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Brief Notes on Conformal Symmetry and Anomaly

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We discuss a few interesting aspects of local conformal symmetry and its violations on classical and quantum level, paying a special attention to the cosmological applications. For the classical particles the massless limit corresponds to the vanishing trace of the Energy-Momentum tensor. Hence the field - particle correspondence tells us that the "correct" field Lagrangian should have the same property. For scalar field this means the conformal version of the nonminimal coupling. At the quantum level the conformal invariance is violated by anomaly and this leads to various consequencies, in particular to the modified equation of state for the photon gas.

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1. Introduction

The local conformal symmetry plays very important role in both classical and quantum theories of gravity. There are two main aspects in the use of this symmetry by theoreticians. At the classical level, conformal transformation is frequently applied for mapping different theories into each other, e.g. by mapping the popular $f(R)$ models into metric-scalar gravity theories. At quantum level, the most remarkable thing about the local conformal symmetry is that it is always violated by anomaly. Despite violation of some symmetry is intuitively seen as a weakness of the theory, this particular feature turns out to be extremely fruitful. The existence and simplicity of conformal anomaly enables one to derive an important part of the vacuum effective action and, therefore, obtain or better understand the origin of the most important applications of quantum field theory in curved space, such as Hawking radiation [1] and Starobinsky inflationary model [2].

In the recent years there were several extensive reviews of conformal symmetry (e.g. [3]) and conformal anomaly [4], including the ones of the present author [5, 6]. In this article we will try to present a little bit different view on the problem, despite we are using mainly the known and published material on the subject.

The paper is organized as follows. In section 2 we consider some approximate simple model for the ideal gas on massive relativistic particles [7] and use it to discuss the role of conformal symmetry in cosmology and, also, the soft violation of conformal symmetry by masses of the particles at the classical level. In section 3, which is very close in content to [5, 6], we briefly review conformal anomaly in the vacuum sector. In section 4 we consider the anomaly induced effective action in case of gravitational and electromagnetic background and obtain a local and covariant representation of such action. In section 5 we briefly discuss possible applications of the abovementioned result to cosmology and draw our conclusions.

2. Massive and massless particles on the cosmological background

It is well known that the ideal gas of massless particles and the ideal gas of massive particles have distinct equations of state. In the cosmological setting the gas of massless particles does not change the law of the expansion of the universe, which behaves like $a \sim t^{1/2}$ with or without massless matter content. Let us consider, following [7] an approximate description of an ideal gas and show how one can interpolate between the massless and massive cases. In fact, the assumption of equal kinetic energies can be viewed as something not artificial, because this property is indeed shared, with a good precision, by the completely degenerated Fermi gas, where all particles “live” at the Fermi surface. One can imagine that the model of, e.g. Dark Matter (DM) where the particle constituents of the gas have such property, would be rather exotic, but maybe not completely impossible.

Consider a single relativistic particle with the rest mass m. The expressions for energy and momentum for such particle have standard form

$$\varepsilon^2 - c^2\mathbf{p}^2 = m^2c^4\,, \quad \text{where} \quad \mathbf{p} = \frac{m\mathbf{v}}{\sqrt{1-v^2/c^2}}\,. \tag{2.1}$$

If the particle is confined in a vessel of a volume V, an elementary consideration shows that the time average of the pressure produced by the particle is

$$P = \frac{1}{3V}\cdot\frac{mv^2}{\sqrt{1-v^2/c^2}}\,. \tag{2.2}$$

For the gas of N such particles which ate distributed according to the Maxwell law, the equation of state was derived about a century ago by Jüttner [8], it involves the ratio of two modified Bessel functions and is a bit complicated. Let us simplify things and obtain an approximate equation of state, by assuming that all these particles has identical kinetic energies, ε. Using this assumption, we arrive at the following equation of state for the gas

$$P = \frac{\rho}{3}\cdot\left[1-\left(\frac{mc^2}{\varepsilon}\right)^2\right] = \frac{\rho}{3}\cdot\left[1-\frac{\rho_d^2}{\rho^2}\right]\,, \quad \text{where} \quad \rho = \frac{N\varepsilon}{V} \tag{2.3}$$

is the energy density and ρ_d^2 is the rest energy density, $\rho_d^2 = Nmc^2/V$.

