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Non-holomorphic corrections to black hole partition functions

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We review our recent work on the construction of duality invariant black hole partition functions for four-dimensional BPS black holes in $N = 2$ supergravity theories.

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1. Introduction

In [1] it was proposed that the entropy of four-dimensional BPS black holes with $N = 2$ supersymmetry is related to a partition function based on a mixed ensemble defined in terms of magnetic charges and electrostatic potentials. Discarding non-holomorphic corrections this partition function equals the modulus square of the topological string partition function. On the basis of this relation it was concluded that the microscopic black hole degeneracies can be retrieved from the topological string partition function by an inverse Laplace transform.

Duality invariance, however, is not manifest in the proposal of [1]. An alternative starting point [2, 3] can be based on an ensemble of electric and magnetic charges, which is manifestly invariant under duality. From this set-up the previous formulation based on the mixed partition function can be reobtained in the semiclassical approximation, but, as it turns out, it is now accompanied by a non-trivial measure factor. Independently, a direct evaluation of the mixed partition function from specific microscopic degeneracy formulae also revealed the presence of a measure factor [4], and it was shown that for large charges these measure factors were in fact equal [3, 5].

Non-holomorphic terms are essential for duality invariance. In [6] a method was presented for incorporating them into black hole partition functions, suggesting that non-holomorphic deformations are possible in the context of special geometry. We briefly review our work in [6] and we refer to [6, 7] for a detailed presentation.

2. The BPS black hole free energy and the partition function

At the field-theoretic level it is known that the attractor equations that determine the values of the moduli at the black hole horizon [8, 9, 10], follow from a variational principle. This variational principle is described in terms of a so-called entropy function. There exists an entropy function for extremal black holes [11, 12], where the attractor mechanism is induced by the restricted space-time geometry of the horizon, and one for BPS black holes [3], where the attractor mechanism follows from supersymmetry enhancement at the horizon. For $N = 2$ supergravity the relation between these entropy functions has been clarified in [13]. To preserve the variational principle when non-holomorphic corrections are present, it follows that these corrections must enter into the BPS free energy in a well-defined way.

We consider charged black holes in the context of $N = 2$ supergravity in four space-time dimensions, which contains $n+1$ abelian vector gauge fields, labeled by indices $I,J = 0,1,\ldots,n$, so that black hole solutions can carry $2(n+1)$ possible electric and magnetic charges. The theory describes the supergravity fields and n vector multiplets (the extra index $I = 0$ accounts for the gauge field that belongs to the supergravity multiplet), and possibly a number of hypermultiplets which will only play an ancillary role. A partition sum over a canonical ensemble of corresponding BPS black hole microstates is defined as follows,

$$Z(\phi,\chi) = \sum_{\{p,q\}} d(p,q)\,\mathrm{e}^{\pi[q_I\phi^I - p^I\chi_I]}\,, \tag{2.1}$$

where $d(p,q)$ denotes the degeneracy of the black hole microstates with given magnetic and electric charges equal to p^I and q_I, respectively. This expression is consistent with electric/magnetic duality, provided that the electro- and magnetostatic potentials (ϕ^I, χ_I) transform as a symplectic vector, just as the charges (p^I, q_I), while the degeneracies $d(p,q)$ transform as functions of the charges under the duality. In case that the duality is realized as a symmetry, then the $d(p,q)$ should be invariant.

Viewing $Z(\phi,\chi)$ as an analytic function in ϕ^I and χ_I, the degeneracies $d(p,q)$ can be retrieved by an inverse Laplace transform,

$$d(p,q) \propto \int \mathrm{d}\phi^I \,\mathrm{d}\chi_I \, Z(\phi,\chi)\, \mathrm{e}^{\pi[-q_I\phi^I + p^I\chi_I]}\,, \tag{2.2}$$

where the integration contours run, for instance, over the intervals $(\phi - \mathrm{i}, \phi + \mathrm{i})$ and $(\chi - \mathrm{i}, \chi + \mathrm{i})$ (we are assuming an integer-valued charge lattice). Obviously, this makes sense as long as $Z(\phi,\chi)$ is formally periodic under shifts of ϕ and χ by multiples of 2i.

