PROCEEDINGS OF SCIENCE

α'-corrections and heterotic black holes

Predrag Dominis Prester*
Department of Physics, University of Rijeka
Omladinska 14, HR-51000, Croatia
E-mail: pprester@phy.uniri.hr

We review some recent results on α'-exact calculations of the entropy and near-horizon geometry of black holes in heterotic string theory.

Black Holes in General Relativity and String Theory
August 24-30 2008
Veli Lošinj, Croatia

*Speaker.

PoS(BHs, GR and Strings)033

Contents

1. Introduction

One of the big tests and challenges for every prospective theory of quantum gravity is to explain the nature of black hole entropy, which is in standard Einstein gravity given by Bekenstein-Hawking formula

$$S_{\rm bh} = S_{\rm BH} = \frac{A_h}{4G_N}\,, \tag{1.1}$$

where A_h denote proper area of the black hole horizon. Indeed, one of the most important successes of string theory is that it is indeed able to provide such statistical explanation by direct counting of microstates, at least for simpler class of *extremal* black holes composed of some number of strings, branes and other non-perturbative objects wrapped on some internal cycle of compactification manifold. There is now large set of extremal black holes for which microscopic statistical derivation of (1.1) was done. Restriction to extremal black holes is for technical reasons - direct microscopic calculations are presently tractable only in a regime of small coupling (where black holes are not present), so one has to use either supersymmetric non-renormalization properties (for BPS states) or attractor mechanism (for non-BPS states) to compare statistical entropy with (1.1). As we compare objects defined in different regimes of coupling (weakly coupled strings/branes with strongly coupled black holes) we need a way to compare the results. Supersymmetry (BPS case) and/or attractor mechanism is providing us with this, but both concepts are intrinsically connected with extremal black holes. Also, microscopic counting is frequently more tractable for BPS configurations. It is believed that by improving our knowledge of nonperturbative string theory we shall be able to extend these results to more physical nonextremal black holes.

In string theory classical gravity description is given by low energy/curvature effective actions which have (infinite number) of higher-derivative corrections, parameterized by the string length parameter α' (so called classical, or "stringy", corrections).[1] A consequence is that classical black hole entropy is not given by simple Bekenstein-Hawking area formula (1.1), but by more complicated Wald formula (discussed in section 4.1). This is giving us an opportunity to make precision tests of correspondence between black holes and weakly coupled strings/branes. The most interesting examples are the cases for which we know microscopic result *exactly* in α'. Of course, to perform full calculation on the gravity side, a priori a knowledge of the complete low energy effective action is required, and as is well-known only a limited knowledge beyond 6-derivative (α'^2) order is available at the moment.

We shall show in this review how one can go, on the gravity side, beyond perturbative calculation of black hole entropy and obtain α'-exact results for the extremal black hole entropies and near-horizon geometries. We concentrate on simple cases of extremal black holes in heterotic string theory with four charges in $D = 4$ dimensions, and those with three charges in $D = 5$ dimensions. The main part of the review is devoted to the calculation taking into account *the full heterotic effective action*, originally developed in [1]. We also review the calculations based on R^2-truncated effective actions (supersymmetric and Gauss-Bonnet), and, in the case of small black

[1] There are also quantum corrections parameterized by string coupling g_s, which we shall not discuss. Let us just mention that by changing a duality frame one sometimes exchanges classical and quantum corrections, so our discussion is not "purely classical").

holes in general D, Lovelock-type action. In addition, some results are presented which were not included in original papers.

2. Black holes - solutions of effective actions

2.1 Low energy effective action in heterotic string theory - lowest order

Low energy effective action (LEEA) of heterotic string theory in ten dimensions is a $\mathscr{N}=1$ supergravity theory (having 16 real supersymmetry generators). We shall be interested in purely bosonic solutions of this LEEA, so we can restrict ouselves to bosonic sector which contains the following fields: dilaton $\Phi^{(10)}$, metric tensor $G^{(10)}_{MN}$, 2-form gauge field $B^{(10)}_{MN}$ and ($SO(32)$ or $E_8 \times E_8$) Yang-Mills gauge field $A^{(10)}_M$. For simplicity, we shall additionaly restrict ourselves to backgrounds for which Yang-Mills field $A^{(10)}_M$ vanishes (later we shall explain how one can use T-duality to reconstruct solutions with non-vanishing Yang-Mills field). In this case, LEEA in the *lowest order* in string length parameter α' and string coupling constant g_s, is given by

$$\mathscr{A}^{(10)}_0 = \frac{1}{16\pi G^{(10)}_N} \int d^{10}x\, e^{-2\Phi^{(10)}} \left[R^{(10)} + 4(\partial\Phi^{(10)})^2 - \frac{1}{12} H^{(10)}_{MNP} H^{(10)MNP} \right] , \qquad (2.1)$$

where $M,N,\ldots = 0,1,\ldots,9$, $G^{(10)}_N$ is 10-dimensional Newton constant and $H^{(10)}_{MNP}$ is a 3-form gauge field strength corresponding to 2-form $B^{(10)}_{MN}$

$$H^{(10)}_{MNP} = \partial_M B^{(10)}_{NP} + \partial_N B^{(10)}_{PM} + \partial_P B^{(10)}_{MN} \,. \qquad (2.2)$$

As the fundamental objects of heterotic string theory in ten dimensions are 1-dimensional (elementary string or F1-brane) and 5-dimensional (NS5-brane), interesting effectively 0-dimensional objects (candidates for black holes) can be obtained through compactification, followed by wrapping of d-branes around d-cycles of compactification manifold. The simplest choice is to take the space to have the topology of $M_D \times T^{10-D}$, where M_D is D-dimensional Minkowski space and T^k is k-dimensional torus.

2.2 Small black holes

For a start, let us take $D=9$ where compactification manifold is circle S^1, parametrized with $0 < x_9 < 2\pi\sqrt{\alpha'}$. Following the rules of Kaluza-Klein compactification, we obtain that the massless fields in 9-dimensions are dilaton Φ, metric $G_{\mu\nu}$, 2-form $B_{\mu\nu}$, modulus (radius of S^1) T, and two $U(1)$ gauge fields A^i_μ, $i=1,2$, defined by

$$\begin{aligned}
&\Phi = \Phi^{(10)} - \frac{1}{2}\ln(G^{(10)}_{99}) , \qquad S = e^{-\Phi} , \qquad T = \sqrt{G^{(10)}_{99}} , \\
&G_{\mu\nu} = G^{(10)}_{\mu\nu} - (G^{(10)}_{99})^{-1} G^{(10)}_{9\mu} G^{(10)}_{9\nu} , \\
&A^{(1)}_\mu = \frac{1}{2}(G^{(10)}_{99})^{-1} G^{(10)}_{9\mu} , \qquad A^{(2)}_\mu = \frac{1}{2} B^{(10)}_{9\mu} , \\
&B_{\mu\nu} = B^{(10)}_{\mu\nu} - 2(A^{(1)}_\mu A^{(2)}_\nu - A^{(1)}_\nu A^{(2)}_\mu)
\end{aligned} \qquad (2.3)$$

PoS(BHs, GR and Strings)033

where $\mu, \nu, \ldots = 0, 1, \ldots, 8$. Using (2.3) in (2.1) one obtains the effective 9-dimensional action

$$\begin{aligned} \mathscr{A}_0 = & \frac{1}{16\pi G_N} \int d^9x \sqrt{-G}\, S \left[R + S^{-2} (\partial_\mu S)^2 - T^{-2} (\partial_\mu T)^2 \right. \\ & \left. - \frac{1}{12} (H_{\mu\nu\rho})^2 - T^2 (F^{(1)}_{\mu\nu})^2 - T^{-2} (F^{(2)}_{\mu\nu})^2 \right], \end{aligned} \tag{2.4}$$

where R is Ricci scalar computed from 9-dimensional metric $G_{\mu\nu}$, $G_N = G_N^{(10)}/(2\pi\sqrt{\alpha'})$ is the effective 9-dimensional Newton constant, and $F^{(a)}_{\mu\nu}$ and $H_{\mu\nu\rho}$ are 2-form and 3-form gauge field strengths defined by

$$\begin{aligned} F^{(a)}_{\mu\nu} &= \partial_\mu A^{(a)}_\nu - \partial_\nu A^{(a)}_\mu , \quad a = 1, 2, \\ H_{\mu\nu\rho} &= \left[\partial_\mu B_{\nu\rho} + 2 \left(A^{(1)}_\mu F^{(2)}_{\nu\rho} - A^{(2)}_\mu F^{(1)}_{\nu\rho} \right) \right] + \text{cyclic permutations of } \mu,\, \nu,\, \rho . \end{aligned} \tag{2.5}$$

It was shown in [2] that action (2.4) has the following asymptoticaly flat solutions

$$\begin{aligned} ds^2 &= G_{\mu\nu} dx^\mu dx^\nu = -g_s^{2\gamma} (F(\rho))^{-1} \rho^{2\beta} dt^2 + g_s^{2\gamma} d\vec{x}^2 , \\ S &= g_s^{-2} (F(\rho))^{1/2} \rho^{-\beta} , \qquad T = R \sqrt{\frac{\rho^\beta + 2|N|}{\rho^\beta + 2|W|}} , \\ A^{(1)}_t &= -\frac{g_s^\gamma}{R} \frac{N}{(\rho^\beta + 2|N|)} , \qquad A^{(2)}_t = -R g_s^\gamma \frac{W}{(\rho^\beta + 2|W|)} , \qquad B_{\mu\nu} = 0 = H_{\mu\nu\rho} , \end{aligned} \tag{2.6}$$

where we introduced

$$\begin{aligned} F(\rho) &\equiv (\rho^\beta + 2|W|)(\rho^\beta + 2|N|) , \\ \rho^2 &\equiv \vec{x}^2 , \qquad \beta \equiv D - 3 = 6 , \qquad \gamma \equiv \frac{2}{D-2} = \frac{2}{7} . \end{aligned} \tag{2.7}$$

We shall keep β and γ (without immediately fixing $D = 9$) because this solution generalizes to arbitrary D (corresponding to compactification on $T^{9-D} \times S^1$).

To fully understand the geometry of solution (2.6) and the physical meaning of parameters g_s, R, N, and W, it is better to pass to a canonical metric tensor $G_{E\,\mu\nu}$ (known as Einstein-frame metric) in which the action (2.4) has a more conventional form

$$\mathscr{A}_0 = \frac{1}{16\pi G_N} \int d^9x \sqrt{-G_E}\, (R_E + \ldots) ,$$

where R_E is Ricci scalar obtained from metric $G_{E\,\mu\nu}$. It is easy to show that relation between "string-frame" and "Einstein-frame" metrics is given by

$$G_{E\,\mu\nu} = S^\gamma G_{\mu\nu} . \tag{2.8}$$

In the limit $\rho \to \infty$ one gets

$$G_{E\,\mu\nu} \to \eta_{\mu\nu} , \qquad S \to g_s^{-2} , \qquad T \to R . \tag{2.9}$$

The first relation shows that metric is asymptotically flat, so it describes a point-like object. The second relation shows that parameter g_s is 9-dimensional string coupling. The last one shows that R

is the radius of compactification circle S^1 measured in string-frame metric, in units of string length parameter $\sqrt{\alpha'}$.

What about N and W? First of all, let us show that they are proportional to *electric* charges connected to gauge fields $A_\mu^{(1)}$ and $A_\mu^{(2)}$. This follows from the asymptotic behavior of corresponding field strengths for $\rho \to \infty$

$$F_{\rho t}^{(1)} = N \frac{4 g_s^\gamma}{R} \frac{\beta \rho^{\beta-1}}{(\rho^\beta + 2|N|)^2} \to N \frac{4 g_s^\gamma}{R} \frac{D-3}{\rho^{D-2}}, \qquad F_{\rho t}^{(2)} = W \frac{R g_s^\gamma}{4} \frac{\beta \rho^{\beta-1}}{(\rho^\beta + 2|W|)^2} \to W \frac{R g_s^\gamma}{4} \frac{D-3}{\rho^{D-2}}, \tag{2.10}$$

(where $D = 9$). Combined with the asymptotic flatness of the metric (2.9), this shows that electric charges are proportional to N and W.

To understand stringy meaning of charges N and W, let us find the mass of our configuration. It can be determined in the standard way from asymptotic behavior of canonical metric

$$G_{E\,tt} \simeq -1 + \frac{16\pi G_N}{(D-2)\Omega_{D-2}} \frac{M}{\rho^{D-3}}, \tag{2.11}$$

where Ω_{D-2} is the volume of the unit $(D-2)$ sphere

$$\Omega_{D-2} \equiv 2\pi^{(D-1)/2} \Big/ \Gamma\left(\frac{D-1}{2}\right)$$

Using (2.8) and (2.6) in (2.11), and puting $D = 9$, we get

$$M = \frac{(D-3)\,\Omega_{D-2}(|N| + |W|)}{8\pi G_N} = \frac{\pi^3}{4G_N}(|N| + |W|). \tag{2.12}$$

If we define rescaled charges n and w in the following way

$$n = \frac{\pi^3}{4G_N} g_s^{-\gamma} \sqrt{\alpha'} R N, \qquad w = \frac{\pi^3}{4G_N} g_s^{-\gamma} \sqrt{\alpha'} \frac{W}{R}, \tag{2.13}$$

the expression for mass (2.12) becomes

$$M = \frac{g_s^\gamma}{\sqrt{\alpha'}} \left(\frac{|n|}{R} + |w| R \right). \tag{2.14}$$

As we explain later in section 3, if we restrict n and w to integer values (2.14) is identical to the mass spectrum of a particular subspace of states of elementary string wound around the circle S^1, with n and w identified with momentum number and winding number, respectively.

Let us concentrate now on the behavior of the solution (2.6) in the region $\rho \to 0$. For this purpose, let us take the limit $\rho\alpha \ll |N|, |W|$. If we introduce rescaled coordinates

$$\vec{y} = g^\gamma \vec{x}, \qquad r = \sqrt{\vec{y}^2} = g^\gamma \rho, \qquad \tau = g^{-\beta\gamma} \frac{(D-3)\Omega_{D-2}}{4\pi} t / \sqrt{nw}, \tag{2.15}$$

the solution (2.6) in this limit becomes

$$\begin{aligned} ds^2 &= -\frac{r^{12}}{4} d\tau^2 + d\vec{y}^2, \\ S &= \frac{4}{\pi^3} \frac{\sqrt{|nw|}}{r^6}, \qquad T = \sqrt{\left|\frac{n}{w}\right|}, \\ F_{r\tau}^{(1)} &= \frac{3}{2n} r^5 \sqrt{|nw|}, \qquad F_{r\tau}^{(2)} = \frac{3}{2w} r^5 \sqrt{|nw|}, \end{aligned} \tag{2.16}$$

Comments on 2-charge small black hole solution:

1. From (2.15) we infer that solution (2.6) describes *black hole* with singular horizon coinciding with null singularity $\rho = 0$.

2. Solution (2.6) is *extremal*: it has vanishing Hawking temperature, and it can be obtained in a particular limit from multiparameter regular black hole solutions (with the same charge content) having regular horizons with non-vanishing temperature and entropy, and with mass satisfying

$$M \geq \frac{g_s^\gamma}{\sqrt{\alpha'}} \left(\frac{|n|}{R} + |w| R \right) , \tag{2.17}$$

   where unequality is saturated for solutions (2.6).

3. After including fermionic degrees of freedom (of heterotic string theory), it can be shown that solution (2.6) for $nw \geq 0$ is 1/2-BPS, i.e., it is annihilated by half of supersymmetry generators $(16/2 = 8)$. In fact, (2.17) relation already looks like BPS condition, but additional labor is needed to establish that only when n and w have the same sign solution is supersymmetric. This is a consequence of the fact that $N = 1$ SUSY in the heterotic string theory is purely right-handed.

4. Solution (2.6) was generalised to compactifications on general k-dimensional tori T^k, including non-vanishing 2-form $B_{\mu\nu}$ and Yang-Mills field. When all gauge fields are purely electrically charged, one obtains $(10-k)$-dimensional extremal black holes with the same properties as (2.6) (singular horizon, 1/2-BPS, etc.)

5. From (2.15) we see that near the horizon solution is completely determined by charges n and w and is independent of asymptotic values of moduli g_s and R. This is an example of the *attractor mechanism*.

We shall be interested primarily in the entropy of black holes. When the gravitational part of an action has the simple Einstein form, as is the case for action (2.1) (after passing to Einstein-frame metric), then the black hole entropy is given by Bekenstein-Hawking formula

$$S_{\mathrm{bh}} = \frac{A_h}{4G_N} , \tag{2.18}$$

where A_h is the proper area of the black hole horizon measured by Einstein-frame metric. For the black hole solution (2.6), using (2.8) we obviously get that A_h *vanishes*, so

$$S_{\mathrm{bh}} = 0 , \tag{2.19}$$

For the reasons we explain later, such solutions are called *small* black holes. As it has two charges, we shall refer to solution (2.6) as 2-charge small black hole.

2.3 Large black holes in $D = 4$ and $D = 5$

2.3.1 Compactifications on tori

As we mentioned at the end of section 2.1, if we compactify heterotic string theory on torus T^k, where $k \geq 5$, we can obtain more complicated point-like objects, e.g., by wrapping NS5-branes on such torii, possibly in addition to elementary strings. Now we shall show that one can indeed find black hole solutions of effective actions with the charge content resembling to such states obtained by wrapping strings and branes.

Let us analyze briefly massless field content in bosonic sector after such Kaluza-Klein compactification to $D = 10-k$ dimensions. First of all, non-abelian gauge group ($E_8 \times E_8$ or $SO(32)$) breaks down to abelian subgroup $U(1)^{16}$. This gives 16 U(1) gauge fields and $16 \times k$ scalar fields. From 10-dimensional metric tensor one obtains D-dimensional metric tensor, k U(1) gauge fields and $k(k+1)/2$ scalars. From 2-form B_{MN} one obtains 2-form $B_{\mu\nu}$, k U(1) gauge fields and $k(k-1)/2$ scalars. Altogether, we have dilaton Φ, metric $G_{\mu\nu}$, 2-form $B_{\mu\nu}$, $16+2k = 36-2D$ KK U(1) gauge fields, and $k(k+16) = (10-D)(26-D)$ scalar fields ($\mu, \nu = 0,\dots,D-1$). This looks extremely complicated, but help comes from the following two observations (details with references can be found in [3])

1. Compactification on torus leaves all supersymmetry generators (16 in heterotic theory) unbroken. If one is searching for BPS solutions (which are preserving some supersymmetry), one can use BPS conditions (which are first-order equations) to greatly simplify calculations.

2. The tree-level effective action has $O(10-D, 26-D)$ symmetry (loop-corrections break this symmetry to $O(10-D, 26-D, \mathbf{Z})$, which is a T-duality group of heterotic theory). This symmetry can be used to reduce the complexity of the problem. For example, one can use it to put majority of charges to zero. After solving for such generating solutions, one simply obtains all other solutions (with arbitrary charge assignments) by applying symmetry transformations on generating solutions. For example, in $D = 5$ ($k = 5$) case, generating solution has only 3 non-vanishing charges, and in $D = 4$ ($k = 6$) case, the number of relevant charges is 5.

2.3.2 3-charge large black holes in $D = 5$

We focus now on the case $k = 5$ ($D = 5$), and consider a compactification on $T^4 \times S^1$ (obviously a special case of T^5) in which T^4 is completely factorized and flat. In the language of the previous paragraph, it means that all KK gauge fields (and corresponding charges) which are obtained from T^4-indices are taken to be zero. We again take S^1 to be parametrized by $0 < x_9 < 2\pi\sqrt{\alpha'}$. Non-vanishing dynamical massless fields in 5-dimensions are then dilaton Φ, metric $G_{\mu\nu}$, 2-form $B_{\mu\nu}$, modulus (radius of S^1 in α'-units) T, and two $U(1)$ gauge fields A^i_μ, $i = 1,2$, defined by

$$\begin{aligned}
&\Phi = \Phi^{(10)} - \frac{1}{2}\ln(G^{(10)}_{99}), \qquad S = e^{-\Phi}, \qquad T = \sqrt{G^{(10)}_{99}}, \\
&G_{\mu\nu} = G^{(10)}_{\mu\nu} - (G^{(10)}_{99})^{-1} G^{(10)}_{9\mu} G^{(10)}_{9\nu}, \\
&A^{(1)}_\mu = \frac{1}{2}(G^{(10)}_{99})^{-1} G^{(10)}_{9\mu}, \qquad A^{(2)}_\mu = \frac{1}{2} B^{(10)}_{9\mu}, \\
&B_{\mu\nu} = B^{(10)}_{\mu\nu} - 2(A^{(1)}_\mu A^{(2)}_\nu - A^{(1)}_\nu A^{(2)}_\mu)
\end{aligned} \tag{2.20}$$

where now $\mu, \nu, \ldots = 0,1,\ldots,4$. Using (2.20) in (2.1) one obtains the effective 5-dimensional action

$$\mathscr{A}_0 = \frac{1}{16\pi G_N} \int d^5x \sqrt{-G}\, S \Big[R + S^{-2} (\partial_\mu S)^2 - T^{-2} (\partial_\mu T)^2$$

$$-\frac{1}{12}(H_{\mu\nu\rho})^2 - T^2\,(F^{(1)}_{\mu\nu})^2 - T^{-2}\,(F^{(2)}_{\mu\nu})^2\Bigg], \tag{2.21}$$

where R is Ricci scalar computed from 5-dimensional metric $G_{\mu\nu}$, and $G_N = G_N^{(10)}/(2\pi\sqrt{\alpha'}\mathscr{V})$ is the effective 5-dimensional Newton constant ($\mathscr{V}$ is volume of T^4).

$F^{(a)}_{\mu\nu}$ and $H_{\mu\nu\rho}$ are 2-form and 3-form gauge field strengths defined by

$$\begin{aligned}
F^{(a)}_{\mu\nu} &= \partial_\mu A^{(a)}_\nu - \partial_\nu A^{(a)}_\mu\,, \quad a=1,2\,,\\
H_{\mu\nu\rho} &= \left[\partial_\mu B_{\nu\rho} + 2\left(A^{(1)}_\mu F^{(2)}_{\nu\rho} - A^{(2)}_\mu F^{(1)}_{\nu\rho}\right)\right] + \text{cyclic permutations of } \mu,\, \nu,\, \rho\,.
\end{aligned} \tag{2.22}$$

As the torus T^4 trivially factorizes, and enters only in reparametrization of the Newton constant, the obtained expressions are the same as those from the begging of section 2.2, the only difference is that now theory is effectively 5-dimensional. We note that truncated action (2.21) has reduced supersymmetry (from $\mathscr{N}=4$ to $\mathscr{N}=2$), and has T-duality group generated by only one element, given by $T \to 1/T$, $A^{(1)}_\mu \leftrightarrow A^{(2)}_\mu$.