Let us notice that $w = P/\rho$ tends to $1/3$ in the ultra-relativistic limit $\varepsilon \to \infty$ and to zero in the non-relativistic limit $\varepsilon \to mc^2$. Indeed this property is shared by Maxwell and (degenerate or not) Fermi-Dirac and Bose-Einstein relativistic distributions, and moreover, the relative numerical difference between (2.3) and the Maxwell case does not exceed 2.5% [7].

One of the important things here is that the massless case is characterized by $w = P/\rho = 1/3$. As a result the trace of the Energy-Momentum tensor vanish. This has very special significance in cosmology. Consider the first of the Friedmann equations

$$H^2 = \left(\frac{\dot{a}}{a}\right) = \frac{8\pi G}{3}\rho - \frac{k}{a^2}\,, \tag{2.4}$$

where $H = \dot{a}/a$ is a Hubble constant and ρ is the overall energy density. At this stage we are interested in the pure radiation content of the Universe, so ρ is the radiation density, $\rho = \rho_r$. It is obvious that the value of ρ_r defines the speed of the expansion of the Universe, parameterized by the Hubble parameter. From the other side, there is another Friedmann equation, which defines acceleration of the Universe expansion,

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p)\,. \tag{2.5}$$

And this equation has vanishing *r.h.s.* for the radiation content, when $w = 1/3$. As a result the acceleration is not sensitive to the presence or absence of the electromagnetic radiation, or to any other ideal gas of massless particles.

One can see this situation from different viewpoint. If we are interested in the evolution of the conformal factor of the Universe, $a(t)$, the most natural thing is to parametrize the metric as

$$g_{\mu\nu} = \bar{g}_{\mu\nu}\cdot e^{2\sigma}\,, \qquad \sigma = \ln a \tag{2.6}$$

where $\bar{g}_{\mu\nu}$ is the fiducial metric with fixed determinant. For the cosmological metric, using spherical coordinates, we have

$$\bar{g}_{\mu\nu} = \text{diag}\left(1, -\frac{1}{1-kr^2}, -r^2\sin^2\theta, -r^2\right).$$

Furthermore, $a = a(t)$ in the cosmological case, that means the conformal factor of the metric depends only on time but not on the space coordinates.

There is a useful relation which is valid for any functional $A[g_{\mu\nu}]$ of the metric and maybe other fields (which we do not show here for brevity),

$$-\frac{2}{\sqrt{-g}}\,g_{\mu\nu}\,\frac{\delta A[g_{\mu\nu}]}{\delta g_{\mu\nu}} = -\frac{1}{\sqrt{-g}}\,e^{-4\sigma}\,\frac{\delta A[\bar{g}_{\mu\nu}\,e^{2\sigma}]}{\delta\sigma}\Big|_{\bar{g}_{\mu\nu}\to g_{\mu\nu},\sigma\to 0}. \tag{2.7}$$

If we replace the action of some theory in curved space at the place of $A[g_{\mu\nu}]$, then the *l.h.s.* of the above relation is nothing else than the trace of the corresponding Energy-Momentum tensor. In order to remove the effect of other fields variables, it is sufficient to use the corresponding equations of motion.

It is obvious, at this point, that the vanishing trace of the Energy-Momentum tensor implies that the conformal factor of the metric decouples from the matter in a sense the dynamical equation for this factor does not depend on the presence or absence of the corresponding matter, exactly as we have already seen above. Of course, this does not mean that the first integral of this dynamical equations does depend on the presence of radiation, that is why we observe such dependence in eq. (2.4). In general, the situation when the coordinate-dependent conformal factor decouples from matter or from other components of the metric of the space-time, is called local conformal symmetry. As we shall see in what follows, the local conformal symmetry is never perfect. On the contrary, it is always violated in one or another way, on both classical and quantum level.

