can extend this to a parallel pseudo-orthonormal frame,

$$ds^2 = 2E^+E^- + \delta_{ab}E^aE^b \tag{2.2}$$

The profile of the plane-wave

$$ds^2 = 2dudv + A_{ab}(u)x^ax^bdu^2 + d\vec{x}^2 \tag{2.3}$$

that results upon taking the Penrose limit of this geometry with respect to γ is

$$A_{ab}(x^+) = -R_{a+b+}|_{\gamma(x^+)} \tag{2.4}$$

where on the right hand side we have frame components of the curvature tensor of the original metric. Thus the Penrose limit is actually encoding some physical information about the original metric. In particular the Penrose limit space-time gives an exact description of the space-time along the geodesic, this also being the leading behaviour of the Fermi coordinate expansion around the null geodesic in the original space-time [6].

The geometric significance of the wave-profile $A_{ab}(x^+)$ is that it is the transverse null geodesic deviation matrix along γ of the original metric

$$\frac{d^2}{du^2}Z^a = A_{ab}(u)Z^b \tag{2.5}$$

where Z is the transverse geodesic deviation vector. Physically we see that the Penrose limit retains precisely the tidal forces along the corresponding null geodesic in the original geometry.

Using the Penrose limit to study singularities one makes a remarkable observation. In all cases for which a non-trivial Penrose limit exists the limit space-time is a member of a very special class of plane wave space-times - the singular homogeneous plane waves (SHPW's) - this under very mild restrictions on the stress-energy tensor that is the source for the Einstein equations.

For example, for a spherically symmetric time-dependent metric

$$ds^2 = -f(t)dt^2 + g(t)dr^2 + t^2d\Omega_d^2 \tag{2.6}$$

one can show that for the geodesic $\gamma(u) = (t(u), r(u), \Omega(u))$ the non-trivial components of the PL metric are

$$A_{11} = (t\dot{t}\sqrt{fg})^{-1}\partial_u^2(t\dot{t}\sqrt{fg}) \tag{2.7}$$

$$A_{ab} = \delta_{ab}\left(\frac{\ddot{t}(u)}{t(u)} - \frac{L^2}{t(u)^4}\right) \tag{2.8}$$

In the context of investigations of the Cosmic Censorship Hypothesis, Szekeres and Iyer [13] studied a large class of four-dimensional spherically symmetric metrics that they dubbed "metrics with power-law type singularities". This class encompasses the near singularity behaviour of practically all known singular spherically symmetric metrics. The singularities in these metrics are time-like, null and space-like. The analysis of the PL for the null and time-like singularities in the class of Szekeres-Iyer geometries probably does not give a complete picture of the nature of the singularities so we will focus just on the (more interesting) case of space-like singularities.

One finds that for all of these geometries

$$t(u) = u^a \tag{2.9}$$

where the exponent a depends on the details of the singularity. Clearly this can give rise to various power law behaviours as one can easily see from the expression for A_{ab} above.

For the PL metric one can prove the following;

Penrose Limits of spherically symmetric space-like or time-like singularities of power-law type satisfying (but not saturating) the Dominant Energy Condition (DEC) are singular homogeneous plane waves with profile

$$A_{ab}(u) = -\omega_a^2 \delta_{ab} u^{-2}. \tag{2.10}$$

There is a qualitative difference in behaviour between SHPW's with $\omega^2 > 1/4$ and those with $\omega^2 \leq 1/4$ and it turns out that for all geometries satisfying the above conditions the resulting frequencies squared ω_a^2 are bounded from above by $1/4$ unless one is on the border to an extreme equation of state. Apart from this very interesting universal SHPW behaviour of the PL limit geometry, the bound on frequencies is also quite intriguing. In studies of scalar fields near singularities, it is precisely for $\omega^2 \leq 1/4$ that one scalar field theory near a singularity becomes more tractable [5].

3. Singular Homogeneous Plane Waves

As we have seen then, the plane wave geometry that arises when one takes the Penrose limit of a singular geometry is generically quite a special and symmetric space-time - a Singular Homogeneous Plane Wave. These metrics have a maximal group of isometries with the algebraic structure of a Heisenberg algebra plus an extension rising from a scaling of the coordinates and thus in addition to being Plane waves they are also homogeneous space-times.

Plane waves can be represented in two different coordinate systems, the Rosen and the Brinkmann coordinates. Each being useful for studying different aspects of the geometries. The Brinkmann coordinates are unique, the only non-trivial component of the metric being equal to the only non-trivial component of the curvature tensor (the $A_{ab} = -R_{a+b+}$).