Extremal black hole solutions of the action (2.21) were reviewed in [3]. We are primarily interested here in the black hole entropy, which is given by Bekenstein-Hawking formula (2.18). From (2.18) it is obvious that one just needs near-horizon behavior of the solution, so we shall from now on concentrate on it (and avoid writing solutions in the whole space, which are usually cumbersome). Near-horizon behavior of the mentioned black hole solution is given by

$$\begin{aligned}
&ds^2 \equiv G_{\mu\nu}dx^\mu dx^\nu = \frac{\alpha'|N|}{4}\left(-r^2dt^2 + \frac{dr^2}{r^2}\right) + \alpha'|N|\,d\Omega_3^2\,,\\
&S = \frac{4G_N}{\alpha'^{3/2}\pi}\,\frac{\sqrt{|nw|}}{|N|}\,, \qquad T = \sqrt{\left|\frac{n}{w}\right|}\\
&F^{(1)}_{rt} = \frac{1}{n}\sqrt{\alpha'|nwN|}\,, \qquad F^{(2)}_{rt} = \frac{1}{w}\sqrt{\alpha'|nwN|}\,, \qquad H_{234} = 2\alpha' N\sqrt{g_3}\,,
\end{aligned} \tag{2.23}$$

and all other components of the fields are zero. Here $d\Omega_3^2$ denotes the metric on the unit 3-sphere S^3 (with coordinates x^i, $i=2,3,4$), and g_3 is a determinant of its metric tensor.

Solution (2.23) contains three parameters, n, w, and N, which play the roles of charges. First of all, electric charges n and w are appearing in the same way as in the small black hole solution (2.6), and so they have the same interpretation in stringy theory, i.e., they are momentum and winding number of the elementary string around S^1 circle.[2] As for N, let us first show that it is the *magnetic* charge corresponding to 2-form gauge field $B_{\mu\nu}$.

Generally, an object will be magnetically charged under a $(p-1)$-form gauge field (with p-form field strength A_p) according to formula (up to a constant factor, depending on conventions)

$$Q_m = \frac{1}{\Omega_p}\int_{S^p} A_p \tag{2.24}$$

where integration is over any p-sphere which encircles the object, and Ω_p is the volume of the unit p-sphere. From (2.24) directly follows that in D dimensions only $d = D - p - 2$ dimensional

[2] Indeed, for $N=0$ the solution with near-horizon behavior (2.23) becomes a 5-dimensional generalization of small black hole solution (2.6).

objects can be magnetically charged under $(p-1)$-form gauge field. So, in $D=5$ only point-like objects ($d=0$) can be magnetically charged under 2-form gauge field $B_{\mu\nu}$. Indeed, for solution (2.23) application of (2.24) gives

$$Q_m = \frac{1}{\Omega_3}\int_{S^3} H = 2\alpha' N \qquad (2.25)$$

As already mentioned, we expect that the 3-charge black hole solution with near-horizon behavior (2.23) describes configuration in which, beside the elementary string, we also have NS5-branes wound around $T^4 \times S^1$. We shall show that N is properly normalized to represent the number of NS5-branes.[3]

Let us make few comments on the near-horizon solution (2.23):

1. Geometry of solution is $\mathrm{AdS}_2 \times S^3$. The isometry group of such geometry is $SO(2,1)\times SO(4)$, and this symmetry group is respected by complete solution, not just metric. This is what is expected for near-horizon geometry of static spherically symmetric *extreme* black hole in $D=5$ (simple example is 5-dimensional Reisner-Nordstrom solution).

2. When all charges are finite and non-vanishing, black hole horizon, given by $r=0$, is *regular*, with all curvature invariants being finite and well defined on it. Such black holes are called *large* extremal black holes.

3. It is completely determined by charges n, w and N, and is independent of asymptotic values of moduli. This is another example of the attractor mechanism.

4. It is by itself exact solution of equations of motion.

5. For $nw>0$ solution is supersymmetric, but this time is only 1/4-BPS (it breaks 3/4 of supersymmetry generators) [7].

6. There is a supersymmetry *enhancement* near the horizon. Though solution in full space breaks 1/2 of $\mathcal{N}=2$ SUSY present in truncated action (2.21), the near-horizon solution is completely supersymmetric.

7. As mentioned above, for $N=0$ our 3-charge black hole reduces to 2-charge small black hole of section 2.2. Taking $N\to 0$ in the near-horizon solution (2.23) is not well defined, which is expected because near-horizon geometry of 2-charge small black holes, given in (2.16), is not $\mathrm{AdS}_2\times S^3$ but geometry with singular horizon.

From (2.23) (and using (2.8)) we can easily calculate Bekenstein-Hawking black hole entropy

$$S_{\mathrm{bh}} = \frac{A_h}{4G_N} = 2\pi\sqrt{|nwN|}\,. \qquad (2.26)$$

As expected for large black hole, the entropy is finite.

[3] This can be also understood from the viewpoint in which NS5-brane is dual to elementary string, and duality exchanges electric and magnetic charges (that is why elementary string is electrically charged on $B_{\mu\nu}$, and NS5-brane magnetically).

2.3.3 4-charge large black holes in $D = 4$

To obtain the last example of black holes that we study in this review, one needs to compactify one dimension more on the circle, which we denote $\widehat{S}^1$. This means that the compactification manifold is now $T^4 \times \widehat{S}^1 \times S^1$, and that after KK procedure the theory is effectively $D = 4$ dimensional. As explained in section 2.3.1, KK procedure leads to effective action with large number of fields, but which are organized according to $O(6,22)$ symmetry group of T-duality.

Again, we want to simplify things as much as possible, which means taking as many of the degrees of freedom to be zero or trivially constant. As before we take torus T^4 to be "trivial", and furthermore that 2-form $B_{\mu\nu} \equiv B^{(10)}_{\mu\nu}$ (where now $\mu,\nu = 0,1,2,3$) and KK scalars $G^{(10)}_{89}$ and $B^{(10)}_{89}$ all *vanish*. We again take S^1 to be parametrized by $0 < x_9 < 2\pi\sqrt{\alpha'}$, and $\widehat{S}^1$ by $0 < x_8 < 2\pi\sqrt{\alpha'}$. This leaves us with the following massless fields

$$\begin{aligned}
&\Phi = \Phi^{(10)} - \frac{1}{4}\ln(G^{(10)}_{99}) - \frac{1}{4}\ln(G^{(10)}_{88}) , \\
&S = e^{-2\Phi} , \qquad T = \sqrt{G^{(10)}_{99}} , \qquad \widehat{T} = \sqrt{G^{(10)}_{88}} , \\
&G_{\mu\nu} = G^{(10)}_{\mu\nu} - (G^{(10)}_{99})^{-1} G^{(10)}_{9\mu} G^{(10)}_{9\nu} - (G^{(10)}_{88})^{-1} G^{(10)}_{8\mu} G^{(10)}_{8\nu} , \\
&A^{(1)}_\mu = \frac{1}{2}(G^{(10)}_{99})^{-1} G^{(10)}_{9\mu} , \qquad A^{(2)}_\mu = \frac{1}{2}(G^{(10)}_{88})^{-1} G^{(10)}_{8\mu} , \\
&A^{(3)}_\mu = \frac{1}{2}B^{(10)}_{9\mu} , \qquad A^{(4)}_\mu = \frac{1}{2}B^{(10)}_{8\mu} ,
\end{aligned} \tag{2.27}$$

Putting (2.27) into (2.1) we get the following $D = 4$ effective action

$$\begin{aligned}
\mathscr{A}_0 = \frac{1}{16\pi G_N} \int d^4x \sqrt{-G}\, S \Big[& R + S^{-2}(\partial_\mu S)^2 - T^{-2}(\partial_\mu T)^2 - \widehat{T}^{-2}(\partial_\mu \widehat{T})^2 \\
& - T^2 (F^{(1)}_{\mu\nu})^2 - \widehat{T}^2 (F^{(2)}_{\mu\nu})^2 - T^{-2}(F^{(3)}_{\mu\nu})^2 - \widehat{T}^{-2}(F^{(4)}_{\mu\nu})^2 \Big] ,
\end{aligned} \tag{2.28}$$

where R is Ricci scalar computed from 4-dimensional metric $G_{\mu\nu}$, $G_N = G^{(10)}_N/(4\pi^2\alpha'\mathscr{V})$ is the effective 4-dimensional Newton constant ($\mathscr{V}$ is the volume of T^4), and $F^{(i)}_{\mu\nu}$ are 2-form strengths of gauge fields $A^{(i)}_\mu$. Again, truncation of the theory leads to the reduction of the supersymmetry (now $\mathscr{N} = 2$ instead of $\mathscr{N} = 4$) and T-duality (now reduced to two generators, $T \to 1/T$ and $\widehat{T} \to 1/\widehat{T}$).

In $D = 4$ dimensions 1-form gauge field is Hodge self-dual, which has a consequence that point-like configurations (like black holes) can be electrically and magnetically charged on the same 1-form gauge field. As in the truncated action (2.28) there are 4 1-form gauge fields, we could have altogether 8 charges, 4 electric denoted as $\{Q^{(i)}\} = \{n, \widehat{n}, w, \widehat{w}\}$, and 4 magnetic denoted as $\{P^{(i)}\} = \{N, \widehat{N}, W, \widehat{W}\}$. Correspondingly, we could have 8-charge black hole solutions. Again, to have things as simple as possible (but the line with an idea to obtain large black holes), we put half of this 8 charges to zero. It appears that the choice $\widehat{n} = \widehat{w} = N = W = 0$ will do it.

Extremal black hole solutions of the action (2.28) with such charge content were constructed in [43]. Again, we are interested in the near-horizon behaviour, so we present here just $r \to 0$ limit of the solution

$$ds^2 \equiv G_{\mu\nu}dx^\mu dx^\nu = \frac{\alpha'}{4}|\widehat{N}\widehat{W}| \left(-r^2dt^2 + \frac{dr^2}{r^2} \right) + \frac{\alpha'}{4}|\widehat{N}\widehat{W}| \left(d\theta^2 + \sin^2\theta\, d\phi^2 \right) ,$$

$$S = \frac{8G_N}{\alpha'}\sqrt{\left|\frac{nw}{\widehat{N}\widehat{W}}\right|}, \qquad T = \sqrt{\left|\frac{n}{w}\right|}, \qquad \widehat{T} = \sqrt{\left|\frac{\widehat{W}}{\widehat{N}}\right|}, \tag{2.29}$$
$$F^{(1)}_{rt} = \frac{\sqrt{\alpha'}}{4n}\sqrt{|nw\widehat{N}\widehat{W}|}, \quad F^{(3)}_{rt} = \frac{\sqrt{\alpha'}}{4w}\sqrt{|nw\widehat{N}\widehat{W}|}, \quad F^{(2)}_{\theta\phi} = \frac{\sqrt{\alpha'}}{4}\widehat{N}\sin\theta, \quad F^{(4)}_{\theta\phi} = \frac{\sqrt{\alpha'}}{4}\widehat{W}\sin\theta$$

and all other components of the fields are zero.

What is the meaning of the numbers n, w, $\widehat{N}$ and $\widehat{W}$? Obviously, n, w are electric charges, and $\widehat{N}$, $\widehat{W}$ are magnetic charges (for this just apply (2.24)). What about their stringy interpretation? For n and w everything is the same as in 2-charge and 3-charge solutions we studied before - they are momentum and winding numbers of elementary string wound around circle S^1. Now, from (2.27) follows that $\widehat{N}$ and $\widehat{W}$ are connected to the other circle $\widehat{S}^1$, and they are charges of the so called Kaluza-Klein monopole and H-monopole, respectively. If we follow precise definition for $A^{(4)}_\mu$ in (2.27), we see that $\widehat{N}$ enters in similar way as the magnetic charge N from section 2.3.2, which is suggesting that $\widehat{W}$ should be proportional to number of NS5-branes wrapped around $T^4 \times S^1$.

All in all, it appears that solution (2.29) describes near horizon geometry of an configuration in heterotic string theory consisting of elementary string wound w times around S^1, $\widehat{N}$ KK-monopoles wrapped around $T^4 \times S^1$ (and with "the core" on circle $\widehat{S}^1$), $\widehat{W}$ NS5-branes wrapped around $T^4 \times S^1$, and on top of it there is a momentum on S^1 with momentum number n. In appendix A of [42] one can find a proof that n, w, $\widehat{N}$ and $\widehat{W}$, as we defined them, are indeed properly normalized, i.e., only as integer numbers they have a meaning in string theory interpretation.

Let us make few comments on the near-horizon solution (2.29):

1. Geometry of solution is $\mathrm{AdS}_2 \times S^2$, and the whole solution respects symmetry on isometry group $SO(2,1) \times SO(3)$. This is what is expected for near-horizon geometry of static spherically symmetric *extreme* black hole in $D = 4$.

2. When all charges are finite and non-vanishing, black hole horizon, given by $r = 0$, is *regular*, with all curvature invariants being finite and well defined on it, so this is another example of large extremal black hole.

3. It is completely determined by charges n, w, $\widehat{N}$ and $\widehat{W}$, and is independent of asymptotic values of moduli. This is yet another example of the attractor mechanism.

4. It is by itself exact solution of equations of motion.

5. There are 3 types of solutions: (1) $nw > 0$ and $\widehat{N}\widehat{W} > 0$ solutions are supersymmetric 1/4-BPS (they break 3/4 of the supersymmetry generators) [7], (2) $nw < 0$ and $\widehat{N}\widehat{W} < 0$ solutions are non-BPS, (3) $nw\widehat{N}\widehat{W} < 0$ solutions are non-BPS. In fact, they are examples of 3 possible types of black hole solutions in general classification in $\mathcal{N} = 4$ SUGRA [6].

6. For $\widehat{N} = \widehat{W} = 0$ our 4-charge black hole reduces to 2-charge small black hole of section 2.2. Taking $\widehat{N} \to 0$ and/or $\widehat{W} \to 0$ in the near-horizon solution (2.29) is not well defined, which is expected because near-horizon geometry of 2-charge small black holes, given in (2.16), is not $\mathrm{AdS}_2 \times S^2$ but geometry with singular horizon.

From (2.29) (and using (2.8)) we can easily calculate Bekenstein-Hawking black hole entropy

$$S_{\rm bh} = \frac{A_h}{4G_N} = 2\pi\sqrt{|nw\widehat{N}\widehat{W}|}\,. \tag{2.30}$$

As expected for large black hole, the entropy is finite.

2.3.4 Stringy (α') and quantum (g_s) corrections

Full low energy effective action of heterotic string theory is much more complicated then (2.1). It has the general form

$$\mathscr{A} = \mathscr{A}_0 + \text{ higher derivative terms + string loop corrections} \tag{2.31}$$

String loop corrections, parametrized by string coupling constant g_s, are coming from quantum corrections and generally have perturbative and nonperturbative contributions (a thing well-known already from ordinary QFT). The higher derivative terms are coming from finite size of strings and are parametrized by the (square of) string length parameter α'.

As quantum loop-corrections are much more subtle to deal with (quantum effective actions are either non-local or not manifestly symmetric on dualities), we shall restrict ourselves to classical tree-level analyses. The corresponding effective action is *perturbative* in α'

$$\mathscr{A}_{\rm tree} = \sum_{n=0}^{\infty} \alpha'^n \mathscr{A}_n = \sum_{n=0}^{\infty} \alpha'^n \int dx^D \sqrt{-G}\,\mathscr{L}_n \tag{2.32}$$

and is known incompletely. Only lowest-order and first-correction ($\mathscr{A}_0$ and $\mathscr{A}_1$) are known fully. As string theory is not expected to be equivalent to any QFT, an expansion in (2.32) is believed to be infinite. It is obvious from dimension od α' that $\mathscr{A}_n$ is composed of $2(n+1)$-derivative terms. As the effective theory contains gravity, $\mathscr{A}_n$ can contain powers of Riemann tensor up to $(n+1)$-order and that is why the whole $\mathscr{A}_n$ is sometimes called R^{n+1} part of the action. There is a field redefinition scheme in which dependence of the tree-level Lagrangian on the dilaton field $S(x)$ is of the form

$$\mathscr{L}_{\rm tree} = S\,\mathscr{K}\left(\partial^m S/S\right), \qquad m = 1,2,\ldots. \tag{2.33}$$

This is a manifestation to the fact that effective string coupling is with expectation value of dilaton field by a relation which is in our conventions given by $g_{\rm eff}^2 \propto 1/S$.

Let us now go back to the lowest-order near-horizon solutions (obtained from action $\mathscr{A}_0$) that we presented in previous sections, and analyse what conditions should be applied to the parameters so that perturbative expansions are well-defined. We take first the regular large black hole solutions, using 5-dimensional 3-charge black case as an example. Let us start with loop-corrections which are parameterized by string coupling constant. From solution (2.23) we copy

$$S = \frac{4G_N}{\alpha'^{3/2}\pi}\,\frac{\sqrt{|nw|}}{|N|}\,. \tag{2.34}$$

It is obvious that if we are in the regime in which $|nw| \gg N^2$, then $S \gg 1$ and so $g_{\rm eff} \ll 1$, which means that we can ignore loop-corrections. As we do not want to cope with loop-corrections, we shall assume that we are in such asymptotic regime. As for α'-corrections, note that all 2-derivative scalar monomials which appear in $D=5$ effective action (2.21) (i.e., Ricci scalar R, $T^2(F^{(1)})^2$, $T^{-2}(F^{(2)})^2$ and H^2) when evaluated on the solution (2.23) are proportional to $1/(\alpha' N)$. It is easy to realize that all monomials which contain $2k$ derivatives will be proportional to $1/(\alpha' N)^k$. This means that the terms in the expansion (2.32) will behave like

$$\alpha'^n \mathscr{K}_n \propto \frac{1}{\alpha' N} \frac{1}{N^n} \tag{2.35}$$

We see that expansion in α' is effectively expansion in $1/N$. So, if $N \gg 1$ the α'-corrections will be small and we expect to have well-behaved perturbative expansion. The same analyses can be repeated for the case of 4-dimensional 4-charge large black holes with near-horizon solution given by (2.23), leading to the similar conclusions, with the only difference that the role of N is now played by the product $\widehat{N}\widehat{W}$.

In the case of *small* black hole solutions things are much different. Let us take 2-charge black hole solution reviewed in section 2.2. We saw that this solution has singular horizon, which is obvious from near-horizon behavior (2.16). We see that dilaton is singular on the horizon/singularity $r=0$, which means that $g_{\mathrm{eff}}^2 \propto 1/S = 0$. So, if we are hoping that higher-oreder corrections can regularize the solution, it is obvious that loop-corrections cannot do this because they vanish on the horizon. What about stringy α'-corrections? From near-horizon solution follows that n-th order terms will behave as $\mathscr{K}_n \propto r^{-14(n+1)}$ (in $D=9$), which shows that higher-order α'-terms in the action are more and more singular when evaluated on lowest-order solution. So, α'-corrections are important. Let us assume that they can regulate the solution. As we are dealing with extremal black holes, we expect to obtain $\mathrm{AdS}_2 \times S^{D-2}$ near-horizon geometry, with radii of the order of string length, i.e., $\ell_{A,S}^2 \sim \alpha'$ (indeed, it was shown on explicit examples that inclusion of general R^2-corrections is leading to such behavior [60, 59, 61, 58]). The Ricci scalar is $R \sim 1/\alpha'$. Repeating the above analyses of behavior of higher-order terms in the tree-level Lagrangian we obtain

$$\alpha'^n \mathscr{K}_n \propto \frac{1}{\alpha'}\,, \tag{2.36}$$

where we again assumed that $|n/w| \sim 1$. Though the solution is regular, we see from (2.36) that for small black holes α'-expansion is *not* well-defined as perturbative expansion, and one should find a way to somehow "sum" the complete α'-dependence. It is easy to understand the reason for this - black holes with horizon radius of the order of string length are intrinsically stringy objects, and for such objects we do not expect that low energy (or low curvature) expansion is meaningful.

We shall see in the next section that in some cases (including black holes we analyze here) it is possible to obtain statistical entropies by counting of microstates in string theory *exactly* in α'. If we could calculate α'-corrections to black hole entropies from the gravity side (i.e., by using LEEA), this would be strong test for the validity of such stringy desciption of black holes. On the other hand, if we beleive in such description, we could use the equality of entropies to get some new information on structure of higher-order terms in effective actions. We shall show here how both of this ideas can be succesfully applied on our examples of extremal black holes in heterotic string theory.

Now when we have understanding of all type of corrections in the realm of black holes that we study here, we can fix the values of parameters. We shall use the convention in which $\alpha' = 16$

and $G_N = 2$ throughout the review, with the exception of section 6 in which we take $\alpha' = 1$ and $G_N = \pi/4$, a convention frequently used in the literature for theories in $D = 5$.

3. Stringy description

3.1 Microstate counting

We want to find configurations in heterotic string theory which are in the supergravity limit described by black hole solutions that we analysed in section 2. The simplest case is 2-charge solution, for which we assumed to represent just the elementary string living in $M_9 \times S^1$ spacetime, which is wound w times around circle S^1 and has momentum number n. The simplicity of this case is that these states are purely perturbative, i.e., they exist in the spectrum of a free string.

Let us now analyze these states in the limit of free string with string coupling $g_s \to 0$ such that geometry can be considered flat, i.e., $G^{(10)}_{MN} = \eta_{MN}$. We parametrize circle S^1 with $0 \le x_9 < 2\pi\sqrt{\alpha'}R$. Perturbative string states are characterized by:

- Momentum 9-vector p^μ in the uncompactified directions (M_9 Minkowski space).
- Right-moving and left-moving momenta in the compact direction (S^1) given by

$$p_R = \frac{1}{\sqrt{\alpha'}}\left(\frac{n}{R} + wR\right), \qquad p_L = \frac{1}{\sqrt{\alpha'}}\left(\frac{n}{R} - wR\right). \tag{3.1}$$

- Excitations described by independent right- and left-moving oscillators, of which we will need just total level numbers N_R and N_L (measured here from physical vacuum).

Physical states satisfy the following mass-shell conditions

$$M^2 = p_R^2 + 4N_R/\alpha' \tag{3.2}$$
$$= p_L^2 + 4N_L/\alpha' \tag{3.3}$$

where M is the mass connected to uncompactified dimensions, defined by $M^2 = -p_\mu p^\mu$.

We saw in section 2.2 that for some choices of signatures of charges ($nw > 0$) black holes are 1/2 BPS states, so let us locate such states in the string spectrum. Heterotic string theory has $\mathcal{N} = 1$ supersymmetry, which contains 16 generators in 10 dimensions. It can be shown that states which satisfy the BPS condition

$$M = |p_R| = \frac{1}{\sqrt{\alpha'}}\left|\frac{n}{R} + wR\right| \tag{3.4}$$

are $1/2$-BPS states preserving half of the supersymmetries ($8 = 16/2$). We see that (3.4) is for $nw > 0$ exactly equal to mass of BPS small black holes given in (2.14), which confirms that we are on the right track.[4]

[4] The factor $g_s^{\gamma/2}$ present in (2.14) is because in section 2.2 we defined mass (energy) by using canonical (Einstein-frame) metric, while here we are using string frame metric. From (2.8) and (2.9) follows that they asymptotically they differ by g_s^γ, which gives to the above difference in mass scale.