The two very important observations are in order. First, it is obvious that the violation of the local conformal symmetry may occur in two distinct ways. This symmetry is violated by any, even tiny mass of the particles, because (using our simple model) the masses make the value of w in the relation (2.3) different from $1/3$ and therefore make the trace of the Energy-Momentum tensor nonzero. On the other hand, the same relation is necessary violated by the interaction of absolutely massless particles between themselves and maybe even by the interaction of these particles with some external source.

We do not know whether the first way of violating conformal symmetry holds universally or it does not, this depends on the existence of the nonzero mass of the photon. It might happen that this mass is exactly zero, then the photon represents an example (probably unique) of the theory where the first version of violating local conformal symmetry does not take work. However, it is indeed impossible to avoid the second version of such violation. The point is that there is no single photon in the Universe which is *absolutely* free, and this is in fact requested by rigid local conformal symmetry of the massless particles gas. It is well known that the photons do not interact with each other only in the classical theory, while taking the quantum (QED) effects we meet certain interactions. Of course, this interaction is very week, but it is important that the ideal gas description is in fact an approximation and not an absolutely correct approach. Furthermore, the real photons never move an absolute vacuum, the last is always an approximation, despite it may be a very good-quality approximation. Finally, according to General Relativity photons always interact to gravity, that means there is a coupling between electromagnetic potential and the metric. At the classical level, however, the electromagnetic potential decouples from the conformal factor of the metric, because the classical action of electromagnetic field,

$$S_{em} \,=\, -\frac{1}{4}\int d^4x\sqrt{-g}\,F_{\mu\nu}F^{\mu\nu}\,. \tag{2.8}$$

possesses local conformal invariance. The last means this action does not change under simultaneous transformation of the metric and of the vector A_μ, namely

$$g_{\mu\nu} \to g'_{\mu\nu} = g_{\mu\nu}\,e^{2\sigma}\,, \quad A_\mu \to A'_\mu = A_\mu\,. \tag{2.9}$$

Let us note that the difference between conformal weight and dimension for the vector field is due to the vector field definition in curved space-time,

$$A_\mu = A_b\,e^b_\mu\,, \quad e^b_\mu\,e^a_\nu\,\eta_{ab} = g_{\mu\nu}\,, \quad e^b_\mu\,e^a_\nu\,g^{\mu\nu} = \eta^{ab}\,. \tag{2.10}$$

For other known kinds of the fields (scalars, fermions, higher derivative scalars and spinors) there is similar correspondence between the dimension and conformal weight, in these cases there is no need to perform consideration like in (2.10).

For scalars the action of the free field with the non-minimal coupling to external metric curvature has the form

$$S_{sc} = \frac{1}{2}\int d^4x\sqrt{-g}\,\Big(g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi + m^2\varphi^2 + \xi R\varphi^2\Big)\,. \tag{2.11}$$

The particular case of the theory which posses local conformal symmetry satisfies the constraints $m = 0$, $\xi = 1/6$. One can rewrite it in the form

$$S \,=\, -\frac{1}{2}\int d^4x\sqrt{-g}\,\phi\,\Delta_2\,\phi\,, \quad \text{where} \quad \Delta_2 \,=\, \Box - R/6\,. \tag{2.12}$$

Another conventional example of conformal field is the massless spinor,

$$S_{1/2} \,=\, \frac{i}{2}\int d^4x\sqrt{-g}\,\big\{\,\bar\psi\gamma^\mu\,\nabla_\mu\psi \,-\, \nabla_\mu\bar\psi\,\gamma^\mu\psi\,\big\}\,. \tag{2.13}$$