In [2, 3] it was proposed to identify the logarithm of $Z(\phi,\chi)$ with a free energy that equals twice the so-called Hesse potential. These expressions can also be written in terms of the usual complex variables Y^I and F_I, where $F_I = \partial F/\partial Y^I$. Then, the electro- and magnetostatic potentials are [14]

$$\phi^I = Y^I + \bar{Y}^I\,, \qquad \chi_I = F_I + \bar{F}_{\bar{I}} \tag{2.3}$$

and the resulting expression for $d(q,p)$ can be written as

$$d(p,q) \propto \int \mathrm{d}(Y+\bar{Y})^I \,\mathrm{d}(F+\bar{F})_I \,\mathrm{e}^{\pi\Sigma(Y,\bar{Y},p,q)} \propto \int \mathrm{d}Y^I \,\mathrm{d}\bar{Y}^I \,\Delta(Y,\bar{Y})\,\mathrm{e}^{\pi\Sigma(Y,\bar{Y},p,q)}\,, \tag{2.4}$$

where $\Delta(Y,\bar{Y})$ denotes the Jacobian associated with the change of integration variables $(\phi,\chi) \to (Y,\bar{Y})$, and Σ denotes the BPS entropy function

$$\Sigma(Y,\bar{Y},p,q) = \mathscr{F}(Y,\bar{Y}) - q_I(Y^I + \bar{Y}^I) + p^I(F_I + \bar{F}_I)\,. \tag{2.5}$$

Here p^I and q_I couple to the corresponding magneto- and electrostatic potentials (c.f. (2.3)) at the horizon in a way that is consistent with electric/magnetic duality. Furthermore, $\mathscr{F}(Y,\bar{Y})$ represents the free energy alluded to earlier. The black hole attractor equations follow from the variation of Σ with respect to the Y^I,

$$Y^I - \bar{Y}^I = \mathrm{i}p^I\,, \qquad F_I - \bar{F}_I = \mathrm{i}q_I\,. \tag{2.6}$$

These equations determine the values of the Y^I at the black hole horizon in terms of the charges. Thus, stationary points of Σ satisfy the attractor equations.

The expression (2.4) is duality invariant provided that Σ is duality invariant. Since Σ depends on the Weyl background field Υ (which takes the value $\Upsilon = -64$ at the horizon) through F, the duality invariance of Σ actually requires F to contain non-holomorphic terms. As shown in [6], these can be incorporated by requiring that in their presence, the variational principle for Σ still gives attractor equations that retain the form (2.6). This is possible provided the function F is taken to be of the form

$$F = F^{(0)}(Y) + 2\mathrm{i}\,\Omega(Y,\bar{Y},\Upsilon,\bar{\Upsilon})\,, \tag{2.7}$$

with Ω a real homogeneous function of second degree.

The decomposition (2.7) seems to take the form of a consistent non-holomorphic deformation of special geometry, as we now review.

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3. Non-holomorphic deformations of special geometry?

Here we consider some of the more conceptual issues related to the presence of non-holomorphic corrections. Let us consider electric/magnetic dualities on the periods (X^I, F_I), which take the form of $\mathrm{Sp}(2n)$ rotations. Here we do not assume that the F_I are holomorphic functions or sections. Hence we have holomorphic and anti-holomorphic coordinates X^I and $\bar X^{\bar I}$, while the F_I may depend on both X^I and $\bar X^I$. To avoid ambiguous notation we will use anti-holomorphic indices $\bar I$ wherever necessary.

Electric/magnetic dualities are defined by monodromy transformations of the periods, defined in the usual way,

$$
\begin{aligned}
X^I &\to \tilde X^I = U^I{}_J X^J + Z^{IJ} F_J \,, \\
F_I &\to \tilde F_I = V_I{}^J F_J + W_{IJ} X^J \,,
\end{aligned}
\tag{3.1}
$$

where U, V, Z and W are the $(n+1)\times(n+1)$ submatrices that constitute an element of $\mathrm{Sp}(2n+2,\mathbb{R})$. As a result the relation between the old and the new fields, X^I and $\tilde X^I$, will no longer define a holomorphic map, and we note,

$$
\frac{\partial \tilde X^I}{\partial X^J} \equiv \mathscr{S}^I{}_J = U^I{}_J + Z^{IK} F_{KJ} \,, \qquad \frac{\partial \tilde X^I}{\partial \bar X^J} = Z^{IK} F_{K\bar J} , \tag{3.2}
$$

where $F_{IJ} = \partial F_I / \partial X^J$ and $F_{I\bar J} = \partial F_I / \partial \bar X^J$. Let us consider the transformation behaviour of F_{IJ} induced by electric/magnetic duality (3.1). Straightforward use of the chain rule yields the relation,

$$
F_{IJ} \to \tilde F_{IJ} = (V_I{}^L \hat F_{LK} + W_{IK})\, [\hat{\mathscr{S}}^{-1}]^K{}_J \,, \tag{3.3}
$$

where

$$
\begin{aligned}
\hat F_{IJ} &= F_{IJ} - F_{I\bar K}\, \bar{\mathscr{Z}}^{\bar K\bar L}\, \bar F_{\bar L J} \,, \\
\hat{\mathscr{S}}^I{}_J &= U^I{}_J + Z^{IK} \hat F_{KJ} \,, \\
\mathscr{Z}^{IJ} &= [\mathscr{S}^{-1}]^I{}_K\, Z^{KJ} \,.
\end{aligned}
\tag{3.4}
$$