Putting (3.4) in the condition (3.2) we obtain $N_R = 0$, i.e., right-moving sector is unexcited for these states. Equating now (3.2) and (3.3), and using (3.1), we can write the condition obtain the condition (3.4) equivalently as

$$N_L = \alpha'(p_R^2 - p_L^2)/4 = nw\,. \tag{3.5}$$

As by definition $N_L \le 0$ it directly follows $nw \le 0$, as expect from supergravity analysis. All states with fixed n, w and $N_L = nw$ are equal candidates to represent (in the free string regime) the BPS 2-charge small black hole with charges n and w. Let us calculate the number of such states. Closed form expression for general n and w is not known, but we saw in section 2.3.4 that we should be interested in the regime $nw \gg 1$ where classical black hole solutions can be reliable (quantum corrections are negligible). It this regime it is quite easy to get asymptotic expression for number of states

$$\Gamma = e^{4\pi\sqrt{N_L}}\,, \qquad N_L \gg 1\,, \tag{3.6}$$

which is obviously a huge number. We can assign the statistical entropy to this ensemble of states using standard microcanonical definition

$$S_{\mathrm{stat}}^{(\mathrm{BPS})} \equiv \ln\Gamma = 4\pi\sqrt{nw}\,, \qquad nw \gg 1\,. \tag{3.7}$$

It is important to emphasize is that though the result (3.7) is asymptotic in nw, it is *exact* in α'.

What about the cases when $nw < 0$ for which we found non-BPS black hole solutions? Though we cannot use supersymmetry here[5], we can by analogy define ensemble of states with fixed n and w, which are unexcited now in left-moving sector, i.e., with $N_L = 0$. Putting this in (3.3) we obtain for the mass

$$M = |p_L| = \frac{1}{\sqrt{\alpha'}}\left|\frac{n}{R} - wR\right|\,, \tag{3.8}$$

which again agrees with black hole mass formula (2.14) for $nw < 0$. So we are on the right track. Putting (3.8) in (3.2) give us now $N_R = -nw$, which forces $nw < 0$. We obtained ensemble of states defined by fixing n, w, and $N_R = -nw$. The asymptotic formula for number of such states is again easily calculated and the result is

$$\Gamma = e^{2\sqrt{2}\pi\sqrt{N_R}}\,, \qquad N_R \gg 1\,, \tag{3.9}$$

which gives statistical entropy

$$S_{\mathrm{stat}}^{(\mathrm{non-BPS})} \equiv \ln\Gamma = 2\sqrt{2}\pi\sqrt{|nw|}\,, \qquad -nw \gg 1\,. \tag{3.10}$$

Now we pass to microscopic (stringy) description for large black holes. As they generally contain non-perturbative objects (like, e.g., $NS5-branes$ and KK monopoles) the corresponding microstates in string theory are also not perturbative. This drastically complicates calculation of statistical entropy by direct counting of string microstates, and only in some special cases closed

[5] Note that this sector, in which only right-movers are excited, is basically the same as the corresponding perturbative sector of type II theory (in which case it is also supersymmetric due to the larger $\mathcal{N} = 2$ SUSY). The degeneracy (for all signs of charges n and w) in type II theory is given by (3.9).

form expressions were obtained. For example, in the case of 5-dimensional 3-charge large black holes (discussed in section 2.3.2) such calculation is still not available.

Fortunately, for 4-dimensional 4-charge large black holes (discussed in section 2.3.2) such calculation was done, though only in the BPS case $nw > 0$, $\widehat{N}\widehat{W} > 0$, where one can use powerful properties of supersymmetry (see [12] for a detailed review). We take heterotic string theory compactified on flat $T^4 \times \widehat{S}^1 \times S^1$, and count 1/4-BPS micro-configurations consisting of elementary string wound w times around S^1, $\widehat{N}$ KK-monopoles wrapped around $T^4 \times S^1$, $\widehat{W}$ NS5-branes wrapped around $T^4 \times S^1$, and a momentum on S^1 with momentum number n. For 1/4-BPS states ($nw > 0$, $\widehat{N}\widehat{W} > 0$) the asymptotic formula for statistical entropy in the regime $nw \gg \widehat{N}\widehat{W}$ is given by

$$S_{\mathrm{stat}}^{(\mathrm{BPS})} = 2\pi\sqrt{nw(\widehat{N}\widehat{W}+4)}\,. \tag{3.11}$$

Again, (3.11) is α'-exact, but, and the cumulative effect of α' corrections is encoded in number 4 inside the square root. This time there is a well defined perturbative expansion in α', as expected from our discussion in section 2.3.4 on properties of corresponding 4-charge large black holes. From (3.11) follows that in lowest order in α' we obtain agreement with result for black hole entropy (2.30). This is one of the many examples in which string theory is giving microscopic explanation for black hole termodynamics, in the most direct and straightforward way without any suspicious assumptions.

What about non-BPS states? In the case $nw < 0$, $\widehat{N}\widehat{W} < 0$ we can use the trick to go to the type-II string theory, where those microstates are supersymmetric. As the microstates are in NS-NS sector and purely right-moving, the bosonic part of this sector is in one-to-one correspondence with that in heterotic theory which means that microcanonical entropies are the same. Counting of microstates gives asymptotically for $|nw| \gg |\widehat{N}\widehat{W}|$

$$S_{\mathrm{stat}}^{(\mathrm{non-BPS})} = 2\pi\sqrt{|nw\widehat{N}\widehat{W}|}\,, \qquad nw < 0\,, \quad \widehat{N}\widehat{W} < 0\,. \tag{3.12}$$

There are no α'-corrections. In the next section we shall give an explanation for this.

As for the non-BPS case with $nw\widehat{N}\widehat{W} < 0$, the α'-exact direct microstate counting was not performed.

Comments on direct microscopic analysis:

1. Microscopic entropies calculated *exactly* in α'. As for quantum corrections, closed form expressions so far typically available only at lowest order (tree-level).

2. Large black holes - Lowest order in α' agreeing with Bekenstein-Hawking entropies of corresponding black holes. In principle one can use perturbative analyses to check agreement at higher orders, by systematically taking into account higher-derivative terms in supergravity effective action.

3. Small black holes - Microscopic entropy intrinsically non-perturbative in α'. To compare it with black hole entropy, full tree-level effective action is needed on the gravity side.

4. Microstate counting is typically performed in the limit of small g_s, such that influence of relevant microscopic configurations on space-time geometry can be neglected. On the other

hand, in supergravity analyses we are dealing with black holes which significantly change the geometry of space-time. Obviously, those are two completely different regimes, so why we should be allowed to compare the two entropies?[6] For BPS black holes there is a direct answer - such states are organized in special shorter multiplets, and the number of states inside the multiplets cannot change when parameters of the theory (such as g_s) are changed continuously (if we assume that nothing violently like phase transitions is happening in the process). So the total number of states, which gives the entropy, is protected by supersymmetry. For non-BPS we cannot use this argument, but we shall show in section 4.2 that attractor mechanism can be used to argue that the entropy should again be unchanged when we "turn effective coupling constant on".

3.2 AdS/CFT methods

In the classic paper [4] Brown and Henneaux have shown that gravity in $D = 3$ dimensions has asymptotic symmetry group containing two independent Virasoro algebras,

$$
\begin{aligned}
&[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12} m(m^2-1)\delta_{m+n}\,, \\
&[\overline{L}_m, \overline{L}_n] = (m-n)\overline{L}_{m+n} + \frac{\overline{c}}{12} m(m^2-1)\delta_{m+n}\,, \\
&[L_m, \overline{L}_n] = 0
\end{aligned}
$$

where $m, n \in Z$ and c and $\overline{c}$ are central charges of corresponding algebras. This is exactly what is present in 2-dimensional conformal field theories. It was argued much later by Maldacena that this is just one example of a more general idea known today as the AdS/CFT conjecture [5], which states that D-dimensional gravity theory which is asymptotically AdS should be equivalent to the conformal field theory (without gravity) which is "living" on the boundary (in asymptotic infinity) of AdS space. The equivalence is of strong/weak type (it connects strongly coupled theory on one side with weakly coupled on the other side) which can be extremely useful as it can be used to study strong-coupling behavior by using perturbative calculations. But, because of this it is hard to prove conjecture, as for this one should be able to calculate in the regime of strong coupling at least on one side, which is typically at the moment not known (that is way it is still a conjecture). One possible exception is $D = 3$ case, where the dual theories are 2-dimensional conformal field theories, for which much more is known of strongly coupling regime then for higher-dimensional CFT's. This is one of the motivations for analyzing cases in which one has asymptotic AdS_3 geometry. For, example,microcanonical entropy (logarithm of number of states) at the level $L_0 = \Delta$, $\overline{L} = \overline{\Delta}$ asymptotically for $\Delta \gg c$, $\overline{\Delta} \gg \overline{c}$ is given by simple Cardy formula

$$
S_{\mathrm{CFT}} = 2\pi\sqrt{\frac{c\Delta}{6}} + 2\pi\sqrt{\frac{\overline{c}\overline{\Delta}}{6}}\,, \tag{3.13}
$$

[6] Mathematically this means the following. Influence of an object with mass M on geometry is proportional to $G_N M$. For string configurations typically considered we have $G_N M \propto g_s^a$, with $a > 0$. Now, to avoid large quantum effects, we take $g_s \ll 1$. No effect on geometry means that $G_N M$ should be small compared to the string scale, i.e., $G_N M \ll \alpha'^{(D-3)/2}$. But if we have large black holes, for which Schwarzschild radius should be much larger then the string length parameter, we have $\sqrt{\alpha'} \ll R_{\mathrm{Sch}} \propto (G_N M)^{1/(D-3)}$. This is obviously an opposite limit from the one above, so we have two completely different regimes. We note that an "intermediate" regime $G_N M \sim \alpha'^{(D-3)/2}$ is fully nonperturbative stringy regime of which very little is known so far.

depending just on central charges, and not on the specific content of the theory.

Interestingly, all (non-singular) solutions that we consider in this review contain such AdS_3 factor in the near-horizon geometry. It can be easily shown that factors $AdS_2 \times S^1$, which appear in all these solutions, are *locally* isometric to AdS_3. They all satisfy

$$R_{MNPQ} = -\ell_A^{-2}\left(G_{MP}G_{NQ} - G_{MQ}G_{NP}\right) \quad \text{for} \quad M,N,P,Q \in \{t,r,x^9\}\,. \tag{3.14}$$

meaning that they are locally maximally symmetric. The geometries would be also globally isometric to AdS_3 if the proper radius of S^1 would be infinitely large. We shall assume that the radius is large enough so that corresponding finite-size effects are negligible and we can take that geometry has AdS_3 factor.[7]

For extremal black holes that we analyse, either Δ or $\overline{\Delta}$ are vanishing, and the one which does not vanish is equal to $|n|$. In the cases where all other charges are positive, BPS case $n > 0$ correspond to $\Delta = 0$ and $\overline{\Delta} = n$, and non-BPS case $n > 0$ to $\Delta = |n|$ and $\overline{\Delta} = 0$. If one could find central charges c and $\bar{c}$ then the entropy would be simply given by Cardy formula (3.13). There are two methods which were used in the literature, (1) direct sigma model calculation [46], (2) indirect by using anomaly inflow arguments [22, 47].

In the first, and historicaly earlier method, one treats heterotic string theory on the relevant compactifications and backgrounds with AdS_3 factors, and then heavily relying on explicit realisations of $(0,4)$ supersymmetry (present in all cases of interest to us) and AdS/CFT correspondence one is able to obtain relevant central charges [46]. For the geometry appearing as near horizon geometry in case of the 4-dimensional 4-charge large black holes from section 2.3.3, when $w > 0$, $\widehat{N}\widehat{W} > 0$ one obtains

$$c = 6w(\widehat{N}\widehat{W} + 2)\,, \qquad \bar{c} = 6w(\widehat{N}\widehat{W} + 4)\,. \tag{3.15}$$

When $w < 0$, $\widehat{N}\widehat{W} > 0$ the only difference is the left $\leftrightarrow$ right interchange, which leads to $c \leftrightarrow \bar{c}$, so

$$c = 6|w|(\widehat{N}\widehat{W} + 4)\,, \qquad \bar{c} = 6|w|(\widehat{N}\widehat{W} + 2)\,. \tag{3.16}$$

When used in Cardy formula (3.13) this gives in the BPS case ($nw > 0$)

$$S_{\mathrm{CFT}}^{(\mathrm{BPS})} = 2\pi\sqrt{nw(\widehat{N}\widehat{W} + 4)}\,, \qquad nw > 0\,, \quad \widehat{N}\widehat{W} > 0\,, \tag{3.17}$$

which exactly agrees with the expression obtained from direct microstate counting (3.11). The virtue of AdS/CFT method is that it can give us the result also in the non-BPS case in which $n < 0$

$$S_{\mathrm{CFT}}^{(\mathrm{non-BPS})} = 2\pi\sqrt{|nw|(|\widehat{N}\widehat{W}| + 2)}\,, \qquad nw\widehat{N}\widehat{W} < 0\,. \tag{3.18}$$

Note that this was the case in which direct counting of microstates was not performed, so this is a new result.

As for the other type of non-BPS states defined by $nw < 0$, $\widehat{N}\widehat{W} < 0$, because AdS_3 background with $\widehat{N}\widehat{W} < 0$ is nonsupersymmetric in the heterotic theory, the method apparently cannot be used.

For the geometry appearing as near horizon geometry in case of the 5-dimensional 3-charge large black holes from section 2.3.2, when $w > 0$ one obtains

$$c = 6w|N|\,, \qquad \bar{c} = 6w(|N| + 2)\,. \tag{3.19}$$

[7] This means that we have to take special limit for the charges, e.g., in our examples $|n| \gg |w|$.

When $w < 0$, the only change is $c \leftrightarrow \bar{c}$. When used in Cardy formula (3.13) we obtain in the BPS case ($nw > 0$)

$$S_{\mathrm{CFT}}^{(\mathrm{BPS})} = 2\pi\sqrt{nw(|N|+2)}\,, \tag{3.20}$$

while in the non-BPS case ($nw < 0$) we obtain

$$S_{\mathrm{CFT}}^{(\mathrm{non-BPS})} = 2\pi\sqrt{|nwN|}\,. \tag{3.21}$$

Both (3.20) and (3.21) are new results, valid α'-exactly.

Later it was shown [22, 47, 23] that when effective 3-dimensional theory on AdS$_3$ has $(0,4)$ (or even smaller $(0,2)$ [24]) supersymmetry, central charges are generally determined purely by the coefficients of Chern-Simons terms. This method of calculating central charges has two virtues: (i) it is general, depending only on symmetries, (ii) as Chern-Simons terms are connected to anomalies and correspondingly 1-loop saturated, their coefficients in many cases can be calculated exactly (at least in α'). In fact, in [22] the power of this method was demonstrated by calculating central charges (3.15) relevant for the entropy of 4-dimensional 4-charge black holes. As for the case relevant for 5-dimensional 3-charge black holes, i.e., (3.19), such calculations were not performed. Let us mention that α'-exact gravity calculations [54] are confirming both results (3.15) and (3.19).

4. Some formalities

4.1 Wald entropy formula

As noted in section 2.3.4, low energy effective actions of string theories, even on tree-level, contain higher-derivatives terms. It is known that in such theories entropy of black hole solutions is not any more given by simple Bekenstein-Hawking formula (2.18). If the theory is *manifestly* diffeomorphism invariant, in which case Lagrangian density is of the form

$$\mathscr{L} = \mathscr{L}(g_{ab}, R_{\mu\nu\rho\sigma}, \nabla_\lambda R_{\mu\nu\rho\sigma}, \ldots, \psi, \nabla_\mu\psi, \ldots)\,, \tag{4.1}$$

where ψ denotes matter fields and dots denote higher-order derivatives, then the black hole entropy is given by Wald formula [41]

$$S_{\mathrm{bh}} = -2\pi \int_{\mathscr{H}} d^{D-2}x\sqrt{h}\,E^{abcd}\eta_{ab}\eta_{cd}\,. \tag{4.2}$$

Here $\mathscr{H}$ is a cross-section of the horizon, η_{ab} denotes binormal to $\mathscr{H}$, $h = \det(h_{ab})$ is determinant of the induced metric on $\mathscr{H}$, and

$$E^{\mu\nu\rho\sigma} = \frac{\partial\mathscr{L}}{\partial R_{\mu\nu\rho\sigma}} - \nabla_{\lambda_1}\frac{\partial\mathscr{L}}{\partial\nabla_{\lambda_1}R_{\mu\nu\rho\sigma}} + \ldots + (-1)^m\nabla_{(\lambda_1}\ldots\nabla_{\lambda_m)}\frac{\partial\mathscr{L}}{\partial\nabla_{(\lambda_1}\ldots\nabla_{\lambda_m)}R_{\mu\nu\rho\sigma}} \tag{4.3}$$

The derivative in 4.3 is taken with $g_{\mu\nu}$ and ∇_μ fixed.

Two important comments on Wald formula:

1. Here it is important to notice that Wald entropy is again purely determined from near-horizon behavior of black hole solution. Because in higher-derivative theories it is generally not possible to find exact solutions in the whole space-time, this property is essential if we are hoping to calculate exact entropies.

2. There are theories (and heterotic theory is an example) which contain also terms which are *not* manifestly diff-covariant, so called (purely gravitational or mixed) Chern-Simons terms. In [11] a generalization of Wald formula to such theories was proposed. However, instead of using this generalised formula, we shall handle Chern-Simons terms in a more direct and quicker way developed in [25].

4.2 Sen's entropy function method

Let us assume that we have a D-dimensional theory with a field content consisting of the metric tensor $G_{\mu\nu}$, neutral scalar fields ϕ_s, and a number of (also neutral) p-form fields (of which some are U(1) gauge fields with corresponding $(p+1)$-form strengths), with the Lagrangian which is *manifestly* gauge and diffeomorphism invariant. We are interested in the near-horizon behavior of the rotationally invariant extremal black holes. One expects that the metric is $AdS_2 \times S^{D-2}$, which has $SO(2,1) \times SO(D-1)$ as an isometry group, and that the whole background respects thissymmetry manifestly.[8] In this case one can apply Sen's entropy function formalism [40] which we now briefly review.

The point is that manifest symmetry under $SO(2,1) \times SO(D-1)$ heavily restricts the near-horizon behavior of the fields. For example, it follows that the only manifestly covariant p-forms (which means strengths in case of gauge fields) which are allowed to be non-vanishing are 2-form (denoted F^I) and $(D-2)$-form (denoted H_m). More completely, the near-horizon behavior is constrained to have the following form

$$\begin{aligned}
&ds^2 = v_1 \left(-r^2 dt^2 + \frac{dr^2}{r^2} \right) + v_2 \, d\Omega_{D-2}^2 \\
&\phi_s = u_s \,, \qquad s = 1, \ldots, n_s \\
&F^I_{rt} = f^I \,, \qquad i = 1, \ldots, n_F \\
&H_m = h_m \epsilon_S \qquad m = 1, \ldots, n_H
\end{aligned} \tag{4.4}$$

where $v_{1,2}$, u_s, e^I and h_m are all constant, and ϵ_S is an induced volume-form on unit sphere S^{D-2}. For F^I and H_m which play the role of gauge field strengths it follows that $e^I = f^I$ and $p_m = h_m$ are the electric fields and magnetic charges, respectively.[9]

It can be shown that for background (4.4) solving of equations of motion is equivalent to extremization of the (algebraic) function $\mathscr{F}$, defined by

$$\mathscr{F}(\vec{v}, \vec{u}, \vec{f}, \vec{h}; \vec{e}, \vec{p}) = \oint_{S^{D-2}} \sqrt{-G}\, \mathscr{L} \,, \tag{4.5}$$

over $\vec{v}$, $\vec{u}$ and $\vec{f}$. We have divided forms into gauge forms (whose corresponding electric field strengths $\vec{e}$ and magnetic charges $\vec{p}$ are taken as *fixed*) and non-gauge whose values are variables denoted as $\vec{f}$ and $\vec{h}$ (in our examples they will auxiliary fields). This means that we have to solve a system of algebraic equations

$$0 = \frac{\partial \mathscr{F}}{\partial \vec{v}} \,, \qquad 0 = \frac{\partial \mathscr{F}}{\partial \vec{u}} \,, \qquad 0 = \frac{\partial \mathscr{F}}{\partial \vec{f}} \,, \qquad 0 = \frac{\partial \mathscr{F}}{\partial \vec{h}} \,. \tag{4.6}$$

[8] In [37, 38, 39] this was proven for broad class of actions in $D = 4$ and 5.

[9] We see that $D = 4$ is a special case in which there are only 2-form strengths, but which can carry both electric and magnetic charges.

If the system happens to be regular, we can solve it for all unknowns and obtain solutions for $\vec{v}$,$\vec{u}$ and $\vec{f}$ as functions of $\vec{e}$ and $\vec{p}$.

It is more common to express solutions not as functions of electric field strengths but as a function of electric charges. It can be easily shown that electric charge (in particular normalization) is given by

$$\vec{q} = \frac{\partial \mathscr{F}}{\partial \vec{e}} \,, \tag{4.7}$$

One of the virtues of the Sen's entropy function method is that it gives straightforwardly the entropy of black hole. Let us define the entropy function $\mathscr{E}$ as Legandre-transform of the function $\mathscr{F}$ with respect to electric field/charge

$$\mathscr{E}(\vec{v},\vec{u},\vec{f},\vec{h},\vec{e};\vec{q},\vec{p}) = 2\pi\,(\vec{q}\cdot\vec{e} - \mathscr{F})\;. \tag{4.8}$$

Then obviously system (4.6) and (4.7) is equivalent to extremization of the entropy function $\mathscr{E}$ with respect to all variables except electric and magnetic charges which are kept fixed, i.e.,

$$0 = \frac{\partial \mathscr{E}}{\partial \vec{v}}\,, \qquad 0 = \frac{\partial \mathscr{E}}{\partial \vec{u}}\,, \qquad 0 = \frac{\partial \mathscr{E}}{\partial \vec{f}}\,, \qquad 0 = \frac{\partial \mathscr{E}}{\partial \vec{h}}\,, \qquad 0 = \frac{\partial \mathscr{E}}{\partial \vec{e}}\,. \tag{4.9}$$

Solving of this system is giving us $\vec{v}$,$\vec{u}$, $\vec{f}$ and $\vec{e}$ directly as functions of charges $\vec{q}$ and $\vec{p}$.

Finally, it was shown in [40] that the the value of the entropy function at the extremum gives the same result as black hole entropy calculated from Wald formula (4.2), i.e.,

$$S_{\mathrm{bh}}(\vec{q},\vec{p}) = \mathscr{E} \quad \text{(evaluated at the solution of (4.9))}\,. \tag{4.10}$$

Comments on the entropy function method:

1. It enormously simplifies calculation of near horizon geometry and entropy, as it turns solving of system of differential equations into solving of system of algebraic equations.

2. Manifest gauge and diffeomorphism invariance neccessary. If there are Chern-Simons terms of any kind (gravitational, gauge, or mixed) additional labor is necessary [12]. One idea is to use dimensional reduction to write such terms in the manifestly covariant form. We shall use this idea in section 5 to handle mixed Chern-Simons term that is present in heterotic theory (pure gauge terms, which also exist, will be vanishing in our examples).