The conformal transformation rules for scalar and spinor are

$$\varphi \to \varphi' = \varphi\,e^{-\sigma/2}\,, \quad \psi \to \psi' = \psi\,e^{-3\sigma/2}\,, \quad \bar\psi \to \bar\psi' = \bar\psi\,e^{-3\sigma/2}\,.$$

The metric is always transformed like in (2.9). Other examples of conformal theories include, typically, higher derivatives and, also, have different transformation laws [5]. Here we shall mention only higher derivative scalar theory, with the action [9, 10]

$$S_4 \,=\, \int d^4x\sqrt{g}\,\chi\,\Delta_4\,\chi\,, \tag{2.14}$$

$$\text{where} \qquad \Delta_4 \,=\, \Box^2 + 2R^{\mu\nu}\nabla_\mu\nabla_\nu - \frac{2}{3}R\Box + \frac{1}{3}R_{;\mu}\,\nabla^\mu\,. \tag{2.15}$$

The conformal transformation law for this scalar is $\chi \to \chi' = \chi$. The importance of the model (2.14) is based on its use for deriving and integrating conformal anomaly. We shall discuss this point in the next section.

Let us come back to the photon case. At classical level electromagnetic field gains interaction with the conformal factor of the metric σ only through its couplings to matter fields, which are typically massive and hence strongly couple to σ. If the photons moves in vacuum, this interaction is very weak and we can in fact think that the photon is conformal. However, the situation changes dramatically if we take quantum effects onto account. We consider this part of the story in the next section.

3. Conformal anomaly in the semiclassical theory

Consider the derivation of conformal anomaly, including in the photon sector. This anomaly shows up because of the contribution of the electron loop (we consider the one loop approximation only) or due to the loop of other field coupled to the electromagnetic vector field. One can replace electromagnetic vector field by the Yang-Mills one, without essential changes in the local conformal anomaly. Therefore, without losing the generality we shall always speak about the electromagnetic field and photon.

The first step is to consistently formulate the action of the theory on classical curved background. In our case the background includes also the vector potential but we need also the pure gravitational sector. The standard criteria for the action of external metric field are (see, e.g. [11, 12]) as follows:
a) locality of the vacuum action,
b) renormalizability and
c) what one can call simplicity, e.g. we assume there are no $\left[m^{-1}\right]$ parameters or, in other words, we include the minimal set of terms which satisfy *a)* and *b)* conditions.

The action of vacuum which satisfies these necessary conditions has the form

$$S_{vac} = S_{EH} + S_{HD}\,, \tag{3.1}$$

where S_{EH} is the Einstein-Hilbert action with cosmological term and

$$S_{HD} = \int d^4x\sqrt{-g}\,\left\{a_1 C^2 + a_2 E + a_3 \Box R + a_4 R^2\right\}\,. \tag{3.2}$$

Here and below we use the following notations

$$E = R^2_{\mu\nu\alpha\beta} - 4R^2_{\alpha\beta} + R^2\,. \tag{3.3}$$

is the Gauss-Bonnet term (Euler density in $n = 4$). We avoid using the letter G to denote this quantity because it may be confused with the Newton constant.

In the case of conformal theory at the one-loop level it is sufficient to consider the simplified vacuum action

$$S_{conf} = \int d^4x\sqrt{-g}\left\{a_1 C^2 + a_2 E + a_3 \Box R - \frac{1}{4e^2}F_{\mu\nu}F^{\mu\nu}\right\}\,, \tag{3.4}$$

where we included the electromagnetic term at once. Let us emphasize that it is not *impossible* to add the Einstein-Hilbert action, cosmological constant or the $\int\sqrt{-g}R^2$ term here. The statement is that these terms are *not really necessary* at the one-loop level. In fact, beyond the one-loop approximation the $\int\sqrt{-g}R^2$ terms becomes also necessary, this means the conformal theory is not consistent beyond one loop [13]. In case of broken symmetry and generated masses of the matter fields (e.g. through the Coleman-Weinberg mechanism), other mentioned terms may also become necessary.