In Brinkmann coordinates the metric for SHPW's has the unique form,

$$ds^2 = -2dz^+dz^- + \frac{A_{ab}z^az^b}{(z^+)^2}(dz^+)^2 + \frac{f_{ab}z^a}{z^+}dz^+dz^b + d\vec{z}^2 \tag{3.1}$$

with constant symmetric A_{ab} and anti-symmetric f_{ab} whilst in Rosen coordinates we find

$$ds^2 = -2dy^+dy^- + g_{ij}(y^+)dy^i dy^j \tag{3.2}$$

with less obvious restrictions on the g_{ij}. It is evident from the Brinkmann form of the metric that there is a singularity at $z^+ = 0$ at finite affine distance corresponding to divergent tidal forces, and so the metrics are geodesically incomplete.

The Brinkmann coordinates contain the essential information about the structure of the metric in a unique way, however the full symmetries are somewhat difficult to see in this coordinate system. In the Rosen coordinates these symmetries are very simple and it is also thus easy to see what is required to extend the Heisenberg algebra of isometries of the generic plane wave to the isometries required for the geometry to additionally be homogeneous [8].

The Heisenberg algebra is generated (in Rosen coordinates) by the commuting translations, $Z = \partial_{y^-}$ and $Q_{(i)} = \partial_{y^i}$, together with the "hidden" Heisenberg dual translations, $P_{(i)}$.

The additional isometry is that of a scale invariance in Brinkmann coordinates, $(z^+, z^-) \rightarrow (\lambda z^+, \lambda^{-1} z^-)$, and the corresponding Killing vector is,

$$X = z^+\partial_{z^+} - z^-\partial_{z^-}. \tag{3.3}$$

String theory in the background of a SHPW is solvable and as such provides an example of a non-trivial (time-dependent), singularity with a large isometry group. In principle this means that one can hope to study more deeply the behaviour of string theory in the presence of singularities and in time-dependant backgrounds using strings in SHPW's as a solvable toy model.

4. Sen-Seiberg and Discrete Light-Cone Quantization

The basis for our discussion of the DLCQ (Discrete Light-Cone Quantization) of string/M-theory, is the treatment by Sen [9] and Seiberg [10] reviewed in [11]. One considers the metric

$$ds^2 = -2dy^+dy^- + \ldots = -(dy^0)^2 + (dy^9)^2 + \ldots \tag{4.1}$$

and looks for a limit which enables one to realise the light-like compactification, $y^- \sim y^- - 2\pi R$ as a limit of space-like compactifications. To achieve this we do a boost,

$$x^\pm \equiv y'^\pm = e^{\pm\beta} y^\pm \tag{4.2}$$

together with the identification $x^9 \sim x^9 + 2\pi R_s$. With $e^\beta = \sqrt{2}R/R_s$ we then find

$$y^+ \sim y^+ + \pi R_s^2/R,\; y^- \sim y^- - 2\pi R \tag{4.3}$$

Beginning then with N units of momentum in the x^9 direction, $p'_9 = N/R_S$, we find after the boost ($R_s \rightarrow 0$) that,

$$p^+ \equiv -p_- = \frac{N}{R}. \tag{4.4}$$

In a sector with N units of light-cone momentum, and with a mass scale m, the DLCQ Hamiltonian is defined by the limit

$$H_N^{DLCQ}(m,R) = \lim_{R_s \rightarrow 0} i\partial_{y^+} \tag{4.5}$$

On dimensional grounds when the energies and lengths are scaled by $\lambda = R/R_s$ the original Hamiltonian scales as,

$$\frac{R}{R_s} H_N(m, R_S) = H_N\left(\frac{R}{R_s} m, \frac{R_s^2}{R}\right) \equiv H_N(\hat{m}, \hat{R}_s). \tag{4.6}$$

With this rescaling the DLCQ is realised by taking the limit,

$$H_N^{DLCQ}(m, R) = \lim_{\hat{R}_s \to 0} H_N(\hat{m}, \hat{R}_s). \tag{4.7}$$

We see that the discrete light-cone quantization of M-theory is attained by making the above identifications and transformations with $m = m_P$ and then taking $R_s = R_{11} \to 0$ and thus the non-perturbative physics of M-theory is described by the BFSS matrix quantum mechanics of N D0-branes in type IIA string theory.