3. When one has just the solution with symmetries expected of near-horizon solution, it is not guaranteed that for every such solution there is indeed full black hole solution with such near-horizon behavior. However, for large black holes treated here we know that such correspondence exist in the lowest-order (because there are explicit complete solutions), and as corrections to the near horizon geometry are regular we can expect that this correspondence continues to apply at least perturbatively. For small black holes, in which $\alpha' \to 0$ limit is singular, we cannot be that sure.

4.3 Field redefinitions

We shall be dealing with tree-level effective action of heterotic string theory which, as discussed in section 2.3.4 has infinite expansion in derivatives (parametrized by α') (2.32). For such theories, there is no uniquely preferred choice for fields. If we start with some set of fields ϕ_i, we can always make a *field redefinitions* of the type

$$\phi_i \to \phi_i' = \phi_i + \sum_{n=1}^{\infty} \alpha'^n f_i^{(n)}[\phi] \,. \tag{4.11}$$

The Lagrangian, written in transformed fields ϕ_i', will have generally different form (except for the lowest order 2-derivative part which obviously stays unchanged). It can be shown that perturbative properties are not changed. Important examples are S-matrix and Wald entropy, which are both perturbatively invariant on field redefinitions. One can use field redefinitions to make Lagrangian (or some part of it) look "nicer" or more "symmetric" which can simplify some calculations. In the following sections we shall use the field redefinition freedom.

We emphasize the following points, which are sometimes overlooked in the literature:

1. Monomials in Lagrangians are divided into two groups, (a) those whose coefficients are unchanged by any field redefinition (*unambiguous terms*), (b) those whose coefficients change under some field redefinitions *ambiguous terms*). Though by using field redefinitions we can *individually* kill every ambiguous term, it is not generally the case that we can kill them all *simultaneously*. Ambiguous terms which can be killed simultaneously are called *irrelevant*. For some Lagrangians there are ambiguous terms which are *relevant*. It should be emphasized that relevant ambiguous terms are *equally important* as unambiguous terms (which are obviously relevant). Indeed, as shown in [29], the heterotic effective action is giving us a nice example, in which already at first-order in the α'-expansion (4-derivative terms) relevant ambiguous terms are present.

2. If we are interested in *non*perturbative results, we have to be more careful when applying field redefinitions. For example, higher derivatives, typically present in $f_i^{(n)}[\phi]$ in (4.11), are generally introducing new degrees of freedom which obviously means that transformed Lagrangian will not describe the same theory. In such cases we have to be sure that field redefinition is *regular*, not introducing new degrees of freedom or some other anomalies. This is why we have to be careful with the choice of field redefinition scheme when we treat small black holes (which are α'-nonperturbative objects, as we argued before).[10]

5. Full heterotic effective action

In this section we mainly review the results from [1].

[10] Though we emphasize again that α'-expansion of effective action is, well, effective, meaning that normally we are not expecting it to make sense outside the perturbative regime in which corrections are small.

PoS(BHs, GR and Strings)033

5.1 The action and one conjecture

We now start with systematic analyses of α'-corrections from the gravity side, which means starting from the tree-level effective action of (appropriately compactified) heterotic string theory, solving for extremal black holes and calculating the corresponding entropies. To simplify the calculations, we concentrate immediately on the near-horizon behavior and use Sen's entropy function formalism.

Let us start with what is known about the structure of tree-level effective action of heterotic string theory in $D = 10$ dimensions. It has the general structure

$$\mathscr{S}^{(10)} = \int dx^{10} \sqrt{-G^{(10)}} \mathscr{L}^{(10)} = \sum_{n=0}^{\infty} \int dx^{10} \sqrt{-G^{(10)}} \mathscr{L}_n^{(10)} , \tag{5.1}$$

Again, as we explained at the beginning of section 2.1, we are interested in such configurations for which the only non-vanishing elementary fields are string metric $G_{MN}^{(10)}$, dilaton $\Phi^{(10)}$ and 2-form $B_{MN}^{(10)}$. In this case every $\mathscr{L}_n^{(10)}$ is a function of the string metric $G_{MN}^{(10)}$, Riemann tensor $R_{MNPQ}^{(10)}$, dilaton $\Phi^{(10)}$, 3-form gauge field strength $H_{MNP}^{(10)}$ and the covariant derivatives of these fields. 10-dimensional space-time indices are denoted as $M, N, \ldots = 0, 1, \ldots, 9$. The term $\mathscr{L}_n^{(10)}$ has $2(n+1)$ derivatives, and is multiplied with a factor of α'^n.

Ten-dimensional Lagrangian can be decomposed in the following way

$$\mathscr{L}^{(10)} = \mathscr{L}_{01}^{(10)} + \Delta\mathscr{L}_{\text{CS}}^{(10)} + \mathscr{L}_{\text{other}}^{(10)} . \tag{5.2}$$

The first term in (5.2), explicitly written, is

$$\mathscr{L}_{01}^{(10)} = \frac{e^{-2\Phi^{(10)}}}{16\pi G_{10}} \left[R^{(10)} + 4\left(\partial\Phi^{(10)}\right)^2 - \frac{1}{12} H_{MNP}^{(10)} H^{(10)MNP} \right] , \tag{5.3}$$

where G_{10} is 10-dimensional Newton constant. 3-form gauge field strength is not closed, but instead given by

$$H_{MNP}^{(10)} = \partial_M B_{NP}^{(10)} + \partial_N B_{PM}^{(10)} + \partial_P B_{MN}^{(10)} - 3\alpha' \overline{\Omega}_{MNP}^{(10)} , \tag{5.4}$$

where $\overline{\Omega}_{MNP}^{(10)}$ is the gravitational Chern-Simons form

$$\overline{\Omega}_{MNP}^{(10)} = \frac{1}{2} \overline{\Gamma}^{(10)R}{}_{MQ} \partial_N \overline{\Gamma}^{(10)Q}{}_{PR} + \frac{1}{3} \overline{\Gamma}^{(10)R}{}_{MQ} \overline{\Gamma}^{(10)Q}{}_{NS} \overline{\Gamma}^{(10)S}{}_{PR} \quad (\text{antisym. in } M, N, P) \tag{5.5}$$

Bar on the geometric object means that it is calculated using a modified connection

$$\overline{\Gamma}^{(10)P}{}_{MN} = \Gamma^{(10)P}{}_{MN} - \frac{1}{2} H^{(10)P}{}_{MN} \tag{5.6}$$

in which H plays the role of a torsion. It is believed that Chern-Simons terms appear exclusively through Eq. (5.4).

If in (5.4) the Chern-Simons form $\overline{\Omega}_{MNP}^{(10)}$ would be absent, then (5.3) would be equal to (2.1), i.e., we would have $\mathscr{L}_{01}^{(10)} = \mathscr{L}_0^{(10)}$ in (5.1). Its presence introduces non-trivial α'-corrections. Beside, as shown in [27], supersymmetrization (on-shell completion of $\mathscr{N} = 1$ SUSY) of the Chern-Simons term introduces a (probably infinite) tower of terms in the effective action (with increasing number of derivatives), denoted by $\Delta\mathscr{L}_{\text{CS}}^{(10)}$ in (5.2). The first two non-vanishing terms (in expansion in α') are

$$\Delta\mathscr{L}_{\text{CS},1}^{(10)} = \frac{\alpha'}{8}\frac{e^{-2\Phi^{(10)}}}{16\pi G_{10}}\overline{R}^{(10)}_{MNPQ}\overline{R}^{(10)MNPQ} \tag{5.7}$$

and

$$\Delta\mathscr{L}_{\text{CS},3}^{(10)} = -\frac{\alpha'^3}{64}\frac{e^{-2\Phi^{(10)}}}{16\pi G_{10}}\left(3\,T_{MNPQ}T^{MNPQ} + T_{MN}T^{MN}\right) \tag{5.8}$$

where

$$T_{MNPQ} \equiv \overline{R}^{(10)\;RS}_{[MN}\,\overline{R}^{(10)}_{PQ]RS}\,, \qquad T_{MN} \equiv \overline{R}^{(10)\;QR}_{MP}\,\overline{R}^{(10)P}{}_{NQR}\,. \tag{5.9}$$

Though higher terms present in $\Delta\mathscr{L}_{\text{CS}}^{(10)}$ were not explicitly constructed, it was argued in [27] that α'^n contribution should be a linear combination of monomials containing n Riemann tensors $\overline{R}_{MNPQ}$ calculated from the connection with torsion as given in (5.6). This is the key information for us. All large black hole near-horizon solutions that we construct and analyze here have the property that $\overline{R}_{MNPQ}$ evaluated on them *vanishes*, which means that all these terms, including (5.7) and (5.8), will be irrelevant in our calculations (giving vanishing contribution to equations of motion and entropy).

It is well-known that, beside terms connected with Chern-Simons term by supersymmetry, additional terms appear in the effective action starting from α'^3 (8-derivative) order. In (5.2) we have denoted them with $\mathscr{L}_{\text{other}}^{(10)}$. One well-known example is R^4-type (unambiguous) term multiplied by $\zeta(3)$, which appears in all string theories. Unfortunately, the knowledge of structure of $\mathscr{L}_{\text{other}}^{(10)}$ is currently highly limited, and only few terms have been unambiguously calculated.

From now on, we are going to neglect contributions coming from $\mathscr{L}_{\text{other}}^{(10)}$. One motivation is following from AdS$_3$/CFT$_2$ correspondence and anomaly inflow arguments of [22]. There was argued (from 3-dimensional perspective) that for geometries having AdS_3 factor only Chern-Simons terms are important for calculations of central charges (from which one can calculate the black hole entropy). $\mathscr{L}_{\text{other}}^{(10)}$ neither contains Chern-Simons terms nor is connected by supersymmetry to them, it should be irrelevant in such calculations. AdS$_3$/CFT$_2$ argument is sometimes used to explain successes of R^2-truncated actions (supersymmetric and/or Gauss-Bonnet) in calculations of entropies of BPS black holes in $D=4$ and 5. However, AdS$_3$/CFT$_2$ argument is relying on supersymmetry and can be confidently used only when corresponding AdS$_3$ background is supersymmetric (current proofs require $(0,2)$ SUSY). In fact, we shall see that results for black hole entropy show that the argument cannot be used in such non-BPS cases. Also, it would be interesting to have an argument which is not using any other information but the structure of effective action in $D=10$.

In fact, there is such direct argument. If it happens that $\mathscr{L}_{\text{other}}^{(10)}$ could be written in such a way that every monomial in it contains two powers of $\overline{R}_{MNPQ}$, then it would be irrelevant for our calculations and our results would be undoubtedly α'-exact. The argument is the same as the one we used for $\Delta\mathscr{L}_{\text{CS}}^{(10)}$ two paragraphs above. Indeed, this property was conjectured long time ago, see e.g., [29]. There is a stronger form of the conjecture, which claims that $\mathscr{L}_{\text{other}}^{(10)}$ is purely composed of (G^{MN} contracted) products of $\overline{R}_{MNPQ}$, see, e.g., [32]. Though the current status of the conjecture appears to be somewhat controversial – it was disputed in [33, 34], but the most recent detailed calculations [35] of some 8-derivative corrections (some of them recalculating the ones from [33, 34]) are giving results *in agreement* with the strong form of the conjecture.

We shall assume that conjecture (at least in weaker form) is correct, which allows us to neglect $\mathscr{L}^{(10)}_{\rm other}$ part of the Lagrangian, which then is allowing us to calculate α'-*exact* near-horizon solutions and corresponding black hole entropies. As in this way we obtain results for the entropies which are in agreement with microscopic calculations, we can say that our results are speaking in favour of the conjecture (though, of course, are not proving it).

5.2 Manipulating Chern-Simons terms in $D = 6$

All configurations that we analyze in this paper have four spatial dimensions compactified on torus T^4, and are uncharged under Kaluza-Klein 1-form gauge fields originating from four compactified dimensions. Taking from the start that corresponding gauge fields vanish[11] one obtains that the effective action is the same as in the section 5.1, but now considering all fields and variables to be 6-dimensional. Effectively, one just has to replace everywhere (10) with (6) and take indices corresponding to 6-dimensional space-time, i.e., $M,N,\ldots = 0,1,\ldots,5$). To shorten the expressions, we immediately fix the values of Newton constant and α', which in our normalization take values $G_6 = 2$ and $\alpha' = 16$.

Appearance of gravitational Chern-Simons term in (5.4) introduces two problems. One is that it introduces in the action terms which are not manifestly diff-covariant, and that prevents direct use of Sen's entropy function formalism. A second problem is that due to (5.6) and (5.4) Chern-Simons term is mixed in a complicated way with other α'-corrections. We handle these problems by using the following two-step procedure (introduced in [25]).

First, we introduce an additional 3-form $K^{(6)} = dC^{(6)}$ and put a theory in a classically equivalent form in which Lagrangian is given by

$$\begin{aligned}\sqrt{-G^{(6)}}\,\widetilde{\mathscr{L}}^{(6)} &= \sqrt{-G^{(6)}}\,\mathscr{L}^{(6)} + \frac{1}{(24\pi)^2}\varepsilon^{MNPQRS}K^{(6)}_{MNP}H^{(6)}_{QRS} \\ &\quad + \frac{3\alpha'}{(24\pi)^2}\varepsilon^{MNPQRS}K^{(6)}_{MNP}\overline{\Omega}^{(6)}_{QRS}\,,\end{aligned} \tag{5.10}$$

and where now $H^{(6)}_{MNP}$ should not be treated as a gauge field strength but as an auxiliary 3-form. Antisymmetric tensor density ε^{MNPQRS} is defined by $\varepsilon^{012345} = 1$. As a result, Chern-Simons term is now isolated as a single α'^1-correction, in a way which will eventually allow us to write it in a manifestly covariant form.

Before passing to a second step of the procedure from [25], we need to isolate in (5.10) ordinary Chern-Simons term $\Omega^{(6)}$ (obtained from standard Levi-Civita connection) from the rest by using (5.6). The result is [28]

$$\overline{\Omega}^{(6)}_{MNP} = \Omega^{(6)}_{MNP} + \mathscr{A}^{(6)}_{MNP} \tag{5.11}$$

where

$$\begin{aligned}\mathscr{A}^{(6)}_{MNP} &= \frac{1}{4}\partial_M\left(\Gamma^{(6)R}_{NQ}H^{(6)Q}_{RP}\right) + \frac{1}{8}H^{(6)R}_{MQ}\nabla_N H^{(6)Q}_{RP} - \frac{1}{4}R^{(6)\,QR}_{MN}H^{(6)}_{PQR} \\ &\quad + \frac{1}{24}H^{(6)R}_{MQ}H^{(6)S}_{NR}H^{(6)Q}_{PS} \quad \text{(antisymmetrized in } M,N,P).\end{aligned} \tag{5.12}$$

[11] Such truncation is expected to be consistent.

Notice that when (5.12) is plugged in (5.11), and this into (5.10), which is then integrated to obtain the action, contribution from the first term in (5.12) will, after partial integration, have a factor $dK^{(6)}$ which vanishes because $K^{(6)}$ is by definition exact form. We now see that $\mathscr{A}^{(6)}$ gives manifestly covariant contribution to the action.

Now we are ready to write 6-dimensional action

$$\mathscr{S}^{(6)} = \int dx^6 \sqrt{-G^{(6)}}\, \widetilde{\mathscr{L}}^{(6)} \tag{5.13}$$

in the form we are going to use extensively in the paper. Using (5.2) and the above analysis, Lagrangian can be written in the following form

$$\widetilde{\mathscr{L}}^{(6)} = \widetilde{\mathscr{L}}_0^{(6)} + \widetilde{\mathscr{L}}_1^{(6)\prime} + \widetilde{\mathscr{L}}_1^{(6)\prime\prime} + \Delta\mathscr{L}_{\rm CS}^{(6)} + \mathscr{L}_{\rm other}^{(6)} \,. \tag{5.14}$$

First term is lowest order (α'^0) contribution given by (5.3)

$$\widetilde{\mathscr{L}}_0^{(6)} = \frac{e^{-2\Phi^{(6)}}}{32\pi}\left[R^{(6)} + 4\left(\partial\Phi^{(6)}\right)^2 - \frac{1}{12}H_{MNP}^{(6)}H^{(6)MNP}\right] + \frac{\varepsilon^{MNPQRS}}{(24\pi)^2\sqrt{-G^{(6)}}}K_{MNP}^{(6)}H_{QRS}^{(6)} \tag{5.15}$$

For later convenience we have separated first-order terms in three parts. One is given by

$$\widetilde{\mathscr{L}}_1^{(6)\prime} = \frac{\varepsilon^{MNPQRS}}{12\pi^2\sqrt{-G^{(6)}}}K_{MNP}^{(6)}\left(\frac{1}{8}H_{QT}^{(6)U}\nabla_R H_{US}^{(6)T} - \frac{1}{4}R_{QR}^{(6)\;TU}H_{STU}^{(6)} + \frac{1}{24}H_{QT}^{(6)U}H_{RU}^{(6)V}H_{SV}^{(6)T}\right) \tag{5.16}$$

The second part, which contains gravitational Chern-Simons term and is not manifestly covariant, is given by

$$\widetilde{\mathscr{L}}_1^{(6)\prime\prime} = \frac{\varepsilon^{MNPQRS}}{12\pi^2\sqrt{-G^{(6)}}}K_{MNP}^{(6)}\Omega_{QRS}^{(6)} \,. \tag{5.17}$$

Finally, the third part is contained in $\Delta\mathscr{L}_{\rm CS}^{(6)}$ (5.7). In [25] it was shown how to rewrite (5.17) in the manifestly covariant form for the particular type of the backgrounds which includes those we shall analyze in this paper.

All in all, we shall start from the reduced action with Lagrangian given by

$$\widetilde{\mathscr{L}}_{\rm red}^{(6)} = \widetilde{\mathscr{L}}_0^{(6)} + \widetilde{\mathscr{L}}_1^{(6)\prime} + \widetilde{\mathscr{L}}_1^{(6)\prime\prime} \,, \tag{5.18}$$

and check if the near horizon solutions satisfy the condition $\overline{R}_{MNPQ} = 0$. If this is satisfied, it follows immediately that they are also solutions of the action with Lagrangian

$$\widetilde{\mathscr{L}}_{\rm susy}^{(6)} = \widetilde{\mathscr{L}}_{\rm red}^{(6)} + \Delta\mathscr{L}_{\rm CS}^{(6)} \,, \tag{5.19}$$

and, under the above mentioned assumption on $\mathscr{L}_{\rm other}^{(6)}$, of the full heterotic action (5.14).

5.3 Compactification to $D < 6$

Our main interest are black holes in $D = 5$ and $D = 4$ dimensions, so we consider further compactification on $(6 - D)$ circles S^1. Using the standard Kaluza-Klein compactification we obtain D-dimensional fields $G_{\mu\nu}$, $C_{\mu\nu}$, Φ, $\widehat{G}_{mn}$, $\widehat{C}_{mn}$ and $A_\mu^{(i)}$ ($0 \le \mu,\nu \le D-1$, $D \le m,n \le 5$, $1 \le i \le 2(6-D)$):

$$\begin{aligned}
&\widehat{G}_{mn} = G^{(6)}_{mn}\,, \quad \widehat{G}^{mn} = (\widehat{G}^{-1})^{mn}\,, \quad \widehat{C}_{mn} = C^{(6)}_{mn}\,,\\
&A_\mu^{(m-D+1)} = \frac{1}{2}\widehat{G}^{nm}G^{(6)}_{n\mu}\,, \quad A_\mu^{(m-2D+7)} = \frac{1}{2}C^{(6)}_{m\mu} - \widehat{C}_{mn}A_\mu^{(n-D+1)}\,,\\
&G_{\mu\nu} = G^{(6)}_{\mu\nu} - \widehat{G}^{mn}G^{(6)}_{m\mu}G^{(6)}_{n\nu}\,,\\
&C_{\mu\nu} = C^{(6)}_{\mu\nu} - 4\widehat{C}_{mn}A_\mu^{(m-D+1)}A_\nu^{(n-D+1)} - 2(A_\mu^{(m-D+1)}A_\nu^{(m-2D+7)} - A_\nu^{(m-D+1)}A_\mu^{(m-2D+7)})\\
&\Phi = \Phi^{(6)} - \frac{1}{2}\ln\mathscr{V}_{6-D}\,,
\end{aligned} \tag{5.20}$$

There is also (now auxiliary) field $H^{(6)}_{MNP}$ which produces D-dimensional fields $H_{\mu\nu\rho}$, $H_{\mu\nu m}$, $H_{\mu mn}$ and H_{mnp}. As in [25], we take for the circle coordinates $0 \le x^m < 2\pi\sqrt{\alpha'} = 8\pi$, so that the volume $\mathscr{V}_{6-D}$ is

$$\mathscr{V}_{6-D} = (8\pi)^{6-D}\sqrt{\widehat{G}}\,. \tag{5.21}$$

The gauge invariant field strengths associated with $A_\mu^{(i)}$ and $C_{\mu\nu}$ are

$$F^{(i)}_{\mu\nu} = \partial_\mu A^{(i)}_\nu - \partial_\nu A^{(i)}_\mu\,, \qquad 1 \le i,j \le 2(6-D)\,, \tag{5.22}$$

$$K_{\mu\nu\rho} = \left(\partial_\mu C_{\nu\rho} + 2A^{(i)}_\mu L_{ij} F^{(j)}_{\nu\rho}\right) + \text{cyclic permutations of } \mu,\, \nu,\, \rho\,, \tag{5.23}$$

where

$$L = \begin{pmatrix} 0 & I_{6-D} \\ I_{6-D} & 0 \end{pmatrix}\,, \tag{5.24}$$

I_{6-D} being a $(6-D)$-dimensional identity matrix.

For the black holes we are interested in, we have

$$A^{(i)}_\mu L_{ij} F^{(j)}_{\nu\rho} = 0\,. \tag{5.25}$$

Normally, the next step would be to perform the Kaluza-Klein reduction on the 6-dimensional low-energy effective action to obtain a D-dimensional effective action, which can be quite complicated. We shall follow a more efficient procedure [25] – we go to D dimensions just to use the symmetries of the action to construct an ansatz for the background ($AdS_2 \times S^{D-2}$ in our case), and then perform an uplift to 6 dimensions (by inverting (5.20)) where the action is simpler and calculations are easier.

5.4 4-dimensional 4-charge black holes in heterotic theory

Now we want to use the formalism we developed above for studying α'-corrections to the near-horizon geometry of extremal 4-dimensional 4-charge black holes appearing in the heterotic string theory compactified on $T^4 \times S^1 \times \widehat{S}^1$ introduced in section 2.3.3. One can obtain an effective 4-dimensional theory by putting $D = 4$ in (5.20) (using the formulation of the 6-dimensional action from section 5.2) and taking as non-vanishing only the following fields: string metric $G_{\mu\nu}$, dilaton Φ, moduli $T_1 = (\widehat{G}_{44})^{1/2}$ and $T_2 = (\widehat{G}_{55})^{1/2}$, four Kaluza-Klein gauge fields $A_\mu^{(i)}$ ($0 \le \mu,\nu \le 3$,

$1 \le i \le 4$) coming from $G^{(6)}_{MN}$ and 2-form potential $C^{(6)}_{MN}$, and two auxiliary 2-forms $D^{(n)}_{\mu\nu}$ ($n = 1,2$) coming from $H^{(6)}_{MNP}$ (which is now, as explained in section 5.2, an auxiliary field).