Consider the derivation of the conformal anomaly. This issue was addressed in many papers, in particular we have recently reviewed it in [5]. Hence there is no need to enter into full details again and we shall mainly take care of the electromagnetic sector which was not considered in [5].

We assume the theory includes the metric $g_{\mu\nu}$ and the vector potential A_μ as background fields and also some quantized matter fields Φ which do contribute to the effective action (including the one of the background) via the loop corrections. We denote, furthermore, k_Φ the conformal weight of the fields, in particular for the vector potential A_μ it is $k_A = 0$, as it was explained above.

As we have already discussed above, the Noether identity for the local conformal symmetry

$$\left[-2g_{\mu\nu}\frac{\delta}{\delta g_{\mu\nu}} + k_\Phi\,\Phi\,\frac{\delta}{\delta\Phi}\right] S(g_{\mu\nu},\Phi) = 0 \qquad (3.5)$$

produces $T^\mu_\mu = 0$ on the mass shell.

At quantum level S_{conf} has to be replaced by the effective action of vacuum $\Gamma_{vac}(g_{\mu\nu},A_\mu)$. At the one-loop level in the relevant sector we have (see [12] for the introduction and further references)

$$\bar{\Gamma}^{(1)}_{div} = \frac{1}{\varepsilon}\,\mu^{n-4}\int d^nx\sqrt{-g}\,\left\{\beta_1 C^2 + \beta_2 E + \beta_3\Box R + \frac{\beta_e}{2e^3}F_{\mu\nu}F^{\mu\nu}\right\}, \qquad (3.6)$$

where ε is the dimensional regularization parameter, $\varepsilon = n-4$, $\beta_{1,2,3}$ are beta-functions for the corresponding effective charges $a_{1,2,3}$ and β_e is the beta-function for the electromagnetic charge e in the minimal subtraction (MS) scheme or renormalization. Since $\beta_e \propto e^3$, the coefficient in the last term does not depend on e. For this reason we shall denote, in what follows, $\tilde{\beta} = \beta_e/e^3$.

Consider the high energy (or UV) limit, when the mass of the quantum field (e.g. of electron) is negligible. In this situation the (MS) scheme is reliable and we arrive at the following leading-log behavior of the electromagnetic sector:

$$\tilde{\beta}\,F^{\mu\nu}\,\ln\left(\frac{\Box}{\mu^2}\right)F_{\mu\nu}\,. \qquad (3.7)$$

Similar asymptotic behavior takes place also in the gravitational sector of the theory. For instance, the Weyl term has similar formfactor,

$$\beta_1\,C^{\mu\nu\alpha\beta}\,\ln\left(\frac{\Box}{\mu^2}\right)C_{\mu\nu\alpha\beta}\,. \qquad (3.8)$$

Of course, the d'Alembertian operator in both cases is the covariant one.

The presence of the term (3.8) has been confirmed by a direct covariant calculations [14] in the physical (momentum subtraction equivalent) renormalization scheme. The $\overline{\text{MS}}$-scheme based procedure described above can be successfully applied to derive the quantum corrections to the classical action of gravity and, e.g., scalar field [15, 12] (see also [16] for an alternative consideration).

In fact, the expressions (3.7) and (3.8) are sufficient to derive the corresponding parts of the conformal anomaly, even in case of a local conformal symmetry. For this end, let us apply the conformal parametrization of the metric (2.9) and the differential relation (2.7). Consider the case of (3.7) as an example, (3.8) is completely analogous. If we replace the parametrization (2.9) into (3.7), the only place where the σ field shows up is the $\Box$. This operator becomes

$$\Box = e^{-2\sigma}\,\left[\bar{\Box} + \mathscr{O}(\partial\sigma)\right]\,, \tag{3.9}$$

where the operator $\bar{\Box}$ is constructed with the metric $\bar{g}_{\mu\nu}$ and the explicit form of the terms $\mathscr{O}(\partial\sigma)$ is in fact irrelevant for us. When we apply (2.7), only the first term in the bracket (3.9) is important, because the other terms vanish after we set $\sigma \to 0$. Of course, the logarithmic dependence makes

$$\ln\frac{\Box}{\mu^2} = -2\sigma + \ln\frac{\bar{\Box} + \mathscr{O}(\partial\sigma)}{\mu^2}\,. \tag{3.10}$$