As was shown in [15], $\mathscr{Z}^{IJ}$ is a symmetric matrix by virtue of the fact that the duality matrix belongs to $\mathrm{Sp}(2n+2,\mathbb{R})$. For the same reason $[\hat{\mathscr{S}}^{-1}]^I{}_K Z^{KJ}$ is also symmetric in (I,J). Observe that $\mathscr{Z}^{IJ}$ satisfies the equation,

$$
\delta \mathscr{Z}^{IJ} = -\mathscr{Z}^{IK} \delta F_{KL} \mathscr{Z}^{LJ} \,. \tag{3.5}
$$

Let us now assume that F_{IJ} is symmetric in I and J. This symmetry implies that the F_I can be written as the holomorphic derivatives of some function $F(X,\bar X)$. It is of interest to determine whether this symmetry is preserved under duality. In general this is not the case. However, when we assume that

$$
F_{I\bar J} = \pm \bar F_{\bar J I} \,, \tag{3.6}
$$

then $\hat F_{IJ}$ will also be symmetric. In that case one can derive from (3.3) that $\tilde F_{IJ}$ must be symmetric as well, so that the $\tilde F_I$ can be expressed as the holomorphic derivatives of some function $\tilde F(\tilde X, \bar{\tilde X})$ with respect to $\tilde X^I$. This is a first indication that non-holomorphic deformations satisfying (3.6) can be consistent with the special geometry transformations of the periods. Henceforth we will assume that (3.6) holds. Observe that terms in F that depend exclusively on $\bar{X}^{\bar{I}}$ are not determined by the above arguments.

Let us further assume that the function F depends on some auxiliary real parameter η and consider partial derivatives with respect to it. A little calculation shows that $\partial_\eta F_I$ transforms in the following way,

$$\partial_\eta \tilde{F}_I = [\hat{S}^{-1}]^J{}_I \left[\partial_\eta F_J - F_{J\bar{K}}\, \bar{\mathscr{Z}}^{\bar{K}\bar{L}}\, \partial_\eta \bar{F}_{\bar{L}}\right] , \tag{3.7}$$

where the η-derivative in $\partial_\eta \tilde{F}_I(\tilde{X}, \bar{\tilde{X}}; \eta)$ is a partial derivative that does not act on the arguments $\tilde{X}^I$ and their complex conjugates, and likewise, in $\partial_\eta F_I(X, \bar{X}; \eta)$ the arguments X^I and their complex conjugates are kept fixed. Let us now assume that the function $F(X, \bar{X}; \eta)$ decomposes into a holomorphic function of X^I and a purely imaginary function that depends on X^I, its complex conjugates, and on the auxiliary parameter η,

$$F(X, \bar{X}; \eta) = F^{(0)}(X) + 2\mathrm{i}\,\Omega(X, \bar{X}; \eta) , \tag{3.8}$$

where Ω is real, just as in (2.7). For this class of functions we have the following identities,

$$F_{I\bar{J}} = -\bar{F}_{\bar{J}I}, , \qquad \partial_\eta F_{\bar{I}} = -\partial_\eta \bar{F}_{\bar{I}}, \tag{3.9}$$

so that we must adopt the minus sign in (3.6). With this result we can establish that

$$\partial_\eta \tilde{F}(\tilde{X}, \bar{\tilde{X}}; \eta) = \partial_\eta F(X, \bar{X}; \eta) , \tag{3.10}$$

up to terms that no longer depend on X^I and $\bar{X}^{\bar{I}}$. Ignoring such terms on the ground that they are not relevant for the vector multiplet Lagrangian, this implies that the first derivative of the function F with respect to some auxiliary parameter transforms as a function under electric/magnetic duality. Of course, it is crucial that we assumed the decomposition (3.8) so that η appears only in the non-holomorphic component Ω of F.

When the electric/magnetic duality defines a symmetry, then it follows that $\partial_\eta F$ must be invariant under this symmetry. The above arguments, when applied to the free energy for BPS black holes, imply that the free energy and hence also Σ are duality invariant [6].

We stress that the effective action encoded in a non-holomorphic function F is not fully known. Although the arguments presented above indicate that non-holomorphic deformations are possible within the context of special gometry, a lot of work remains to be done in order to establish the full consistency and the implications of this approach.

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