The black holes we are interested in are charged purely electrically with respect to $A^{(1)}_\mu$ and $A^{(3)}_\mu$, and purely magnetically with respect to $A^{(2)}_\mu$ and $A^{(4)}_\mu$. As discussed before, from the heterotic string theory viewpoint, these black holes should correspond to 4-charge states in which, beside fundamental string wound around the S^1 circle (with coordinate x^4), and with nonvanishing momentum on it, there are also Kaluza-Klein and H-monopoles (NS5-branes) wound around $T^4 \times S^1$ (with a "nut" on $\widehat{S}^1$).

For extremal black holes one expects $AdS_2 \times S^2$ near-horizon geometry [37, 38, 39] which in the present case is given by:

$$
\begin{aligned}
& ds^2 \equiv G_{\mu\nu} dx^\mu dx^\nu = v_1 \left(-r^2 dt^2 + \frac{dr^2}{r^2} \right) + v_2 (d\theta^2 + \sin^2\theta \, d\phi^2) \,, \\
& e^{-2\Phi} = u_S \,, \qquad T_1 = u_1 \,, \qquad T_2 = u_2 \,, \\
& F^{(1)}_{rt} = \widetilde{e}_1 \,, \qquad F^{(3)}_{rt} = \frac{\widetilde{e}_3}{16} \,, \qquad F^{(2)}_{\theta\phi} = \frac{\widetilde{p}_2}{4\pi} \sin\theta \,, \qquad F^{(4)}_{\theta\phi} = \frac{\widetilde{p}_4}{64\pi} \sin\theta \,, \\
& D^{(1)\,rt} = \frac{2 u_1^2 h_1}{v_1 v_2 u_S} \,, \qquad D^{(2)\,\theta\phi} = - \frac{8\pi u_2^2 h_2}{v_1 v_2 u_S \sin\theta} \,.
\end{aligned}
\tag{5.26}
$$

Here v_1, v_2, u_S, u_n, $\widetilde{e}_i$ and h_n ($n = 1,2$, $i = 1,\ldots,4$) are unknown variables fixed by equations of motion and values of electric charges $\widetilde{q}_{1,3}$. Somewhat unusual normalization for $h_{1,2}$ is introduced for later convenience.

Once we have a background obeying the full group of symmetries of $\mathrm{AdS}_2 \times S^2$ space, we can use Sen's entropy function formalism reviewed in section 4.2. We could calculate the entropy function from

$$
\mathscr{E} = 2\pi \left(\sum_I \widetilde{q}_I \widetilde{e}_I - \int_{S^2} \sqrt{-G}\, \widetilde{\mathscr{L}} \right) \,, \tag{5.27}
$$

where $\widetilde{q}_I$ are electric charges, and $\widetilde{\mathscr{L}}$ is the effective Lagrangian in four dimensions. For this, we would need to calculate $\widetilde{\mathscr{L}}$ by doing dimensional reduction from six to four dimensions, which would give us quite a complicated effective Lagrangian.

Instead of this, it is much easier to perform calculation of the entropy function $\mathscr{E}$ directly in six dimensions were we already know the action. For this, we have to lift the background to six dimensions, which for (5.26) gives

$$
\begin{aligned}
& ds_6^2 \equiv G^{(6)}_{MN} dx^M dx^N = ds^2 + u_1^2 \left(dx^4 + 2\widetilde{e}_1 r dt \right)^2 + u_2^2 \left(dx^5 - \frac{\widetilde{p}_2}{2\pi} \cos\theta \, d\phi \right)^2 \,, \\
& K^{(6)}_{tr4} = \frac{\widetilde{e}_3}{8} \,, \qquad K^{(6)}_{\theta\phi 5} = - \frac{\widetilde{p}_4}{32\pi} \sin\theta \,, \\
& H^{(6)tr4} = \frac{4 h_1}{v_1 v_2 u_S} \,, \qquad H^{(6)\theta\phi 5} = \frac{16\pi h_2}{v_1 v_2 u_S \sin\theta} \,, \\
& e^{-2\Phi^{(6)}} = \frac{u_S}{64\pi^2 u_1 u_2} \,.
\end{aligned}
\tag{5.28}
$$

Instead of $\widetilde{\mathscr{L}}$ and G we now use in (5.27) the six dimensional Lagrangian $\widetilde{\mathscr{L}}^{(6)}$ given in (5.14)-(5.17) and the determinant $G^{(6)}$

$$\mathscr{E} = 2\pi \left(\sum_I \widetilde{q}_I \widetilde{e}_I - \int_{S^2\times S^1\times\widehat{S}^1} \sqrt{-G^{(6)}}\, \widetilde{\mathscr{L}}^{(6)} \right) . \tag{5.29}$$

This is obviously equivalent to (5.27). Equations of motion turn into extremization of the entropy function (5.27) over variables $\{\varphi_a\} = \{v_1, v_2, u_S, u_n, \widetilde{e}_i, h_n\}$,

$$0 = \left.\frac{\partial \mathscr{E}}{\partial \varphi_a}\right|_{\varphi=\bar{\varphi}} . \tag{5.30}$$

The black hole entropy is given by the value of the entropy function at the extremum

$$S_{\mathrm{bh}} = \mathscr{E}(\bar{\varphi}) , \tag{5.31}$$

which is a function of electric and magnetic charges only.

As we discussed in sections 5.1 and 5.2, we concentrate on the part of the action connected by 10-dimensional supersymmetry with Chern-Simons term (obtained by neglecting $\mathscr{L}^{(6)}_{\mathrm{other}}$ in (5.14)). For the moment we also neglect $\Delta\mathscr{L}^{(6)}_{\mathrm{CS}}$, for which we show a posteriori that it does not contribute to the near-horizon solutions and the entropies. This means that we start with the reduced Lagrangian $\widetilde{\mathscr{L}}^{(6)}_{\mathrm{red}}$ defined by (5.18), (5.15), (5.16) and (5.17). Putting (5.28) in (5.18), and then this into the entropy function (5.29), we obtain

$$\mathscr{E} = \mathscr{E}_0 + \mathscr{E}_1' + \mathscr{E}_1'' , \tag{5.32}$$

where

$$\begin{aligned} \mathscr{E}_0 &= 2\pi \left[\widetilde{q}_1\widetilde{e}_1 + \widetilde{q}_3\widetilde{e}_3 - \int d\theta\, d\phi\, dx^4 dx^5 \sqrt{-G^{(6)}}\, \widetilde{\mathscr{L}}^{(6)}_0 \right] \\ &= 2\pi \left[\widetilde{q}_1\widetilde{e}_1 + \widetilde{q}_3\widetilde{e}_3 - \frac{1}{8} v_1 v_2 u_S \left(-\frac{2}{v_1} + \frac{2}{v_2} + \frac{2u_1^2\widetilde{e}_1^2}{v_1^2} + \frac{128\pi^2 u_2^2 h_2 (2\widetilde{e}_3 - h_2)}{v_1^2 u_S^2} \right.\right. \\ &\qquad \left.\left. - \frac{u_2^2 \widetilde{p}_2^2}{8\pi^2 v_2^2} - \frac{8u_1^2 h_1 (2\widetilde{p}_4 - h_1)}{v_2^2 u_S^2} \right) \right] , \end{aligned} \tag{5.33}$$

and

$$\begin{aligned} \mathscr{E}_1' &= -2\pi \int d\theta\, d\phi\, dx^4 dx^5 \sqrt{-G^{(6)}}\, \widetilde{\mathscr{L}}^{(6)\prime}_1 \\ &= -4\pi v_1 v_2 u_S \left(\frac{8192\pi^4 u_2^4 \widetilde{e}_3 h_2^3}{v_1^4 u_S^4} + \frac{8u_2^4 \widetilde{e}_3 h_2 \widetilde{p}_2^2}{v_1^2 v_2^2 u_S^2} - \frac{128\pi^2 u_2^2 \widetilde{e}_3 h_2}{v_1^2 v_2 u_S^2} \right. \\ &\qquad \left. + \frac{32 u_1^4 \widetilde{p}_4 h_1^3}{v_2^4 u_S^4} + \frac{8u_1^4 \widetilde{e}_1^2 h_1 \widetilde{p}_4}{v_1^2 v_2^2 u_S^2} - \frac{8u_1^2 \widetilde{p}_4 h_1}{v_1 v_2^2 u_S^2} \right) . \end{aligned} \tag{5.34}$$

With $\mathscr{E}_1''$, defined by

$$\mathscr{E}_1'' = -2\pi \int d\theta\, d\phi\, dx^4 dx^5 \sqrt{-G^{(6)}}\, \widetilde{\mathscr{L}}^{(6)\prime\prime}_1 , \tag{5.35}$$

the situation is a bit tricky because of the presence of Chern-Simons density in (5.17). This means that $\widetilde{\mathscr{L}}^{(6)\prime\prime}_1$ is not manifestly diffeomorphism covariant, and one cannot apply directly Sen's entropy function formalism. Fortunately, this problem was solved in [25] where it was shown how for the class of the metrics, to which (5.28) belongs, one can write $\mathscr{E}_1''$ in a manifestly covariant form.

Next, notice that the background (5.51) has a form of a product of two 3-dimensional backgrounds, the first one is on (t,r,x^5) space ($\mathrm{AdS}_2 \times S^1$) and the second one on (θ,ϕ,x^4) space ($S^2 \times \widehat{S}^1$). From this follows

$$\mathscr{E}_1'' = \frac{1}{6\pi} \int d\theta d\phi dx^4 dx^5 \varepsilon^{ijk} \varepsilon^{abc} \left(K^{(6)}_{ijk} \Omega^{(6)}_{abc} - \Omega^{(6)}_{ijk} K^{(6)}_{abc} \right) , \tag{5.36}$$

where $\{a,b,c\} = \{t,r,4\}$ and $\{i,j,k\} = \{\theta,\phi,5\}$, and the convention for the antisymmetric tensor densities is

$$\varepsilon^{tr4} = 1 , \qquad \varepsilon^{\theta\phi 5} = 1 . \tag{5.37}$$

Furthermore, Kaluza-Klein compactification is performed on x^4 and x^5 which leaves us with 4-dimensional effective space. So, for our purposes it would be enough to have result which is manifestly covariant in two reduced 2-dimensional spaces (AdS_2 and S^2).

In three dimensions it is known [8, 9] that for the metrics of the "Kaluza-Klein form"

$$ds^2 = \phi(x) \left[g_{mn}(x) dx^m dx^n + (dy + 2A_m(x) dx^m)^2 \right] , \tag{5.38}$$

where $0 \le m,n \le 1$, we have (modulo total derivative terms)

$$\varepsilon^{\alpha\beta\gamma} \Omega_{\alpha\beta\gamma} = \frac{1}{2} \varepsilon^{mn} \left[R^{(2)} F_{mn} + 4 g^{m'p'} g^{q'q} F_{mm'} F_{p'q'} F_{qn} \right] , \tag{5.39}$$

where $F_{mn} = \partial_m A_n - \partial_n A_m$, ε^{mn} is antisymmetric with $\varepsilon^{01} = 1$, and $R^{(2)}$ is a Ricci scalar obtained from g_{mn}. (5.39) gives us the desired *manifestly covariant form* (in the reduced 2-dimensional space) for the gravitational Chern-Simons term.

Using (5.39) for $\mathrm{AdS}_2 \times S^1$ and $S^2 \times \widehat{S}^1$ separately, it is now easy to obtain [25]

$$\mathscr{E}_1'' = -(8\pi)^2 \left[\frac{\widetilde{p}_4}{4\pi} \left(\frac{u_1^2}{v_1} \widetilde{e}_1 - 2 \frac{u_1^4}{v_1^2} \widetilde{e}_1^3 \right) + \widetilde{e}_3 \left(\frac{u_2^2}{v_2} \frac{\widetilde{p}_2}{4\pi} - 2 \frac{u_2^4}{v_2^2} \left(\frac{\widetilde{p}_2}{4\pi} \right)^3 \right) \right] . \tag{5.40}$$

We are now ready to find near-horizon solutions, by solving the system (5.30), and black hole entropy from (5.31). As we want to compare the results with the statistical entropy obtained in string theory by counting of microstates, it is convenient to express charges $(\widetilde{q},\widetilde{p})$ in terms of (integer valued) charges naturally appearing in the string theory. By comparing the lowest-order solution (one uses just (5.33), which is an easy exercise) with the near-horizon solution (2.29) we obtain

$$\widetilde{q}_1 = \frac{n}{2} , \qquad \widetilde{p}_2 = 4\pi \widehat{N} , \qquad \widetilde{q}_3 = -4\pi \widehat{W} , \qquad \widetilde{p}_4 = -\frac{w}{2} , \tag{5.41}$$

where n and w are momentum and winding number of string wound along circle S^1, and $\widehat{N}$ and $\widehat{W}$ are Kaluza-Klein monopole and H-monopole charges associated with the circle $\widehat{S}^1$. Why we expect that α'-corrections will not introduce corrections in (5.41)? Standard answer would be that in the Lagrangian α'-corrections are higher-derivative corrections, which in asymptotic infinity are vanishing to fast to give contribution for charges. However, this argument does not work for Chern-Simons terms, which could give such corrections. If this happens values of electric charge may depend on where we define them, near the horizon, or in infinity. Fortunately, for our $D = 4$ black holes this is not happening, but later we shall see that this happens in $D = 5$ case.

Using (5.32)-(5.41) in (5.30), we obtain quite a complicated algebraic system, naively not expected to be solvable analytically. Amazingly, we have found analytic near-horizon solutions for all values of charges, corresponding to BPS and non-BPS black holes.[12] While in BPS case analytic solutions are expected because one can use BPS conditions to drastically simplify calculations, in non-BPS case in theories which involve higher-derivative corrections analytic solutions are typically not known[13].

We mentioned in section 2.3.3 that from the supersymmetry point of view there are three types of solutions, differing in relative signs of products nw and $\widehat{N}\widehat{W}$. We now analyze them separately, by writing α'-exact near-horizon solutions explicitly for representative cases of each type. We nore that expressions for entropies are representative independent.

The type-1 consists of supersymmetric 1/4-BPS large black holes for which $nw > 0$, $\widehat{N}\widehat{W} > 0$. For clarity of presentation, we take $n, w, \widehat{N}\widehat{W} > 0$ as a representative of this type. The near-horizon solution is then given by

$$
\begin{aligned}
& v_1 = v_2 = 4(\widehat{N}\widehat{W} + 2)\,, \qquad u_S = \sqrt{\frac{nw}{\widehat{N}\widehat{W} + 4}}\,, \\
& u_1 = \sqrt{\frac{n(\widehat{N}\widehat{W} + 2)}{w(\widehat{N}\widehat{W} + 4)}}\,, \qquad u_2 = \sqrt{\frac{\widehat{W}}{\widehat{N}}\left(1 + \frac{2}{\widehat{N}\widehat{W}}\right)}\,, \\
& \widetilde{e}_1 = \frac{1}{n}\sqrt{nw(\widehat{N}\widehat{W} + 4)}\,, \qquad \widetilde{e}_3 = h_2 = -\frac{\widehat{N}}{8\pi}\sqrt{\frac{nw}{\widehat{N}\widehat{W} + 4}}\,, \qquad h_1 = -\frac{w}{2}\,.
\end{aligned}
\tag{5.42}
$$

For the entropy of type-1 black holes we obtain

$$
S_{\mathrm{bh}}^{\mathrm{BPS}} = 2\pi\sqrt{nw(\widehat{N}\widehat{W} + 4)}\,, \qquad nw > 0\,, \quad \widehat{N}\widehat{W} > 0 \tag{5.43}
$$

This is exactly what one obtains by microstate counting in string theory (3.11), in the limit $nw \gg \widehat{N}\widehat{W}$, which corresponds to tree-level approximation on gravity side.

The type-2 consists of non-supersymmetric large black holes for which $nw < 0$, $\widehat{N}\widehat{W} < 0$. For clarity of presentation, we take $n, \widehat{N} < 0$ and $w, \widehat{W} > 0$ as a representative of this type. The near-horizon solution is then given by[14]

$$
\begin{aligned}
& v_1 = v_2 = 4|\widehat{N}\widehat{W}|\,, \qquad u_S = \sqrt{\left|\frac{nw}{\widehat{N}\widehat{W}}\right|}\,, \qquad u_1 = \sqrt{\left|\frac{n}{w}\right|}\,, \qquad u_2 = \sqrt{\left|\frac{\widehat{W}}{\widehat{N}}\right|}\,, \\
& \widetilde{e}_1 = \frac{1}{n}\sqrt{|nw\widehat{N}\widehat{W}|}\,, \qquad \widetilde{e}_3 = h_2 = -\frac{\widehat{N}}{8\pi}\sqrt{\left|\frac{nw}{\widehat{N}\widehat{W}}\right|}\,, \qquad h_1 = -\frac{w}{2}\,.
\end{aligned}
\tag{5.44}
$$

For the entropy of type-2 black holes we obtain

$$
S_{\mathrm{bh}}^{\mathrm{non-BPS}} = 2\pi\sqrt{|nw\widehat{N}\widehat{W}|}\,, \qquad nw < 0\,, \quad \widehat{N}\widehat{W} < 0\,. \tag{5.45}
$$

[12] The way we constructed solutions was indirect - we managed to conjecture them from perturbative calculations (which we did up to α'^4), and then checked them by putting into exact equations. For some special sets of charges we then numerically checked that there are no other physically acceptable solutions.

[13] However, one exception can be found in [20].

[14] While this review was in preparation this solution was presented in [71].

Again, agreement with statistical calculation in string theory (3.12) is exact in α'. Note that here both the near-horizon solution (5.44) and the black hole entropy (5.45) are α'-*uncorrected*. We shall comment this later.

The type-3 consists of non-supersymmetric large black holes for which $nw\widehat{N}\widehat{W} < 0$. For clarity of presentation, we take $n < 0$, $w, \widehat{N}\widehat{W} > 0$ as a representative of this type. The near-horizon solution is then given by

$$\begin{aligned} v_1 &= v_2 = 4(\widehat{N}\widehat{W}+2)\,, & u_S &= \sqrt{\frac{|n|w}{\widehat{N}\widehat{W}+2}}\,, \\ u_1 &= \sqrt{\frac{|n|}{w}}\,, & u_2 &= \sqrt{\frac{\widehat{W}}{\widehat{N}}\left(1+\frac{2}{\widehat{N}\widehat{W}}\right)}\,, \\ \widetilde{e}_1 &= \frac{1}{n}\sqrt{|n|w(\widehat{N}\widehat{W}+2)}\,, & \widetilde{e}_3 &= h_2 = -\frac{\widehat{N}}{8\pi}\sqrt{\frac{|n|w}{\widehat{N}\widehat{W}+2}}\,, \qquad h_1 = -\frac{w}{2}\,. \end{aligned} \tag{5.46}$$

For the entropy of type-3 black holes we obtain

$$S_{\mathrm{bh}}^{\mathrm{non-BPS}} = 2\pi\sqrt{|nw|(|\widehat{N}\widehat{W}|+2)}\,, \qquad nw\widehat{N}\widehat{W} < 0 \tag{5.47}$$

Again, agreement with statistical calculation in string theory (3.18) is exact in α'.

Now we have to check that $\overline{R}^{(6)}_{MNPQ}$ vanishes when evaluated on our solutions. From (5.6) one gets

$$\overline{R}^{(6)M}{}_{NPQ} = R^{(6)M}{}_{NPQ} + \nabla_{[P}H^{(6)M}{}_{Q]N} - \frac{1}{2}H^{(6)M}{}_{R[P}H^{(6)R}{}_{Q]N}\,. \tag{5.48}$$

It is easy to show that all three solutions, (5.42), (5.46) and (5.46), when used in 6-dimensional background (5.28) give

$$\overline{R}^{(6)}_{MNPQ} = 0\,. \tag{5.49}$$

As explained in section 5.1, from this follows that inclusion of the term $\Delta\mathscr{L}^{(6)}_{\mathrm{CS}}$ does not change neither the near-horizon solutions (5.42), (5.44) and (5.46) nor the corresponding black hole entropies (5.43), (5.45) and (5.47), which means that all our results would be obtained if we started with the more complicated supersymmetric Lagrangian (5.19), constructed by supersymmetrizing gravitational Chern-Simons term.

5.5 5-dimensional 3-charge black holes in heterotic theory

Here we consider the 5-dimensional spherically symmetric 3-charge extremal black holes which appear in the heterotic string theory compactified on $T^4 \times S^1$. One can obtain an effective 5-dimensional theory by putting $D = 5$ in (5.20) (again using the formulation of the 6-dimensional action from section 5.2) and taking as non-vanishing only the following fields: string metric $G_{\mu\nu}$, dilaton Φ, modulus $T = (\widehat{G}_{55})^{1/2}$, two Kaluza-Klein gauge fields $A^{(i)}_\mu$ ($0 \le \mu, \nu \le 4$, $1 \le i \le 2$) coming from $G^{(6)}_{MN}$ and 2-form potential $C^{(6)}_{MN}$, the 2-form potential $C_{\mu\nu}$ with the strength $K_{\mu\nu\rho}$, one Kaluza-Klein auxiliary two form $D_{\mu\nu}$ coming from $H^{(6)}_{MNP}$, and auxiliary 3-form $H_{\mu\nu\rho}$.

The black holes we are interested in are charged purely electrically with respect to $A^{(i)}_\mu$, and purely magnetically with respect to $K_{\mu\nu\rho}$. From the heterotic string theory viewpoint, these black holes should correspond to 3-charge states in which, beside fundamental string wound around S^1 circle with nonvanishing momentum on it, there are NS5-branes wrapped around $T^4 \times S^1$.

For extremal black holes we now expect[15] $\mathrm{AdS}_2 \times S^3$ near-horizon geometry which in the present case is given by:

$$
\begin{aligned}
& ds^2 \equiv G_{\mu\nu} dx^\mu dx^\nu = v_1 \left(-r^2 dt^2 + \frac{dr^2}{r^2} \right) + v_2 d\Omega_3 \,, \\
& F^{(1)}_{rt} = \widetilde{e}_1 \,, \qquad F^{(2)}_{rt} = \frac{\widetilde{e}_2}{4} \,, \qquad K_{234} = \frac{\widetilde{p}}{4}\sqrt{g_3} \,, \\
& D^{rt} = \frac{2u_T^2 h_1}{v_1 v_2^{3/2} u_S} \,, \qquad H^{234} = -\frac{8h_2}{v_1 v_2^{3/2} u_S \sqrt{g_3}} \,, \\
& e^{-2\Phi} = u_S \,, \qquad T = u_T \,.
\end{aligned}
\tag{5.50}
$$

Here g_3 is a determinant of the metric on the unit 3-sphere S^3 (with coordinates x^i, $i = 2,3,4$).