Finally, after applying (2.7) we arrive at

$$< T^\mu_\mu >_{em} = \tilde{\beta} F^2_{\mu\nu} \tag{3.11}$$

in the electromagnetic sector. The general expression involving the purely gravitational sector, have the form (see, e.g., [4] for the historical review and [5] for the technical introduction)

$$< T^\mu_\mu > = \left\{ \beta_1 C^2 + \beta_2 E + a'\Box R + \tilde{\beta} F^2_{\mu\nu} \right\}\,. \tag{3.12}$$

For the anomaly corresponding to the global conformal symmetry, $\sigma = \lambda = const$, we find $a\,' = \beta_3$ [18, 19, 12]. However, in the case of local conformal invariance there is an ambiguity in the parameter a' [11, 4]. The origin and mechanism for this ambiguity has been explained recently in [13].

The anomaly can be derived in many different ways, which mainly differ by the regularization choice [20, 21] (see, e.g. [11] for the list of results in some regularizations). We refer the reader to [5] for the details of the most simple, dimensional-regularization based approach to this calculation [20] and further references. Let us consider here how the same scheme can be applied to the electromagnetic term.

The renormalized one-loop effective action has the form

$$\Gamma_R = S + \bar{\Gamma} + \Delta S\,, \tag{3.13}$$

where $\bar{\Gamma} = \bar{\Gamma}_{div} + \bar{\Gamma}_{fin}$ is the naive quantum correction to the classical action S and ΔS is a counterterm. The classical action is $S = S_{matter} + S_{vac}$, where S_{vac} has the form (3.1). Indeed, only conformal invariant part of the vacuum action must be used in (3.13).

ΔS in (3.13) is an infinite local counterterm which is called to cancel the divergent part of the effective action $\bar{\Gamma}$. It turns out that ΔS is the only source of the noninvariance of the effective action, since naive (albeit divergent and nonlocal) contributions of quantum matter fields are conformal. The anomalous trace is therefore equal to

$$T =< T^\mu_\mu >= -\frac{2}{\sqrt{-g}}\, g_{\mu\nu}\, \frac{\delta\,\Gamma_R}{\delta\, g_{\mu\nu}}\Big|_{D=4} = -\frac{2}{\sqrt{-g}}\, g_{\mu\nu}\, \frac{\delta\,\Delta S}{\delta\, g_{\mu\nu}}\Big|_{D=4}\,. \tag{3.14}$$

The calculation of this expression can be done, in a most simple way, by changing the parametrization of the metric to (2.6). The counterterm ΔS is non-invariant because it is local and because it must be formulated in n spacetime dimensions. At that point we need a transformation laws for the divergent electromagnetic term, which has the form

$$g'_{\mu\nu} = g_{\mu\nu}\, e^{2\sigma(x)}\,, \qquad \int\sqrt{-g'}\, F'^2(n) = \int\sqrt{-g'}\, e^{(n-4)\sigma}\, F^2(n)\,, \tag{3.15}$$

where $F^2 = F^2_{\mu\nu} = F_{\mu\nu}F^{\mu\nu}$. In the simplest case of global conformal factor $\sigma = \lambda = const$ we immediately arrive at the expression (3.12) with $a' = \beta_3$. However in the local case $\sigma = \sigma(x)$ the situation is more complicated. We refer the reader to Refs. [13, 17, 5] for the recent and (in our opinion) complete discussion of this issue.