One can apply this same construction to IIA string theory leading to a non-perturbative description that utilizes N D-strings in type IIB string theory in a limit that corresponds to a matrix string action. We will discuss in slightly more detail below and in particular the duality transformations that one must perform on the plane wave spacetimes.

Before we go to the general case of SHPW's we will first take a quick look at the half-way situation discussed in the Big Bang matrix string paper of Craps, Sethi and Verlinde [12]. In this case the geometry (a simplified model of the null big-bang) is flat-space with a null linear dilaton in IIA string theory. It is amusing to observe that this geometry, when lifted to 11-dimensions is precisely a SHPW.

Due to the fact that the dilaton depends on one of the light-cone coordinates we can non-longer use the approach of Sen and Seiberg and to adapt their approach to such geometries we require that our geometry includes one other compact space-like direction of constant radius.

$$ds^2 = -2dy^+ dy^- + (dy^1)^2 + \dots. \tag{4.8}$$

The DLCQ proceeds as above but the identification is no longer in the spatial part x^9 of the light-cone coordinates, but rather in the additional circle y^1. With some additional adjustments the DLCQ of CSV [12] extends naturally to SHPW's as we will outline in the next section.

5. The Dirac Born-Infeld action of Matrix String Theory

For a solution to type IIA string theory the SHPW metric is,

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -2dz^+ dz^- + \sum_a m_a(m_a - 1)(z^a)^2 \frac{(dz^+)^2}{(z^+)^2} + d\vec{z}^2, \tag{5.1}$$

accompanied by the corresponding dilaton,

$$e^{2\phi} = (z^+)^p, \tag{5.2}$$

where p is related to the m_a via the equations of low energy type IIA string theory.

The series of transformations discussed in the previous section imply a corresponding series of transformations on the space-time metric. The simplest way to derive these transformations is to read them directly from the discussion of the previous chapter (a more detailed discussion can be found in [11]). One performs first a boost and scaling in type IIA string theory and then lifts the theory to M-theory with the 11th dimension having radius $\hat{R}_{11} = \varepsilon \ell_s g_s$. We then reduce back to ten-dimensions along the scaled circle that has radius $\hat{R}_1 = \hat{R}_s = \varepsilon^2 R$, and then finally carry out a T-duality on the x^{11} circle.

Following this procedure one finds that the paramters of the final IIB theory are related to the ℓ_s, g_s and $R_{11} = \ell_s g_s$ of the original IIA theory by (we will denote IIB quantities by a prime),

$$(\ell_s')^2 = \varepsilon(\ell_s)^2 \frac{R_{11}}{R}, \quad g_s' = \varepsilon \frac{R}{R_{11}} \tag{5.3}$$

where $\varepsilon = R_s/R$ and the scaled dual metric-dilaton background is given by,

$$(ds')^2 = \sqrt{g_{11}} e^{-\phi} \big(-2d\hat{x}^+ d\hat{x}^- + \frac{\ell_s^4}{R^2 g_{11}} (d\hat{x}^1)^2 + \dots \big) \equiv e^{\phi'} d\tilde{s}^2 \tag{5.4}$$

$$\phi' = -\phi + \frac{1}{2} \log g_{11} \tag{5.5}$$

We will now determine H_N for N Dstrings in the dual type IIB string theory. We consider fluctuations in the Dirac Born-Infeld action expanded around a classical solution for a D-string in type IIB string theory. The chosen solution corresponds to a simple centre of mass motion of the D-string and the resulting action is independent of the particular constants of the trajectory provided that it starts at $y^+ \to -\infty$ and goes to $y^+ = 0$.

We expand the Abelian Dirac Born-Infeld action around the simple classical solution with non-trivial motion only in (x^+, x^-, x^1),

$$x_c^+ = \tau, \quad x_c^1 = \sigma/\ell_s \tag{5.6}$$

and x_c^- determined from the constraint equation $\partial_\tau x_c^- = b^2 \tilde{g}_{11}/2a$. One finds for the fluctuations of the transverse coordinates $X^i(\tau, \sigma)$,

$$S = \frac{1}{2\pi\ell_s^2} \int d\tau d\sigma [\frac{1}{2} g_{ij}(\tau)(\partial_\tau X^i \partial_\tau X^j - \partial_\sigma X^i \partial_\sigma X^j) + 2\pi^2 \ell_s^4 g_s^2 e^{2\phi(\tau)} F_{\tau\sigma}^2] \tag{5.7}$$