We follow the procedure from section 5.4. Lift of (5.50) to six dimensions gives

$$
\begin{aligned}
& ds_6^2 \equiv G^{(6)}_{MN} dx^M dx^N = ds^2 + u_T^2 \left(dx^5 + 2\widetilde{e}_1 r dt \right)^2 \,, \\
& K^{(6)}_{tr5} = \frac{\widetilde{e}_2}{2} \,, \qquad K^{(6)}_{234} = K_{234} = \frac{\widetilde{p}}{4}\sqrt{g_3} \,, \\
& H^{(6)tr5} = \frac{4h_1}{v_1 v_2^{3/2} u_S} \,, \qquad H^{(6)234} = -\frac{8h_2}{v_1 v_2^{3/2} u_S \sqrt{g_3}} \,, \\
& e^{-2\Phi^{(6)}} = \frac{u_S}{8\pi u_T} \,.
\end{aligned}
\tag{5.51}
$$

Now v_1, v_2, u_S, u_T, $\widetilde{e}_1$, $\widetilde{e}_2$, h_1 and h_2 are unknown variables whose solution is to be found by extremizing the entropy function for the fixed values of electric and magnetic charges $\widetilde{q}_{1,2}$ and $\widetilde{p}$. Entropy function is now given by

$$
\begin{aligned}
\mathscr{E} &= 2\pi \left(\sum_{i=1}^{2} \widetilde{q}_i \widetilde{e}_i - \int_{S^3} \sqrt{-G}\, \mathscr{L} \right) = 2\pi \left(\sum_{i=1}^{2} \widetilde{q}_i \widetilde{e}_i - \int_{S^3} \sqrt{-G^{(6)}}\, \mathscr{L}^{(6)} \right) \\
&= \mathscr{E}_0 + \mathscr{E}'_1 + \mathscr{E}''_1 \,,
\end{aligned}
\tag{5.52}
$$

where

$$
\begin{aligned}
\mathscr{E}_0 = 2\pi \Bigg[& \widetilde{q}_1 \widetilde{e}_1 + \widetilde{q}_2 \widetilde{e}_2 - \frac{\pi}{16} v_1 v_2^{3/2} u_S \Bigg(-\frac{2}{v_1} + \frac{6}{v_2} + \frac{2u_T^2 \widetilde{e}_1^2}{v_1^2} + \frac{32\, h_2 (2\widetilde{e}_2 - h_2)}{v_1^2 u_S^2} \\
& - \frac{8u_T^2 h (2\widetilde{p} - h)}{v_2^3 u_S^2} \Bigg) \Bigg] \,,
\end{aligned}
\tag{5.53}
$$

$$
\mathscr{E}'_1 = -2\pi^2 v_1 v_2^{3/2} u_S \left[\frac{512\, e_2 h_2^3}{v_1^4 u_S^4} + \frac{32\, u_T^4 \widetilde{p} h_1^3}{v_2^6 u_S^4} + \frac{8u_T^4 \widetilde{p} h_1 \widetilde{e}_1^2}{v_1^2 v_2^3 u_S^2} - \frac{8u_T^2 \widetilde{p} h_1}{v_1 v_2^3 u_S^2} - \frac{96\, \widetilde{e}_2 h_2}{v_1^2 v_2 u_S^2} \right] ,
\tag{5.54}
$$

[15] In $D = 5$ there is no explicit proof that extremal asymptotically flat black holes must have $\mathrm{AdS}_2 \times S^3$ near-horizon geometry. However, for the large black holes analyzed here one knows that lowest order solutions, which were fully constructed, have such near-horizon behavior, and from continuity one expects the same when α'-corrections are included. Again, the situation is not that clear for small black holes, which we shall discuss later.

$$\mathscr{E}_1'' = -8\pi^2 \widetilde{p} \left(\frac{u_T^2}{v_1} \widetilde{e}_1 - 2 \frac{u_T^4}{v_1^2} \widetilde{e}_1^3 \right) . \tag{5.55}$$

Again, to obtain (5.55) we had to deal with gravitational Chern-Simons term, which is done following the procedure reviewed in section 5.4.[16]

We are now ready to find near-horizon solutions, by solving the system (5.30), and black hole entropy from (5.31). As we want to compare the results with the statistical entropy obtained in string theory by counting of microstates, it is convenient to express charges $(\widetilde{q}, \widetilde{p})$ in terms of (integer valued) charges naturally appearing in the string theory. By comparing the lowest-order solution (one uses just (5.53), which is an easy exercise) with the near-horizon solution (2.23) we obtain

$$\widetilde{q}_1 = \frac{n}{2}\,, \qquad \widetilde{q}_2 = -16\pi m\,, \qquad \widetilde{p} = -\frac{w}{\pi}\,. \tag{5.56}$$

Here n and w are momentum and winding number of string wound around S^1. Naively, we would expect that m denotes number of NS5-branes wrapped around $T^4 \times S^1$, but we shall see that α'-corrections are introducing here additional shift.

Again, we were able to find analytic solutions to algebraic system for all values of charges. As discussed in section 2.3.2, from the supersymmetry viewpoint there are two types of black hole solutions which differ in sign of product nw. For clarity of presentation, we restrict to $w, m > 0$. Then $n > 0$ ($n < 0$) correspond to 1/4-BPS (non-BPS) black holes.

In the BPS case ($n, w, m > 0$) near-horizon solutions for 3-charge black holes are given by

$$\begin{gathered} v_1 = 4(m+1)\,, \qquad v_2 = 4v_1\,, \qquad u_S = \frac{1}{8\pi}\sqrt{\frac{nw}{(m+1)(m+3)}}\,, \qquad u_T = \sqrt{\frac{n(m+1)}{w(m+3)}}\,, \\ \widetilde{e}_1 = \frac{1}{n}\sqrt{nw(m+3)}\,, \qquad \widetilde{e}_2 = h_2 = -\frac{1}{32\pi}\sqrt{\frac{nw}{m+3}}\,, \qquad h_1 = -\frac{w}{\pi}\,. \end{gathered} \tag{5.57}$$

For the entropy we obtain

$$S_{\rm bh}^{\rm BPS} = 2\pi\sqrt{nw(|m|+3)}\,, \qquad nw > 0\,. \tag{5.58}$$

In the non-BPS case ($n < 0$, $w, m > 0$) we obtain

$$\begin{gathered} v_1 = 4(m+1)\,, \qquad v_2 = 4v_1\,, \qquad u_S = \frac{\sqrt{|n|w}}{8\pi(m+1)}\,, \qquad u_T = \sqrt{\frac{|n|}{w}}\,, \\ \widetilde{e}_1 = \frac{1}{n}\sqrt{|n|w(m+1)}\,, \qquad \widetilde{e}_2 = h_2 = -\frac{1}{32\pi}\sqrt{\frac{|n|w}{m+1}}\,, \qquad h_1 = -\frac{w}{\pi}\,. \end{gathered} \tag{5.59}$$

For the entropy we obtain

$$S_{\rm bh}^{\rm non-BPS} = 2\pi\sqrt{|nw|(|m|+1)}\,, \qquad nw < 0\,. \tag{5.60}$$

[16]For the 3-sphere the Chern-Simons term vanishes

$$\int_{S^3} \varepsilon^{ijk}\Omega_{ijk} = 0\,.$$

This is obvious if one calculates Ω_{ijk} using standard non-covariant formula (5.5). As sphere does not have boundaries, the inclusion of boundary total-derivative terms (which "covariantize" CS term) cannot change the result.

This is exactly equal to the result conjectured in [26] (on the basis of α'^3-order perturbative results).

There is a subtle issue connected to interpretation of charges. Naively, we would expect that charge m should be equal to the number of NS5-branes, which we denote by N. To check this, let us calculate components of 3-form strength $H^{(6)}$ with indices on S^3, evaluated on our solutions (5.57) and (5.59). The result is

$$H^{(6)}_{234} = 32\left(m + \frac{m}{|m|}\right)\sqrt{g_3}\,. \tag{5.61}$$

From (5.61) follows that $(m+1)$ is the magnetic charge (factor of 32 comes from our normalization and is in fact equal to $2\alpha'$, see (2.25)). As magnetic charges have topological origin, and so are not expected to receive perturbative corrections, we conclude that the number of NS5-branes should be given by

$$N = m + \frac{m}{|m|}\,. \tag{5.62}$$

Using this in (5.57) and (5.59) we finally obtain our solutions expressed using "natural" charges of the string theory, i.e., momentum n, winding w and number of NS5-branes N.

In the BPS case near-horizon solution (5.57) becomes

$$\begin{aligned}
&v_1 = 4N\,, \qquad v_2 = 4v_1\,, \qquad u_S = \frac{1}{8\pi}\sqrt{\frac{nw}{N(N+2)}}\,, \qquad u_T = \sqrt{\frac{nN}{w(N+2)}}\,, \\
&\widetilde{e}_1 = \frac{1}{n}\sqrt{nw(N+2)}\,, \qquad \widetilde{e}_2 = h_2 = -\frac{1}{32\pi}\sqrt{\frac{nw}{N+2}}\,, \qquad h_1 = -\frac{w}{\pi}\,,
\end{aligned} \tag{5.63}$$

while the entropy (5.58) is

$$S^{\mathrm{BPS}}_{\mathrm{bh}} = 2\pi\sqrt{nw(|N|+2)}\,, \qquad nw > 0\,. \tag{5.64}$$

Again we see that agreement with statistical calculation (by using AdS/CFT) in string theory (3.20) is exact in α'.

In the non-BPS case near-horizon solution was given with (5.57) which now becomes

$$\begin{aligned}
&v_1 = 4N\,, \qquad v_2 = 4v_1\,, \qquad u_S = \frac{\sqrt{|n|w}}{8\pi N}\,, \qquad u_T = \sqrt{\frac{|n|}{w}}\,, \\
&\widetilde{e}_1 = \frac{1}{n}\sqrt{|n|wN}\,, \qquad \widetilde{e}_2 = h_2 = -\frac{1}{32\pi}\sqrt{\frac{|n|w}{N}}\,, \qquad h_1 = -\frac{w}{\pi}\,,
\end{aligned} \tag{5.65}$$

and the entropy (5.60) is

$$S^{\mathrm{non-BPS}}_{\mathrm{bh}} = 2\pi\sqrt{|nwN|}\,, \qquad nw < 0\,. \tag{5.66}$$

Again, agreement with statistical calculation (by using AdS/CFT) in string theory (3.21) is exact in α'.

Finally, it is easy to check that both BPS and non-BPS near-horizon solutions presented in this section satisfy 6-dimensional relation (5.49), which again means that inclusion of $\Delta\mathscr{L}^{(6)}_{\mathrm{CS}}$ in the action would not change our solutions and entropies (so they are also solutions of the action (5.19)).

PoS(BHs, GR and Strings)033

5.6 Comments on α'-exact calculation using effective action

- Non-BPS solution (5.65) is α'-*un*corrected in our scheme. Now, it was shown that lowest-order BPS solution is an α'-exact solution from the sigma model calculations [45] (corresponding result for 4-charge 4-dimensional black holes was given in [44]). As we use different scheme, our solutions cannot be directly compared to sigma model ones.

- The expressions for black hole entropies (5.64) and (5.66) are in agreement with those obtained from AdS/CFT correspondence, using the results for central charges calculated in [46] (see section 3.2 for more details).

- Important consequence of our calculation is that α'-corrections to entropies and near-horizon solutions is solely coming from Chern-Simons term, for all values of charges. Now, results from [22, 23, 24] show that this should be expected for black holes which are connected to backgrounds which lead to ($\mathscr{N}=2$) supersymmetric AdS_3 gravities (this AdS_3 comes from $AdS_2 \times S^1$). But, we see in our examples that it works for all signs of the charges, *even for those which are connected to nonsupersymmetric AdS_3 gravities*. It would be interesting to see how far one can extend the results from [22, 23, 24].

- Finally, we note that results from this section agree with perturbative calculations up to α'^2-order obtained in [25, 26] by using the 4-derivative effective action derived in [29]. This is expected, as it was shown in [28] that this action is equivalent up to α'^1 with the action used in this section. We note that this action by itself is not working beyond α'^2 order, which means that in the field redefinition scheme used in [29] one has to take into account also terms with more than four derivatives to obtain α'-exact entropies [26].

5.7 Black holes in type-II string theories

If we considered type-II string theories, instead of heterotic, compactified in the same way, with charges restricted to NS-NS sector, there are large black holes corresponding to all cases we discussed up until now. In fact, it happens that α'-exact results can be obtained immediately with no effort. This is because structure of NS-NS sector of tree-level low energy effective action of type-II theories - it differs from the heterotic effective action in that there are no Chern-Simons terms (present in 5.4), and correspondingly, in notation from 5.2, one has $\Delta\mathscr{L}_{CS}=0$ (from this follows that there are no 4-derivative, together with 6-derivative, terms in the action at all). As we saw in the heterotic case that Chern-Simons term was solely responsible for α'-corrections, we can immediately conclude that entropies and near-horizon solutions remain α'-uncorrected for such black holes.

In particular, for type-II 4-charge extremal black holes in $D=4$ (compactification on $S^1\times S^1\times T^4$, NS-NS charged only) the entropy is

$$S_{\rm bh}^{(\rm II)} = 2\pi\sqrt{|nw\widehat{N}\widehat{W}|}, \tag{5.67}$$

while for corresponding 3-charge extremal black holes in $D=5$ (compactification on $S^1\times T^4$, NS-NS charged only) it is

$$S_{\rm bh}^{(\rm II)} = 2\pi\sqrt{|nwN|}. \tag{5.68}$$

The meaning of charges is the same as in the heterotic case. These results are expected from microscopic point of view.

6. R^2 actions with $\mathcal{N}=2$ off-shell SUGRA

6.1 Calabi-Yau compactifications of M-theory

Let us start with M-theory description of string theory, whose low energy effective action is 11-dimensional $N=1$ SUGRA which has maximal supersymmetry (32 generators). It can be consistently reduced to $D=5$ dimensions by Kaluza-Klein compactification on 6-dimensional Calabi-Yau 3-fold. One obtains $N=2$ 5-dimensional SUGRA with the bosonic part of lowest-order effective action given by

$$\begin{aligned}4\pi^2\mathscr{L}_0 \;&=\; 2\partial^a\mathscr{A}_i^{\alpha}\partial_a\mathscr{A}_{\alpha}^i+\mathscr{A}^2\left(\frac{D}{4}-\frac{3}{8}R-\frac{v^2}{2}\right)+\mathscr{N}\left(\frac{D}{2}+\frac{R}{4}+3v^2\right)+2\mathscr{N}_I v^{ab}F^I_{ab}\\ &\quad+\mathscr{N}_{IJ}\left(\frac{1}{4}F^I_{ab}F^{Jab}+\frac{1}{2}\partial_a M^I\partial^a M^J\right)+\frac{e^{-1}}{24}c_{IJK}A^I_aF^J_{bc}F^K_{de}\varepsilon^{abcde}\end{aligned} \tag{6.1}$$

R is Ricci scalar, $\mathscr{A}^2=\mathscr{A}_i^{\alpha}\mathscr{A}_{\alpha}^i$ and $v^2=v_{ab}v^{ab}$. $i=1,2$ is $SU(2)$, and $\alpha=1,2$ is $USp(2)$ index. Also,

$$\mathscr{N}=\frac{1}{6}c_{IJK}M^IM^JM^K,\quad \mathscr{N}_I=\partial_I\mathscr{N}=\frac{1}{2}c_{IJK}M^JM^K,\quad \mathscr{N}_{IJ}=\partial_I\partial_J\mathscr{N}=c_{IJK}M^K \tag{6.2}$$

M^I are moduli (volumes of $(1,1)$-cycles), and constants c_{IJK} as intersection numbers of Calabi-Yau space. Condition $\mathscr{N}=1$ is a condition of real special geometry.

The bosonic field content of the theory is the following. We have Weyl multiplet which contains the fünfbein e_μ^a, the two-form auxiliary field v_{ab}, and the scalar auxiliary field D. There are n_V vector multiplets enumerated by $I=1,\dots,n_V$, each containing the one-form gauge field A^I (with the two-form field strength $F^I=dA^I$), and the scalar M^I. Scalar fields $\mathscr{A}_{\alpha}^i$, which are belonging to the hypermultiplet, can be gauge fixed and the convenient choice is given by $\mathscr{A}^2=-2$, $\partial_a\mathscr{A}_i^{\alpha}=0$.

Action (6.1) is invariant under supersymmetry variations, which when acting on the purely bosonic configurations are given by

$$\begin{aligned}\delta\psi_\mu^i \;&=\; \mathscr{D}_\mu\varepsilon^i+\frac{1}{2}v^{ab}\gamma_{\mu ab}\varepsilon^i-\gamma_\mu\eta^i\\ \delta\xi^i \;&=\; D\varepsilon^i-2\gamma^c\gamma^{ab}\varepsilon^i\mathscr{D}_a v_{bc}-2\gamma^a\varepsilon^i\varepsilon_{abcde}v^{bc}v^{de}+4\gamma\cdot v\eta^i\\ \delta\Omega^{Ii} \;&=\; -\frac{1}{4}\gamma\cdot F^I\varepsilon^i-\frac{1}{2}\gamma^a\partial_a M^I\varepsilon^i-M^I\eta^i\\ \delta\zeta^{\alpha} \;&=\; \left(3\eta^j-\gamma\cdot v\varepsilon^j\right)\mathscr{A}_j^{\alpha}\end{aligned} \tag{6.3}$$

where ψ_μ^i is gravitino, ξ^i auxiliary Majorana spinor (Weyl multiplet), $\delta\Omega^{Ii}$ gaugino (vector multiplets), and ζ^{α} is a fermion field from hypermultiplet.

In [16] a four-derivative part of the action was constructed by supersymmetric completion of the mixed gauge-gravitational Chern-Simons term $A\wedge\mathrm{tr}(R\wedge R)$. The bosonic part of the action is

$$4\pi^2\mathscr{L}_1 \;=\; \frac{c_I}{24}\left\{\frac{e^{-1}}{16}\varepsilon_{abcde}A^{Ia}C^{bcfg}C^{de}{}_{fg}+M^I\left[\frac{1}{8}C^{abcd}C_{abcd}+\frac{1}{12}D^2-\frac{1}{3}C_{abcd}v^{ab}v^{cd}\right.\right.$$

$$+4v_{ab}v^{bc}v_{cd}v^{da} - (v_{ab}v^{ab})^2 + \frac{8}{3}v_{ab}\hat{\mathscr{D}}^b\hat{\mathscr{D}}_c v^{ac} + \frac{4}{3}\hat{\mathscr{D}}^a v^{bc}\hat{\mathscr{D}}_a v_{bc} + \frac{4}{3}\hat{\mathscr{D}}^a v^{bc}\hat{\mathscr{D}}_b v_{ca}$$
$$-\frac{2}{3}e^{-1}\varepsilon_{abcde}v^{ab}v^{cd}\hat{\mathscr{D}}_f v^{ef}\Big] + F^{Iab}\Big[\frac{1}{6}v_{ab}D - \frac{1}{2}C_{abcd}v^{cd} + \frac{2}{3}e^{-1}\varepsilon_{abcde}v^{cd}\hat{\mathscr{D}}_f v^{ef}$$
$$+e^{-1}\varepsilon_{abcde}v^c{}_f\hat{\mathscr{D}}^d v^{ef} - \frac{4}{3}v_{ac}v^{cd}v_{db} - \frac{1}{3}v_{ab}v^2\Big]\Big\} \tag{6.4}$$

where c_I are constant coefficients connected to second Chern class of Calabbi-Yau space, C_{abcd} is the Weyl tensor. $\hat{\mathscr{D}}_a$ is the conformal covariant derivative, which when appearing linearly in (6.4) can be substituted with ordinary covariant derivative $\mathscr{D}_a$, but when taken twice produces additional curvature contributions

$$v_{ab}\hat{\mathscr{D}}^b\hat{\mathscr{D}}_c v^{ac} = v_{ab}\mathscr{D}^b\mathscr{D}_c v^{ac} + \frac{2}{3}v^{ac}v_{cb}R^b_a + \frac{1}{12}v^2 R\,. \tag{6.5}$$

We are interested in extremal black hole solutions of the action obtained by combining (6.1) and (6.4):[17]

$$\mathscr{A} = \int dx^5\sqrt{-g}\mathscr{L} = \int dx^5\sqrt{-g}(\mathscr{L}_0 + \mathscr{L}_1) \tag{6.6}$$

The action (6.6) is quartic in derivatives and generally too complicated for finding complete analytical black hole solutions even in the simplest spherically symmetric case. Again, we shall concentrate just on near-horizon behavior and apply Sen's entropy function formalism. For spherically symmetric extremal black holes near-horizon geometry is expected to be $AdS_2\times S^3$, which has $SO(2,1)\times SO(4)$ symmetry. If the Lagrangian can be written in a manifestly diffeomorphism covariant and gauge invariant way, it is expected that near the horizon the complete background should respect this symmetry. In our case it means that near-horizon geometry should be given by

$$ds^2 = v_1\left(-x^2dt^2 + \frac{dx^2}{x^2}\right) + v_2\,d\Omega_3^2$$
$$F^I_{tr}(x) = -e^I\,, \qquad v_{tr}(x) = V\,, \qquad M^I(x) = M^I\,, \qquad D(x) = D \tag{6.7}$$

where $v_{1,2}$, e^I, M^I, V, and D are constants. All covariant derivatives are vanishing. Following standard procedure we define the entropy function

$$\mathscr{E} = 2\pi\left(q_I e^I - \mathscr{F}\right)\,, \tag{6.8}$$

where q_I are electric charges. $\mathscr{F}$ is given by

$$\mathscr{F} = \int_{S^3} dy^3\sqrt{-g}\,\mathscr{L}\,, \tag{6.9}$$

where right hand side is evaluated on the background (6.7). Explicitly,

$$\mathscr{F}_0 = \frac{1}{4}\sqrt{v_2}\Big[(\mathscr{N}+3)(3v_1 - v_2) - 4V^2(3\mathscr{N}+1)\frac{v_2}{v_1}$$
$$+8V\mathscr{N}_i e^i\frac{v_2}{v_1} - \mathscr{N}_{ij}e^ie^j\frac{v_2}{v_1} + D(\mathscr{N}-1)v_1v_2\Big] \tag{6.10}$$

[17] Our conventions in this section are different from the rest of the text. We take for Newton constant $G_5 = \pi^2/4$ and for the string tension $\alpha' = 1$.

and

$$\mathscr{F}_1 = v_1 v_2^{3/2} \left\{ \frac{c_I e^I}{48} \left[-\frac{4V^3}{3v_1^4} + \frac{DV}{3v_1^2} + \frac{V}{v_1^2} \left(\frac{1}{v_1} - \frac{1}{v_2} \right) \right] \right.$$
$$\left. + \frac{c_I M^I}{48} \left[\frac{D^2}{12} + \frac{4V^4}{v_1^4} + \frac{1}{4} \left(\frac{1}{v_1} - \frac{1}{v_2} \right)^2 - \frac{2V^2}{3v_1^2} \left(\frac{5}{v_1} + \frac{3}{v_2} \right) \right] \right\} , \tag{6.11}$$

Complete function $\mathscr{F}$ is a sum

$$\mathscr{F} = \mathscr{F}_0 + \mathscr{F}_1 . \tag{6.12}$$

Notice that for the background (6.7) all terms containing ε_{abcde} tensor vanish, *including* the mixed Chern-Simons term. This means that we do not have to worry about violation of manifest diffeomorphism invariance, and so we can use straightforwardly Sen's entropy function formalism.