4. Anomaly-induced action of vacuum

One can use conformal anomaly to construct the equation for the finite part of the 1-loop correction to the effective action (we change notations here for the sake of convenience) of the background matric and electromagnetic potential,

$$\frac{2}{\sqrt{-g}}\, g_{\mu\nu}\, \frac{\delta\,\bar{\Gamma}_{ind}}{\delta g_{\mu\nu}} = \frac{1}{(4\pi)^2}\,\Big(aC^2 + bE + c\Box R + \tilde{\beta}F^2_{\mu\nu}\Big)\,. \tag{4.1}$$

The solution of this equation is straightforward [10] (see also generalizations for the theory with torsion [22] and with a scalar field [23]). The simplest possibility is to parametrize metric as in (3.15), separating the conformal factor $\sigma(x)$ and rewrite the eq. (4.1) using (2.7). The solution for the effective action is

$$\begin{aligned}\bar{\Gamma} \,&=\, S_c[\bar{g}_{\mu\nu}, A_\mu] + \frac{1}{(4\pi)^2}\int d^4x\sqrt{-\bar{g}}\,\{a\sigma\bar{C}^2 + \tilde{\beta}\sigma\bar{F}^2_{\mu\nu} + b\sigma(\bar{E} - \frac{2}{3}\bar{\Box}\bar{R}) \\ &+\, 2b\sigma\bar{\Delta}_4\sigma - \frac{1}{12}\,(c + \frac{2}{3}b)[\bar{R} - 6(\bar{\nabla}\sigma)^2 - (\bar{\Box}\sigma)]^2)\}\end{aligned} \tag{4.2}$$

where $S_c[\bar{g}_{\mu\nu}, A_\mu] = S_c[g_{\mu\nu}]$ is an unknown conformal invariant functional of the metric and A_μ, which serves as an integration constant for the eq. (4.1). All quantities wit bars are constructed using the metric $\bar{g}_{\mu\nu}$, in particular

$$\bar{F}^2_{\mu\nu} = \bar{F}_{\mu\nu}\bar{F}_{\alpha\beta}\bar{g}^{\mu\alpha}\bar{g}^{\beta\nu}\,.$$

The solution (4.2) has the merit of being simple, but an important disadvantage is that it is not covariant or, in other words, it is not expressed in terms of original metric $g_{\mu\nu}$. In order to obtain the non-local covariant solution and after represent it in the local form using auxiliary fields, we shall follow [10, 24]. The presence of the $\bar{F}^2_{\mu\nu}$ terms does not require any essential changes compared to the consideration presented in [5], in particular this term can be always taken together with the $\bar{C}^2$ one. So, we present just the final result in the non-local form, which is expressed in terms of the Green function for the operator (2.15),

$$\Delta_{4,x}\, G(x,y) = \delta(x,y)\,.$$

Using these formulas and (2.7) we find, for any $A(g_{\mu\nu}) = A(\bar{g}_{\mu\nu},\sigma)$, the relation

$$\frac{\delta}{\delta\sigma(y)}\int d^4x\sqrt{-g(x)}\,A\;(E-\frac{2}{3}\Box R)\Big|_{g_{\mu\nu}=\bar{g}_{\mu\nu}} = 4\sqrt{-\bar{g}}\bar{\Delta}_4 A = 4\sqrt{-g}\Delta_4 A\,. \tag{4.3}$$

In particular, we obtain

$$\Gamma_{induced} = \Gamma_a + \Gamma_b + \Gamma_c\,, \tag{4.4}$$

where

$$\Gamma_a = \int d^4x\sqrt{-g(x)}\int d^4y\sqrt{-g(y)}\,\frac{1}{4}\left(aC^2(x)+\tilde{\beta}F^2_{\mu\nu}\right)G(x,y)\left(E-\frac{2}{3}\Box R\right)_y\,, \tag{4.5}$$

$$\Gamma_b = \frac{b}{8}\int d^4x\sqrt{-g(x)}\int d^4y\sqrt{-g(y)}\,(E-\frac{2}{3}\Box R)_x\,G(x,y)\,(E-\frac{2}{3}\Box R)_y \tag{4.6}$$

and

$$\Gamma_c = -\frac{c+\frac{2}{3}b}{12(4\pi)^2}\int d^4x\sqrt{-g(x)}\,R^2(x)\,. \tag{4.7}$$