and the Yang-Mills coupling is related to the original dilaton by

$$g_{YM} \sim \frac{1}{g_s \ell_s} e^{-\phi} \tag{5.8}$$

The ε scaling of the coordinates undoes the Penrose scaling of the transverse coordinates and as such the fluctuations are directly related to the coordinates of the original metric. Furthermore the energies $\delta\hat{E}$ of the above fluctuations are finite and equal to the string theory lightcone fluctuations. We need to compare the energies of massive open string modes and bulk closed string modes to see that they are large compared to the YM energies. In the end it turns out that, very nicely, the only ε dependence is in the IIB string length and string coupling, both of which go to zero as $\varepsilon \to 0$.

Decoupling of the massive open and closed string modes is thus demonstrated by the following

$$\lim_{\varepsilon \to 0} (\delta\hat{E}) \ell_s' = \lim_{\varepsilon \to 0} (\delta\hat{E})(\ell_s')(g_s')^{(1/4)} = 0 \tag{5.9}$$

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The DLCQ limit decouples the D-string from the open and closed string modes and so we have derived a non-perturbative description for a sector of string theory with fixed light-cone momentum in the vicinity of a singularity. For more than one D-string one needs to use non-Abelian actions leading to a description of N D-strings by the non-Abelian matrix string action

$$S = \frac{1}{2\pi\ell_s^2}\int d\tau d\sigma[\frac{1}{2}g_{ij}(\tau)(\mathscr{D}_\tau X^i \mathscr{D}_\tau X^j - \mathscr{D}_\sigma X^i \mathscr{D}_\sigma X^j) \tag{5.10}$$

$$+\ 2\pi^2\ell_s^4 g_s^2 e^{2\phi(\tau)} F_{\tau\sigma}^2 + \frac{1}{16\pi^2\ell_s^4 g_s^2} e^{-2\phi(\tau)}[X^i, X^j]^2] \tag{5.11}$$

where the $X^i(\sigma,\tau)$ are hermitian matrix valued fields.

The full details of the physics near the singularity depends in an important way on the behaviour of the string coupling and consequently on the Yang-Mills coupling near the singularity. For the current approach to singularities in string theory the most interesting situation is that in which the string coupling diverges implying that the corresponding Yang-Mills coupling goes to zero and we enter a non-Abelian phase of the Yang-Mills theory in the sense that the quartic term in the potential is not supressed near the singularity meaning that large commutators between coordinates are allowed and

$$[X^i, X^j] \neq 0. \tag{5.12}$$

The singularity of General Relativity is then replaced by a potentially non-singular desciption in terms of non-commuting space-time coordinates and weakly coupled non-abelian Yang-Mills theory.

6. Different Coordinates, Different Yang-Mills, Same space-time

Given that any SHPW will give rise to a non-trivial Yang-Mills theory, and that furthermore any SHPW has a unique representation in Brinkmann coordinates and many different representations in Rosen coordinates, one can find many "different" equivalent Yang-Mills theories, related by coordinate transformations, although on the face of it they do not obviously describe the same physics. Any reduction from Brinkmann coordinates will give you massive scalars coupled to Yang-Mills, while a reduction from any Rosen coordinates will instead lead to a theory with non-trivial time-dependent coupling constants describing massless scalars coupled to Yang-Mills theory. These apparently different theories are related by non-trivial field transformations inherited from the coordinate transformations that take you from one metric to the other.

Notice that in particular when the string coupling is large near the singularity the matrix string picture in Brinkmann coordinates indicates that a tachyonic mode becomes important. Recall that from the previous discussion, for strong coupling near the singularity the appropriate description of physics comes from weakly coupled Yang-Mills theory, and thus one needs to understand the meaning of these negative mass squared scalars to make further progress in the non-perturbative string theory of singularities.

Summary

The universal behaviour of singularities in the Penrose Limit gives rise to a plane wave of increased symmetry and with an upper bound on its strength subject to a simple condition on the stress-energy tensor. This allows us to carry out a more complete analysis of a singularity in string theory and this in principle allows us to evaluate and elaborate different proposals for the resolution of singularities although there still remains much work to be done. In particular we see that for a range of parameters for which the string coupling diverges at the singularity there is a potentially very interesting new phase for which the dynamics of the metric is replaced by a Yang-Mills theory in a non-commutative space. The full consequences of this result have not yet been completely elaborated.

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