Also notice that the entropy function $\mathscr{E}$ is invariant on the transformation defined with

$$q_I \to -q_I , \qquad e^I \to -e^I , \qquad V \to -V , \tag{6.13}$$

with other variables remaining the same. This symmetry follows from CPT invariance. We can use it to obtain new solutions which have opposite signs of all charges from the given one.

Equations of motion are obtained from extremization of $\mathscr{E}$

$$0 = \frac{\partial \mathscr{E}}{\partial v_1} , \quad 0 = \frac{\partial \mathscr{E}}{\partial v_2} , \quad 0 = \frac{\partial \mathscr{E}}{\partial M^I} , \quad 0 = \frac{\partial \mathscr{E}}{\partial V} , \quad 0 = \frac{\partial \mathscr{E}}{\partial D} , \quad 0 = \frac{\partial \mathscr{E}}{\partial e^I} . \tag{6.14}$$

while the black hole entropy is extremal value of $\mathscr{E}$, i.e.,

$$S_{\mathrm{bh}} = \mathscr{E}\big|_{\mathrm{EOM}} . \tag{6.15}$$

It is immediately obvious that though the system (6.14) is algebraic, it is in generic case too complicated to be solved in direct manner. One idea is to try to find some additional information. Such additional information can be obtained from supersymmetry. It is known that there should be 1/2 BPS black hole solutions, for which it was shown in [10] that near the horizon supersymmetry is enhanced fully. This means that in this case we can put all variations in (6.3) to zero, which one can use to express all unknowns in terms of one. As we have off-shell supersymmetry, variations (6.3) do not receive α'-corrections and so the results obtained in this way are the same as in the lowest-order calculation. Vanishing of $\delta\zeta^\alpha$ in (6.3) fixes the spinor parameter η to be

$$\eta^j = \frac{1}{3} (\gamma \cdot v) \varepsilon^j \tag{6.16}$$

Using this, and the condition that ε^i is (geometrical) Killing spinor, in the remaining equations one gets the following conditions

$$v_2 = 4v_1 , \qquad M^I = \frac{e^I}{\sqrt{v_1}} , \qquad D = -\frac{3}{v_1} , \qquad V = \frac{3}{4}\sqrt{v_1} \tag{6.17}$$

As moduli M^I are all by definition positive, from the second equation follows that all e^I must also be positive in this solution. We see that conditions for full supersymmetry are so constraining that they fix everything except one unknown, which we took above to be v_1. To fix it, we just need one equation from (6.14). In our case the simplest is to take equation for D, which gives

$$v_1^{3/2} = \frac{1}{6} c_{IJK} e^I e^J e^K - \frac{c_I e^I}{48} \tag{6.18}$$

We note that higher derivative corrections violate real special geometry condition, i.e., we have now $\mathcal{N} \neq 1$.[18]

As is typical, we are interested in expressing the results in terms of charges, not field strengths. The results can be put in particularly compact form by defining scaled moduli

$$\bar{M}^I \equiv \sqrt{v_1} M^I \,. \tag{6.19}$$

It can be shown [18] that solution for them is implicitly given by

$$8\, c_{IJK} \bar{M}^J \bar{M}^K = q_I + \frac{c_I}{8} \tag{6.20}$$

and that the black hole entropy (6.15) becomes

$$S_{\mathrm{bh}}^{(\mathrm{BPS})} = \frac{8\pi}{3} c_{IJK} \bar{M}^I \bar{M}^J \bar{M}^K \tag{6.21}$$

A virtue of this presentation is that if one is interested only in entropies, then it is enough to consider just equations (6.20) and (6.21). Unfortunately, for generic intersection numbers c_{IJK} it appears impossible to solve (6.20) explicitly and we do not have analytic expression for black hole entropy as function of electric charges q_I.

As for non-BPS solutions, it is not known how to preceede in generic case. However, as we show next, even in the two-derivative approximation (lowest order in α') solutions have been constructed only for some special compactifications.

6.2 Connection to heterotic string theory on $T^4 \times S^1$

It is known that M-theory compactified on $K3 \times T^2$ manifold is equivalent to heterotic string theory compactified on T^5 manifold. As $K3 \times T^2$ space has $SU(2)$ holonomy it breaks less supersymmetry then generic Calabi-Yau (which has $SU(3)$ holonomy) and that is why in this case we obtain $N = 4$ SUSY in $D = 5$ dimensions. In this case the non-vanishing components of c_{IJK} (up to permutation symmetry of indices) and c_I at tree-level are

$$c_{1ij} \equiv c_{ij} \,, \qquad c_1 = 24 \qquad i,j = 2,\ldots,23 \,, \tag{6.22}$$

where c_{ij} is a regular constant matrix whose inverse we denote as c^{ij}. The prepotential is now given by

$$\mathcal{N} = \frac{1}{2} M^1 c_{ij} M^i M^j \,, \qquad i,j = 2,\ldots,23 \,. \tag{6.23}$$

[18] We emphasize that one should be cautious in geometric interpretation of this result. Higher order corrections generally change relations between fields in the effective action and geometric moduli, and one needs field redefinitions to restore the relations. Then correctly defined moduli may still satisfy condition for real special geometry.

In this case, it is easy to show that the black hole entropy corresponding to the (now 1/4-) BPS solution defined by (6.17) and (6.20) is

$$S_{\rm bh}^{\rm (BPS)} = 2\pi\sqrt{\frac{1}{2}(q_1+3)\,q_i c^{ij} q_j}\,. \tag{6.24}$$

What is new here is that for the simpler form of prepotential (6.23) we can also treat (at least some) non-BPS solutions and obtain closed form expression for entropies [20]. As we cannot use BPS conditions here, the question is what can we use instead to simplify complicated system of equations. The idea comes from the observation that the BPS conditions implied that relations for v_2, D and V in (6.17) are uncorrected by higher-derivative terms in the action. Now, it can be shown that there are lowest order non-BPS near-horizon solutions which satisfy

$$v_2 = 4v_1\,, \qquad D = -\frac{1}{v_1}\,, \qquad V = \frac{1}{4}\sqrt{v_1}\,. \tag{6.25}$$

Let us now *assume* that all relations in (6.25) are unchanged under higher-derivative α'-corrections, and take them as an Ansatz.[19] After using Ansatz (6.25), and some additional manipulations, the equations of motion can be reduced to the following system (for general Calabi-Yau compactification)

$$0 = c_{IJK}\left(\bar{M}^J - e^J\right)\left(\bar{M}^K - e^K\right) \tag{6.26}$$

$$\frac{c_I \bar{M}^I}{12} = c_{IJK}\left(\bar{M}^I + e^I\right)\bar{M}^J e^K \tag{6.27}$$

$$v_1^{3/2} = \frac{c_I e^I}{144} - (e)^3 \tag{6.28}$$

$$q_I - \frac{c_I}{72} = -2\,c_{IJK} e^J e^K\,. \tag{6.29}$$

The above system is apparently overdetermined as there is one equation more than the number of unknowns. More precisely, Eqs. (6.26) and (6.27) should be compatible, and this is not happening for generic choice of parameters. So, we do not expect that our Ansatz will work for general Calabi-Yau compactifications (i.e., generic c_{IJK} and c_I).[20]

However, there are cases in which the system is regular and there are physically acceptable solutions. Important examples are prepotentials of the type (6.23). In this case (6.26) becomes

$$0 = \left(\bar{M}^1 - e^1\right)\left(\bar{M}^i - e^i\right)\,, \qquad 0 = \left(\bar{M}^i - e^i\right) c_{ij}\left(\bar{M}^j - e^j\right) \tag{6.30}$$

It is obvious that there is now at least one consistent solution obtained by taking $\bar{M}^i = e^i$ for all i. With this choice all equations in (6.30) are satisfied, and the remaining unknown scaled modulus $\bar{M}^1$ is fixed by Eq. (6.27). For the black hole entropy we obtain

$$S_{\rm bh}^{\rm (non-BPS)} = 2\pi\sqrt{\frac{1}{2}|\hat{q}_1| c^{ij}\hat{q}_i\hat{q}_j}\,, \qquad \hat{q}_I = q_I - \frac{c_I}{72} \tag{6.31}$$

[19] Inspection of equations of motion shows that at least some of relations between M^I and e^I receive α'-corrections and so we exclude them from Ansatz.

[20] Indeed, our efforts to find numerical solutions of the above system for random choices of c_{IJK}, c_I and e^I have all failed.

Again, the influence of higher-order supersymmetric correction is just to shift electric charges $q_I \to \hat{q}_I$, but with the different value for the shift constant than in BPS case.

For our exemplary case of 3-charge black holes of heterotic string theory compactified on $T^4 \times S^1$ we can obtain analytic expressions for near-horizon solutions and entropies for *all* sets of charges corresponding to large black holes. There is a basis in which (tree-level) prepotential (6.23) has the form

$$\mathcal{N} = M^1 M^2 M^3 + \frac{1}{2} M^1 c_{ab} M^a M^b \,, \qquad a,b = 4,\ldots,23\,. \tag{6.32}$$

We remind the reader that parameters c_I are given in (6.22). To obtain 3-charge solutions we take $q_a = 0$ for $a \geq 4$, which by using (6.29) gives $e^a = 0$ for $a \geq 4$. Analysis of systems of equations ((6.20) in BPS case, and (6.26)-(6.27) in non-BPS case) shows that equations for moduli $M^{2,3}$ decouple from those for M^a, $a \geq 4$ (which make singular system, regularized by quantum corrections of prepotential). It follows that for above 3-charge configurations we can effectively work with truncated theory where index $I = 1,2,3$ and prepotential now has simple so called STU-form

$$\mathcal{N} = M^1 M^2 M^3\,. \tag{6.33}$$

For this prepotential in [20] we constructed full set of near-horizon solutions of $N = 2$ supersymmetric R^2 action. We now present the results. As the theory is symmetric under exchange $I = 2$ and $I = 3$ indices,

BPS near-horizon solutions, with $q_1 \geq 0$, $q_{2,3} > 0$ (which satisfy 6.17)), are given by

$$v_1 = \frac{1}{4} \left| \frac{q_2 q_3 (q_1 + \zeta)^2}{q_1 + 3\zeta} \right|^{1/3} \tag{6.34}$$

$$\frac{e^1}{\sqrt{v_1^3}} (q_1 + 3\zeta) = \frac{e^2 q_2}{\sqrt{v_1^3}} = \frac{e^3 q_3}{\sqrt{v_1^3}} = 4 \frac{q_1 + 3\zeta}{q_1 + \zeta} \tag{6.35}$$

$$\frac{M^1 \sqrt{v_1}}{e^1} = \frac{M^2 \sqrt{v_1}}{e^2} = \frac{M^3 \sqrt{v_1}}{e^3} = 1\,, \tag{6.36}$$

together with (6.17). Solutions with $q_1 \leq 0$, $q_{2,3} < 0$, which are also BPS, are easily obtained by applying transformation (6.13). The entropy formula, valid for both type of BPS solutions, is given by

$$S_{\mathrm{bh}}^{(\mathrm{BPS})} = 2\pi \sqrt{|q_2 q_3| (|q_1| + 3)}\,. \tag{6.37}$$

For non-BPS solutions we have 6 possible combinations for picking signs of charges. By using transformation (6.13), and the symmetry of theory under exchange of indices $I = 2$ and $I = 3$, we are left with only two independent choices. We choose those which satisfy (6.25).

Non-BPS solutions with $q_{1,2} > 0$, $q_3 < 0$ are given by

$$v_1 = \frac{1}{4} \left| \frac{q_2 q_3 (q_1 + \zeta/3)^2}{q_1 - \zeta/3} \right|^{1/3} \tag{6.38}$$

$$\frac{e^1}{\sqrt{v_1^3}} \left(q_1 - \frac{\zeta}{3} \right) = \frac{e^2 q_2}{\sqrt{v_1^3}} = \frac{e^3 q_3}{\sqrt{v_1^3}} = 4 \frac{q_1 - \zeta/3}{q_1 + \zeta/3} \tag{6.39}$$

$$\frac{M^3 \sqrt{v_1}}{e^3} = -\frac{q_1 + \zeta}{q_1 - \zeta/3}\,, \qquad \frac{M^1 \sqrt{v_1}}{e^1} = \frac{M^2 \sqrt{v_1}}{e^2} = 1 \tag{6.40}$$

In the non-BPS case $q_{2,3} > 0$, $q_1 < -1$ the only difference from solution above is

$$\frac{M^1\sqrt{v_1}}{e^1} = -\frac{q_1-\zeta/3}{q_1+\zeta}\,, \qquad \frac{M^2\sqrt{v_1}}{e^2} = \frac{M^3\sqrt{v_1}}{e^3} = 1 \tag{6.41}$$

For both cases of above non-BPS solutions the black hole entropy is given by

$$S_{\rm bh}^{(\rm non-BPS)} = 2\pi\sqrt{|q_2 q_3(q_1-1/3)|}\,. \tag{6.42}$$

Before commenting our solutions, it is necessary to make connection to notation used in previous sections. It is known (see, e.g., section 5.1 of [20]) at lowest order (two-derivative) supersymmetric action (6.1) with STU prepotential (6.33) can be put (by making Poincare duality transformation on A^1 gauge field, and then going from Einstein- to string-frame metric) in the form of the heterotic effective action (2.21), where[21]

$$M^1 = S^{2/3}\,, \qquad M^2 = S^{-1/3}T^{-1}\,, \qquad M^3 = S^{-1/3}T \tag{6.43}$$

In addition, connection between gauge fields in two formulations is such that

$$q_1 = m\,, \qquad q_2 = n\,, \qquad q_3 = w\,, \tag{6.44}$$

Using (6.44) in the expressions for black hole entropies (6.37) and (6.42), we obtain in the BPS cases

$$S_{\rm bh}^{(\rm BPS)} = 2\pi\sqrt{|nw|(|m|+3)}\,, \tag{6.45}$$

while in the non-BPS cases

$$S_{\rm bh}^{(\rm non-BPS)} = 2\pi\sqrt{|nw(m-1/3)|}\,. \tag{6.46}$$

For BPS black holes entropy (6.45) *exactly* matches our previous result (5.58) obtained from full heterotic effective action (which matches microscopic result, after taking care of "near-horizon shift" of NS5-brane charge m, see (5.62)). Not only that, but if we use (6.43) to identify[22] $T = (M^1)^{-1/2}(M^2)^{-1}$ and $S = (M^1)^{3/2}$, after taking care of differences in conventions and normalizations we obtain that complete near-horizon solution is in fact *equal* to the solution given in (5.57).

However, in non-BPS sector things are completely different. Comparison of (6.46) with the corresponding entropy obtained from full heterotic effective action given in (5.60), shows complete disagreement (starting already at first order in α').

[21] Note that relations in (6.43) satisfy real geometry condition $\mathcal{N} = M^1M^2M^3 = 1$, which we showed to be violated by higher-derivative terms. This means that interpretations of fields in effective action as geometric moduli of compactification manifold are receiving corrections anf one has to be careful in such identifications.

[22] It is interesting that choice $T = (M^3/M^2)^{1/2}$ (which gives different result when higher-derivative corrections are includeed) leads to $u_T = \sqrt{n/w}$, which means that this could be a correct identification with heterotic string compactification modulus (radius of S^1) [21].

6.3 R^2 supersymmetric actions in $D = 4$

In $D = 4$ dimensions it is also known how to construct off-shell $N = 2$ supersymmetric action with R^2 terms. This action can be used to find analytic near-horizon solutions for BPS spherically symetric extremal black holes. Instead of going through the details of calculation, this time we shall simply present results for entropies for object of our interest, i.e., 4-charge black holes in theory with prepotential corresponding to tree-level heterotic string theory compactified on $T^4 \times \widehat{S}^1 \times S^1$ (whose lowest order solution is discused in section 2.3.3. The details can be found in review [12]. Again, it should be emphasized that the heterotic theory has larger $N = 4$ supersymmetry.

For BPS black holes, whose charges satisfy $nw > 0$, $\widehat{N}\widehat{W} \geq 0$, the black hole entropy is

$$S_{\mathrm{bh}}^{(\mathrm{BPS})} = 2\pi\sqrt{nw(\widehat{N}\widehat{W} + 4)}\,, \tag{6.47}$$

which agrees with result obtained from full heterotic action and with microscopic entropy.

In the case of non-BPS black holes, no analytic results are known. For the case $n < 0$, $w, \widehat{N}, \widehat{W} > 0$, perturbative calculation was performed with the result [15]

$$S_{\mathrm{bh}}^{(\mathrm{non-BPS})} = 2\pi\sqrt{|n|w\widehat{N}\widehat{W}}\left(1 + \frac{5}{8}\frac{1}{\widehat{N}\widehat{W}} - 29\frac{1}{(\widehat{N}\widehat{W})^2} - 1904\frac{1}{(\widehat{N}\widehat{W})^3} + \ldots\right), \tag{6.48}$$

Comparison with result obtained from full heterotic action (which agrees with microscopic result (3.18)) shows disagreement already at lowest α'^1-correction (instead of $5/8$ it should be 1).

6.4 Comments on R^2 supersymmetric actions

1. In the case of R^2 supersymmetric actions in $D = 4$ and $D = 5$ full solutions (in the whole space, not just near-horizon) for BPS black holes where constructed explicitly up to one function (which is satisfying ordinary differential equation). However, the method of construction is not working in non-BPS cases [70].

2. One can obtain closed form results also for generic c_I's.

3. Why such R^2 actions are working α'-exactly for BPS black holes, when it is evident from perturbative results that these actions are incomplete already at 4-derivative (α'^1) order? It has been frequently claimed that this is a consequence of $\mathrm{AdS}_3/\mathrm{CFT}_2$ duality, and a property that in three-dimensional $N = 2$ supersymmetric gravities in AdS_3 the only non-trivial higher-derivative corrections are Chern-Simons terms. However, this is insufficient to prove the statement.

7. Lovelock-type actions

7.1 Pure Gauss-Bonnet correction

Stimulated by successes of supersymmetric R^2-truncated actions in describing BPS black holes in $D = 4$ and 5 in α'-exact manner, the natural question is can the same be obtained with some ever simpler actions. In $D = 4$ the simplest choice is to take for higher-derivative correction just the pure Gauss-Bonnet density. This means that we start with "toy" effective action in D-dimensions given by

$$\mathscr{A} = \mathscr{A}_0 + \mathscr{A}_{\rm GB}\,, \tag{7.1}$$

where $\mathscr{A}_0$ is a corresponding lowest-order (2-derivative) action, and $\mathscr{A}_{\rm GB}$ is a 4-derivative α'^1-correction given by[23]

$$\mathscr{A}_{GB} = \frac{1}{32\pi}\frac{1}{8}\int d^Dx\sqrt{-G}\,S\left(R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2\right) \tag{7.2}$$

This choice has two notable properties:

1. Such term appears in heterotic effective action on 4-derivative ($\alpha'1$) level (but note there are also other 4-derivative terms).

2. By itself Gauss-Bonnet density is topological in $D = 4$, which means that it is giving contribution to equations of motion just because it is multiplied by the dilaton field S in the Lagrangian. Because of this it gives the simplest contribution to entropy function compared with other 4-derivative possibilities.

3. It produces normal second-order field equations in all dimensions.

Property 1 suggests that the 4-dimensional case is probably simplest to treat, so we start with our 4-charge near-horizon geometries first. In this case we already know that $\mathscr{A}_0$ is given in (2.28). We are interested in near-horizon geometry for which already learned it has to have $\mathrm{AdS}_2 \times S^2$ form, i.e.,

$$\begin{aligned}
& ds^2 = v_1\left(-r^2dt^2 + \frac{dr^2}{r^2}\right) + v_2(d\theta^2 + \sin^2\theta\, d\phi^2)\,, \\
& S = u_S\,, \qquad T = u_1\,, \qquad \widehat{T} = u_2\,, \\
& F^{(1)}_{rt} = e_1\,, \quad F^{(3)}_{rt} = e_3\,, \quad F^{(2)}_{\theta\phi} = \frac{p_2}{4\pi}\,, \quad F^{(4)}_{\theta\phi} = \frac{p_4}{4\pi}\,.
\end{aligned} \tag{7.3}$$

The function $\mathscr{F}$ has two contributions

$$\begin{aligned}
\mathscr{F}_0(\vec{u},\vec{v},\vec{e},\vec{p}) &\equiv \int d\theta d\phi\,\sqrt{-G}\,\mathscr{L}_0 \\
&= \frac{1}{8}v_1v_2u_S\left[-\frac{2}{v_1} + \frac{2}{v_2} + \frac{2u_1^2e_1^2}{v_1^2} + \frac{2e_3^2}{u_1^2v_1^2} - \frac{u_2^2p_2^2}{8\pi^2v_2^2} - \frac{u_2^{-2}p_4^2}{8\pi^2v_2^2}\right]\,,
\end{aligned} \tag{7.4}$$

and

$$\mathscr{F}_{\rm GB}(\vec{u},\vec{v},\vec{e},\vec{p}) \equiv \int d\theta d\phi\,\sqrt{-G}\,\mathscr{L}_0 = -2u_S\,. \tag{7.5}$$

By using formalism developed in section 4.2 one easily obtains near-horizon solutions [42]

$$\begin{aligned}
& v_1 = v_2 = 4|\widehat{N}\widehat{W}| + 8\,, \qquad u_S = \sqrt{\frac{|nw|}{|\widehat{N}\widehat{W}| + 4}} \\
& u_1 = \sqrt{\left|\frac{n}{w}\right|}\,, \qquad u_2 = \sqrt{\left|\frac{\widehat{W}}{\widehat{N}}\right|} \\
& e_1 = \frac{1}{n}\sqrt{|nw|(|\widehat{N}\widehat{W}| + 4)}\,, \qquad e_3 = \frac{1}{w}\sqrt{|nw|(|\widehat{N}\widehat{W}| + 4)}\,.
\end{aligned} \tag{7.6}$$

[23]The conventions here are (again) $G_N = 2$, $\alpha' = 16$.

Results are expressed in terms of charges

$$n = 2q_1 , \quad w = 2q_3 , \quad \widehat{N} = 4\pi p_2 , \quad \widehat{W} = 4\pi p_4 . \tag{7.7}$$

where (see appendix [42]), w and n are winding and momentum numbers along S^1, and $\widehat{N}$ and $\widehat{W}$ are numbers of Kaluza-Klein monopoles and NS5-branes wrapped around $T^4 \times S^1$.