The nonlocal expressions for the anomaly induced effective action can be presented in a local form using two auxiliary scalar fields φ and ψ [24]. Let us give just a final result which has an extra electromagnetic terms compared to the one described in [5]

$$\Gamma = S_c[g_{\mu\nu}] - \frac{3c+2b}{36(4\pi)^2}\int d^4x\sqrt{-g(x)}\,R^2(x) + \int d^4x\sqrt{-g(x)}\,\Big\{\frac{1}{2}\,\varphi\Delta_4\varphi - \frac{1}{2}\,\psi\Delta_4\psi$$
$$+\;\varphi\left[\frac{\sqrt{b}}{8\pi}\,(E-\frac{2}{3}\,\Box R) - \frac{1}{8\pi\sqrt{b}}\left(aC^2+\tilde{\beta}F^2_{\mu\nu}\right)\right] + \frac{1}{8\pi\sqrt{b}}\,\psi\left(aC^2+\tilde{\beta}F^2_{\mu\nu}\right)\Big\}\,. \tag{4.8}$$

The local covariant form (4.8) is dynamically equivalent to the non-local covariant form. The complete definition of the Cauchy problem in the theory with the non-local action requires defining the boundary conditions for the Green functions $G(x,y)$, which shows up independently in the two terms (4.5) and (4.6). The same can be achieved, in the local version, by imposing the boundary conditions on the two auxiliary fields φ and ψ.

5. Discussions: some applications of anomaly-induced effective action

The applications of purely gravitational part of the conformal anomaly have been extensively discussed, e.g., in [11, 4, 5] and [6]. Let us give just a few observations about the possible importance of the electromagnetic terms.

It is well known that the conformal anomaly is relevant for understanding and to some extent deriving the Hawking radiation (see, e.g., [4, 25, 6] and references therein). In particular, the second auxiliary scalar introduced in [24] proved useful for applications. In particular, the vacuum states of the black hole (Boulware, Hartle-Hawking and Unruh) can be classified through the choice of initial conditions for the two auxiliary fields [25] (see also [26] for the treatment of the Reissner-Nordstrom case). Let us stress that this can not be accomplished by using only one field φ. Therefore the correspondence with other approaches to Hawking radiation indicates that our considerations about the correctness of introducing the second auxiliary scalar are correct. It would be interesting to look for the classical solution of the black-hole type on the metric-electromagnetic background and explore the corresponding quantum effects along the same line.

Another important application of the anomaly-induced effective action of gravity is the model of anomaly-induced inflation [27, 23], or Modified Starobinsky Model. In this case, the behaviour of conformal factor of the metric is not affected by the presence of the second auxiliary scalar. However, for investigating the evolution of gravitational waves specifying the initial data for both scalars is essential and the situation is close to the one in the black hole case. Let us note that our results show that the background (that means physical and not virtual) electromagnetic field does inevitably produce an extra terms in the conformal anomaly. If we remember the discussion of the section 2, this means that the quantum effects produce such interaction between electromagnetic field and metric, that the photons do not form an ideal massless gas anymore. This is indeed an important point, because the anomaly results in the slight change of the equation of state for the photon gas and this will of course modify the evolution of the Universe in the radiation dominated period. We are going to come back to this issue in a special publication.

In conclusion, the conformal invariant theories are not consistent at quantum level. In fact, the local conformal symmetry may be only approximate, however it is a very useful tool for calculating quantum corrections. Despite the conformal anomaly is very well studied subject, there are still many interesting problems to solve in this area.

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