For the black hole entropy one obtains

$$S_{\rm bh} = 2\pi \sqrt{|nw|(|\widehat{N}\widehat{W}| + 4)} \tag{7.8}$$

We see that for BPS black holes satisfying $nw > 0$, $\widehat{N}\widehat{W} \geq 0$ Wald entropy (7.8) α'-exactly matches the microscopic statistical entropy (3.11). That this is not coincidental shows comparison of the near-horizon solution (7.6) with the corresponding solution obtained from α'-complete heterotic effective action presented in (5.42) - the only differences are the expressions for two radii $u_{1,2}$, which in (7.6) manifestly satisfy known T-dualities. Thus we expect that the solutions are equivalent, and the difference can be attributed to field redefinitions.

For non-BPS black holes entropy (7.8) obviously differs from microscopic statistical results.

What about 3-charge heterotic black holes in $D = 5$? As shown in section 7 of Ref. [20], for large black holes (all three charges n, w, N nonvanishing) one gets the entropy which differs from statistical results, though in BPS case ($n, w, N > 0$) the result gives correct first α'-correction. For small BPS black hole, given by $N = 0$ and $nw > 0$, we again obtain Wald entropy agreeing with statistical result

$$S_{\rm stat}^{\rm (BPS)} = 4\pi \sqrt{nw} . \tag{7.9}$$

Now, the real question here is *why the sole Gauss-Bonnet correction, which is not complete correction to effective actions even at α'^1-order, in some cases gives the results which are α'-exact?* This question still begs for an answer.

7.2 Small black holes in general dimensions

We have observe in previous subsections that adding just the Gauss-Bonnet term as higher-derivative correction to lowest order heterotic effective actions is producing correct results for the entropy of small BPS black holes in $D = 4$ and 5. What about $D > 5$? We start from heterotic string compactified on $S^1 \times T^{9-D}$ and wounded on S^1. Taking as nonvanishing charges only winding w and momentum n (both on S^1) we obtain, in analogy to (2.4), the truncated D-dimensional effective action which is at lowest order given by

$$\mathscr{A}_0 = \frac{1}{32} \int d^D x \sqrt{-G}\, S \left[R + S^{-2} (\partial_\mu S)^2 - T^{-2} (\partial_\mu T)^2 - T^2 (F^{(1)}_{\mu\nu})^2 - T^{-2} (F^{(2)}_{\mu\nu})^2 \right] , \tag{7.10}$$

where T is the modulus of S^1.

For the higher-derivative terms we could try again with pure Gauss-Bonnet correction, i.e., with action of the form (7.1)-(7.2). However, it was shown in [58] that this works just for four- and five-dimensional small BPS black holes, while for $D > 5$ one gets Wald entropy different from (7.9). But, in $D = 6$ we note that another Euler density appears, which is of 6-derivative type. More general, in D dimensions there are $[D/2]$ different generalized Euler densities[24] E_n

$$E_n = \frac{1}{2^n}\,\delta^{\rho_1\sigma_1\ldots\rho_m\sigma_m}_{\mu_1\nu_1\ldots\mu_m\nu_m}\,R^{\mu_1\nu_1}{}_{\rho_1\sigma_1}\cdots R^{\mu_m\nu_m}{}_{\rho_m\sigma_m}\,, \qquad n=2,\ldots,[D/2] \tag{7.11}$$

where $\delta^{\beta_1\ldots\beta_k}_{\alpha_1\ldots\alpha_k}$ is totally antisymmetric product of k Kronecker deltas, normalized to take values 0 and ±1, and $[x]$ denote integer part of x. Normalization in (7.11) is such that $E_1 = R$ and $E_2 = (R_{\mu\nu\rho\sigma})^2 - 4(R_{\mu\nu})^2 + R^2$. Euler densities E_n are in many respects generalisation of the Einstein term. Especially, E_n is a topological density in $D = 2n$ dimensions. Also note that E_n vanish identically for $m > [D/2]$.

We now see that in $D > 5$ instead of pure Gauss-Bonnet type action it is more "natural" to consider more general Lovelock type action[25]

$$\mathscr{A} = \mathscr{A}_0 + \sum_{n=2}^{[D/2]} \lambda_n \int d^D x \sqrt{-g}\, S\, E_n\,. \tag{7.12}$$

where λ_n are some so far undetermined coefficients, and $\mathscr{A}_0$ is given in (7.10). It is obvious that n-th term consists of $2n$-derivative terms, i.e., it describes α'^{n-1}-correction. In $D = 4$ and 5 (7.12) reduces to the form (7.1). Of course, the action (7.12) is not (truncated) effective action of heterotic string theory. We shell return to this point later. Actions (7.12) have many attractive and notable properties, e.g., they lead to normal second order equations of motion.

In [50] it was shown that there is *unique fixed* choice of coefficients λ_n which does the job we are seeking to – if we take

$$\lambda_n = \frac{1}{4^{n-1} n!}\,, \tag{7.13}$$

in action (7.12) then small extremal 2-charge black hole solutions have the Wald entropy given by [50]

$$S_{\mathrm{bh}} = 4\pi\sqrt{|nw|}\,, \tag{7.14}$$

for all number of dimensions D. Evidently, (7.14) agrees with the microscopic statistical result of string theory in BPS case $nw > 0$ (7.9) (and again differs in non-BPS cases), but now for all D.

Let us also present near-horizon solutions. In D dimensions we expect geometry to be of the $\mathrm{AdS}_2 \times S^{D-2}$ form, so we make the following Ansatz

$$\begin{aligned} ds^2 &= v_1\left(-r^2 dt^2 + \frac{dr^2}{r^2}\right) + v_2\, d\Omega_{D-2}\,, \\ S &= u_S\,, \quad T = u_T\,, \quad F^{(1)}_{rt} = e_1\,, \quad F^{(2)}_{rt} = e_2\,. \end{aligned} \tag{7.15}$$

where $d\Omega_k$ denotes standard metric on the unit k-dimensional sphere, and $v_{1,2}$, $u_{S,T}$, and $e_{1,2}$ are constants to be determined from equations of motion. Using as before entropy function formalism

[24] Sometimes called extended Gauss-Bonnet densities.

[25] Originally Lovelock [69] considered pure gravity actions without dilaton field. It is easy to check that multiplication by scalar field does not change any of important properties of Lovelock actions, except that term containing $E_{D/2}$ (for D even) becomes non-topological.

we obtain the following near-horizon solution

$$v_1 = 8\,, \qquad v_2 = f(D)\,, \qquad u_S = \frac{16\pi}{\Omega_{D-2} f(D)}\sqrt{|nw|}$$
$$u_T = \sqrt{\left|\frac{n}{w}\right|}\,, \qquad e_1 = \frac{2}{n}\sqrt{|nw|}\,, \qquad e_2 = \frac{2}{w}\sqrt{|nw|}\,, \tag{7.16}$$

where $f(D)$ is the real positive root[26] of a particular $[D/2]$-th order polynomial. We note that the AdS_2 radius is $\ell_A \equiv \sqrt{v_1} = \sqrt{\alpha'/2}$ for all D.

7.3 Comments on Lovelock-type action

Some questions and comments on Lovelock-type action defined by (7.12)-(7.13):

1. Why this action works just for small BPS black holes in $D > 4$ (and large and small BPS black holes in $D = 4$ where it reduces to the pure Gauss-Bonnet type)? Again, the real question here is why should it work for any type at all. Still unknown.

2. What is the connection between this action and the low energy effective action of the heterotic string theory (HLEEA)? It is obviously much simpler then HLEEA, as, e.g., it has finite number of higher-derivative corrections which are purely gravitational. However, there are some notable similarities. HLEEA also contains Gauss-Bonnet term (i.e., second Euler density E_2) with the same coefficient $\lambda_2 = 1/8$. The same is also true for the 8-derivative term proportional to E_4, where coefficient is $\lambda_4 = 1/1526$. But, something odd is happening with the E_3 term. It is completely absent in HLEEA, while in our action it appears with $\lambda_3 = 1/96$. It is amusing that this terms, with the same coefficient, appears in bosonic string theory. *It is important to keep in mind that for small black holes, which are intrinsically stringy, one does not expect for low energy/curvature effective action to be usable.*

3. Is this action just a trival construct to obtain (7.14), void of any other meaning? We believe not, for the following reasons. First, note that this action is unique, with the same form for all number of dimensions D, and with coefficients in front of higher-derivative terms looking stringy (as discussed above). Second, note that fixing of coefficients λ_n is not "one for one", but "one for two". Fixing of λ_2 has to work *simultaneously* in $D = 4$ and $D = 5$, then fixing of λ_2 has to do the job both in $D = 6$ and $D = 7$, etc. Also, as discussed in section 7.1, it works also for 8-charge *large* black holes in $D = 4$ (where it boils down to Gauss-Bonnet type action).

4. What then could be the meaning of this action? One of the most attractive possibilities is that it describes some new type of effectiveness in string theory, present in black hole near-horizon analyses.

5. What about corresponding 2-charge small black holes in type-IIA theory compactified on $S^1 \times T^{9-D}$, which are 1/4-BPS states? In this case it is known that string microstate counting gives for statistical entropy
$$S_{\mathrm{small}} = 2\sqrt{2}\pi\sqrt{|nw|}\,. \tag{7.17}$$

[26] We numerically checked up to $D = 9$ that there is unique real positive root.

PoS(BHs, GR and Strings)033

Formally, one can obtain this result in all D by taking for coefficients λ_n in (7.12)

$$\lambda_n^{(\mathrm{II})} = \frac{\lambda_n^{(\mathrm{het})}}{2} = \frac{2}{4^n n!} \,. \tag{7.18}$$

From this follows that AdS$_2$ radius is now $\ell_A = \sqrt{\alpha'}/2$. Note that LEEA of type-II string theory does not contain Gauss-Bonnet, nor any other 4-derivative term.

References

[1] P. Dominis Prester and T. Terzić, "α'-exact entropies for BPS and non-BPS extremal dyonic black holes in heterotic string theory from ten-dimensional supersymmetry," JHEP **0812** (2008) 088 [arXiv:0809.4954 [hep-th]].

[2] A. W. Peet, "Entropy And Supersymmetry Of D-Dimensional Extremal Electric Black Holes Versus String States," Nucl. Phys. B **456** (1995) 732 [arXiv:hep-th/9506200].

[3] D. Youm, "Black holes and solitons in string theory," Phys. Rept. **316** (1999) 1 [arXiv:hep-th/9710046].

[4] J. D. Brown and M. Henneaux, "Central Charges in the Canonical Realization of Asymptotic Symmetries: An Example from Three-Dimensional Gravity," Commun. Math. Phys. **104** (1986) 207.

[5] O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri and Y. Oz, "Large N field theories, string theory and gravity," Phys. Rept. **323** (2000) 183 [arXiv:hep-th/9905111].

[6] B. L. Cerchiai, S. Ferrara, A. Marrani and B. Zumino, "Duality, Entropy and ADM Mass in Supergravity," arXiv:0902.3973 [hep-th].

[7] S. Ferrara and J. M. Maldacena, "Branes, central charges and U-duality invariant BPS conditions," Class. Quant. Grav. **15** (1998) 749 [arXiv:hep-th/9706097].

[8] G. Guralnik, A. Iorio, R. Jackiw and S. Y. Pi, "Dimensionally reduced gravitational Chern-Simons term and its kink," Annals Phys. **308** (2003) 222 [arXiv:hep-th/0305117].

[9] B. Sahoo and A. Sen, "BTZ black hole with Chern-Simons and higher derivative terms," JHEP **0607** (2006) 008 [arXiv:hep-th/0601228].

[10] A. H. Chamseddine, S. Ferrara, G. W. Gibbons and R. Kallosh, "Enhancement of supersymmetry near 5d black hole horizon," Phys. Rev. D **55**, 3647 (1997). [arXiv:hep-th/9610155].

[11] Y. Tachikawa, "Black hole entropy in the presence of Chern-Simons terms," Class. Quant. Grav. **24** (2007) 737 [arXiv:hep-th/0611141].

[12] A. Sen, "Black Hole Entropy Function, Attractors and Precision Counting of Microstates," arXiv:0708.1270 [hep-th].

[13] G. Lopes Cardoso, B. de Wit and T. Mohaupt, "Corrections to macroscopic supersymmetric black-hole entropy," Phys. Lett. B **451** (1999) 309 [arXiv:hep-th/9812082].

[14] K. Behrndt, G. Lopes Cardoso and S. Mahapatra, "Exploring the relation between 4D and 5D BPS solutions," Nucl. Phys. B **732** (2006) 200 [arXiv:hep-th/0506251].

[15] B. Sahoo and A. Sen, "Higher derivative corrections to non-supersymmetric extremal black holes in $N = 2$ supergravity," JHEP **0609** (2006) 029 [arXiv:hep-th/0603149].

[16] K. Hanaki, K. Ohashi and Y. Tachikawa, "Supersymmetric Completion of an R^2 Term in Five-Dimensional Supergravity," Prog. Theor. Phys. **117** (2007) 533 [arXiv:hep-th/0611329].

[17] A. Castro, J. L. Davis, P. Kraus and F. Larsen, "String Theory Effects on Five-Dimensional Black Hole Physics," arXiv:0801.1863 [hep-th].

[18] A. Castro, J. L. Davis, P. Kraus and F. Larsen, "5D Black Holes and Strings with Higher Derivatives," JHEP **0706** (2007) 007 [arXiv:hep-th/0703087].

[19] M. Alishahiha, "On R^2 corrections for 5D black holes," JHEP **0708** (2007) 094 [arXiv:hep-th/0703099].

[20] M. Cvitan, P. Dominis Prester, S. Pallua and I. Smolić, "Extremal black holes in $D = 5$: SUSY vs. Gauss-Bonnet corrections," JHEP **0711** (2007) 043 [arXiv:0706.1167 [hep-th]].

[21] M. Cvitan, P. Dominis Prester, A. Ficnar, S. Pallua and I. Smolić, "Five-dimensional black holes in heterotic string theory," Fortsch. Phys. **56** (2008) 406 [arXiv:0711.4962 [hep-th]].

[22] P. Kraus and F. Larsen, "Microscopic black hole entropy in theories with higher derivatives," JHEP **0509** (2005) 034 [arXiv:hep-th/0506176].

[23] J. R. David, B. Sahoo and A. Sen, "AdS_3, Black Holes and Higher Derivative Corrections," JHEP **0707** (2007) 058 [arXiv:0705.0735 [hep-th]].

[24] P. Kaura and B. Sahoo, "Boundary S-matrix in a $(2,0)$ theory of AdS_3 Supergravity," arXiv:0809.4603 [hep-th].

[25] B. Sahoo and A. Sen, "α'-Corrections to Extremal Dyonic Black Holes in Heterotic String Theory," JHEP **0701** (2007) 010 [arXiv:hep-th/0608182].

[26] M. Cvitan, P. Dominis Prester and A. Ficnar, "α'^2-corrections to extremal dyonic black holes in heterotic string theory," JHEP **0805** (2008) 063 [arXiv:0710.3886 [hep-th]].

[27] E. A. Bergshoeff and M. de Roo, "The Quartic Effective Action Of The Heterotic String And Supersymmetry," Nucl. Phys. B **328** (1989) 439.

[28] W. A. Chemissany, M. de Roo and S. Panda, "α'-Corrections to Heterotic Superstring Effective Action Revisited," JHEP **0708** (2007) 037 [arXiv:0706.3636 [hep-th]].

[29] R. R. Metsaev and A. A. Tseytlin, "Order α' (Two Loop) Equivalence of the String Equations of Motion and the Sigma Model Weyl Invariance Conditions: Dependence on the Dilaton and the Antisymmetric Tensor," Nucl. Phys. B **293** (1987) 385.

[30] D. J. Gross and J. H. Sloan, "The Quartic Effective Action for the Heterotic String," Nucl. Phys. B **291** (1987) 41.

[31] G. Policastro and D. Tsimpis, "R^4, purified," Class. Quant. Grav. **23** (2006) 4753 [arXiv:hep-th/0603165].

[32] A. Kehagias and H. Partouche, "On the exact quartic effective action for the type IIB superstring," Phys. Lett. B **422** (1998) 109 [arXiv:hep-th/9710023].

[33] S. Frolov, I. R. Klebanov and A. A. Tseytlin, "String corrections to the holographic RG flow of supersymmetric $SU(N) \times SU(N+M)$ gauge theory," Nucl. Phys. B **620** (2002) 84 [arXiv:hep-th/0108106].

[34] K. Peeters, P. Vanhove and A. Westerberg, "Chiral splitting and world-sheet gravitinos in higher-derivative string amplitudes," Class. Quant. Grav. **19** (2002) 2699 [arXiv:hep-th/0112157].

[35] D. M. Richards, "The One-Loop H^2R^3 and $H^2(DH)^2R$ Terms in the Effective Action," arXiv:0807.3453 [hep-th].

[36] H. Ooguri, A. Strominger and C. Vafa, "Black hole attractors and the topological string," Phys. Rev. D **70** (2004) 106007 [arXiv:hep-th/0405146].

[37] H. K. Kunduri, J. Lucietti and H. S. Reall, "Near-horizon symmetries of extremal black holes," Class. Quant. Grav. **24** (2007) 4169 [arXiv:0705.4214 [hep-th]].

[38] D. Astefanesei and H. Yavartanoo, "Stationary black holes and attractor mechanism," Nucl. Phys. B **794** (2008) 13 [arXiv:0706.1847 [hep-th]].

[39] K. Goldstein and R. P. Jena, "One entropy function to rule them all," JHEP **0711** (2007) 049 [arXiv:hep-th/0701221].

[40] A. Sen, "Black hole entropy function and the attractor mechanism in higher derivative gravity," JHEP **0509** (2005) 038 [arXiv:hep-th/0506177].

[41] R. M. Wald, "Black hole entropy is the Noether charge," Phys. Rev. D **48** (1993) 3427 [arXiv:gr-qc/9307038];
V. Iyer and R. M. Wald, "Some properties of Noether charge and a proposal for dynamical black hole entropy," Phys. Rev. D **50** (1994) 846 [arXiv:gr-qc/9403028].

[42] A. Sen, "Entropy function for heterotic black holes," JHEP **0603** (2006) 008 [arXiv:hep-th/0508042].

[43] M. Cvetic and D. Youm, "Dyonic BPS saturated black holes of heterotic string on a six torus," Phys. Rev. D **53** (1996) 584 [arXiv:hep-th/9507090].

[44] M. Cvetic and A. A. Tseytlin, "Solitonic strings and BPS saturated dyonic black holes," Phys. Rev. D **53** (1996) 5619 [Erratum-ibid. D **55** (1997) 3907] [arXiv:hep-th/9512031].

[45] A. A. Tseytlin, "Extreme dyonic black holes in string theory," Mod. Phys. Lett. A **11** (1996) 689 [arXiv:hep-th/9601177].

[46] D. Kutasov, F. Larsen and R. G. Leigh, "String theory in magnetic monopole backgrounds," Nucl. Phys. B **550** (1999) 183 [arXiv:hep-th/9812027].

[47] P. Kraus, "Lectures on black holes and the AdS(3)/CFT(2) correspondence," arXiv:hep-th/0609074.

[48] S. Hohenegger, C. A. Keller and I. Kirsch, "Heterotic AdS_3/CFT_2 duality with $(0,4)$ spacetime supersymmetry," Nucl. Phys. B **804** (2008) 193 [arXiv:0804.4066 [hep-th]].

[49] A. Castro, J. L. Davis, P. Kraus and F. Larsen, "5D attractors with higher derivatives," JHEP **0704** (2007) 091 [arXiv:hep-th/0702072].

[50] P. Prester, "Lovelock type gravity and small black holes in heterotic string theory," JHEP **0602** (2006) 039 [arXiv:hep-th/0511306].

[51] J. L. Cardy, "Operator Content Of Two-Dimensional Conformally Invariant Theories," Nucl. Phys. B **270** (1986) 186;
H. W. J. Bloete, J. L. Cardy and M. P. Nightingale, "Conformal Invariance, the Central Charge, and Universal Finite Size Amplitudes at Criticality," Phys. Rev. Lett. **56** (1986) 742.

[52] M. R. Garousi and A. Ghodsi, "On Attractor Mechanism and Entropy Function for Non-extremal Black Holes/Branes," JHEP **0705** (2007) 043 [arXiv:hep-th/0703260].

[53] F. W. Shu and X. H. Ge, "Entropy function and higher derivative corrections to entropies in (anti-)de Sitter space," JHEP **0808** (2008) 021 [arXiv:0804.2724 [hep-th]].

[54] P. Dominis Prester, to appear.

[55] J. T. Liu and P. Szepietowski, "Higher derivative corrections to R-charged AdS_5 black holes and field redefinitions," arXiv:0806.1026 [hep-th].

[56] R. G. Cai and D. W. Pang, "Entropy function for 4-charge extremal black holes in type IIA superstring theory," Phys. Rev. D **74** (2006) 064031 [arXiv:hep-th/0606098].

[57] L. Cornalba, M. S. Costa, J. Penedones and P. Vieira, "From fundamental strings to small black holes," JHEP **0612** (2006) 023 [arXiv:hep-th/0607083].

[58] A. Sen, "Stretching the horizon of a higher dimensional small black hole," JHEP **0507** (2005) 073 [arXiv:hep-th/0505122].

[59] A. Sen, "How does a fundamental string stretch its horizon?," JHEP **0505** (2005) 059 [arXiv:hep-th/0411255].

[60] A. Dabholkar, R. Kallosh and A. Maloney, "A stringy cloak for a classical singularity," JHEP **0412** (2004) 059 [arXiv:hep-th/0410076].

[61] V. Hubeny, A. Maloney and M. Rangamani, "String-corrected black holes," JHEP **0505** (2005) 035 [arXiv:hep-th/0411272].

[62] A. Dabholkar and S. Murthy, "Fundamental Superstrings as Holograms," JHEP **0802** (2008) 034 [arXiv:0707.3818 [hep-th]].

[63] J. M. Lapan, A. Simons and A. Strominger, "Nearing the Horizon of a Heterotic String," arXiv:0708.0016 [hep-th].

[64] P. Kraus, F. Larsen and A. Shah, "Fundamental Strings, Holography, and Nonlinear Superconformal Algebras," JHEP **0711** (2007) 028 [arXiv:0708.1001 [hep-th]].

[65] M. Alishahiha, F. Ardalan, H. Ebrahim and S. Mukhopadhyay, "On 5D Small Black Holes," JHEP **0803** (2008) 074 [arXiv:0712.4070 [hep-th]].

[66] M. Alishahiha and S. Mukhopadhyay, "On Six Dimensional Fundamental Superstrings as Holograms," arXiv:0803.0685 [hep-th].

[67] M. J. Duff, "Near-horizon brane-scan revived," arXiv:0804.3675 [hep-th].

[68] L. Y. Hung and L. I. Uruchurtu, "Type II Small Stringy Black Holes, Probe Branes and Higher Derivative Interactions," JHEP **0803** (2008) 043 [arXiv:0712.2860 [hep-th]].

[69] D. Lovelock, "The Einstein tensor and its generalizations," J. Math. Phys. **12** (1971) 498.

[70] A. Shah, "A Note on Non-BPS Black Holes in 5D," arXiv:0903.2380 [hep-th].

[71] S. Bellucci, S. Ferrara, A. Shcherbakov and A. Yeranyan, "Black hole entropy, flat directions and higher derivatives," arXiv:0906.4910 [hep-th].