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Corrections to the Bekenstein-Hawking entropy in the brick wall approach

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The brick wall model is a semi-classical approach to understand the microscopic origin of black hole entropy. In this approach, the entropy of the black hole arises due to the canonical entropy of matter fields outside the black hole event horizon, evaluated at the Hawking temperature. Usually, in the brick wall model, the density of states and the resulting canonical entropy of the matter fields are calculated at the leading order (in terms of $\hbar$) in the WKB approximation. We extend this approach and compute the brick wall entropy of a quantum scalar field around static and spherically symmetric black holes at the higher orders in the WKB approximation. We find that the brick wall model generally predicts corrections to the Bekenstein-Hawking entropy in all spacetime dimensions. We compare our results with the sub-leading contributions to the black hole entropy that has been obtained from various other approaches.

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1. Introduction

Black hole entropy assumes considerable importance due to the fact that it may provide us with an insight to the microscopic structure of the gravitational theory through the microcanonical, Boltzmann relation $S = (k_{\rm B} \ln \Omega)$, where Ω is the total number of quantum states that are accessible to a black hole described by a small set of classical parameters. The different approaches that have been adopted in the literature to understand the microscopic origin of black hole entropy can be broadly classified into two categories. (i) Count the "microstates" by assuming a fundamental structure like D-branes, spin networks or conformal symmetry [1, 2, 3, 4]. (ii) Associate the black hole entropy to the quantum fields propagating in a fixed black hole spacetime, and count the microstates of these quantum fields [5, 6, 7, 8, 9, 10]. Interestingly, while all these approaches arrive at the leading Bekenstein-Hawking term, they, generally, seem to lead to different sub-leading contributions. For instance, (i) the prefactor to the logarithmic corrections obtained using the spin-networks and conformal symmetry [11, 12, 13, 14, 15] are different from the one obtained using the statistical fluctuations around thermal equilibrium [16], and (ii) the power-law corrections obtained using the Noether charge approach [8] are different from those via entanglement of the modes between inside and outside the horizon [17]. In other words, even though different degrees of freedom lead to the universal Bekenstein-Hawking entropy — *quite naturally*— they lead to different sub-leading terms. This indicates that the key to the understanding of the statistical mechanical interpretation of Bekenstein-Hawking entropy may lie in the origin of the sub-leading contributions. Physically, it is natural to expect corrections to Bekenstein-Hawking entropy. The Bekenstein-Hawking entropy is a semi-classical result, and there are strong indications that it is valid for large black holes (i.e. when horizon radius is much larger than the Planck length]). However, it is not clear whether this relation will continue to hold for, say, Planck size black holes. Besides, there is no reason to expect that the Bekenstein-Hawking entropy to be the complete answer in a correct theory of quantum gravity.

The brick wall approach is a semi-classical approach, wherein the background geometry is assumed to be a fixed classical background in which quantum fields propagate. The entropy of the black hole is identified with the statistical mechanical entropy arising from a thermal bath of quantum fields propagating outside the horizon. The entropy computed in this way turns out to be proportional to the area of the horizon. This approach has been very popular in obtaining the leading order to the black hole entropy in different dimensions (for an incomplete list of references, see Refs. [20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35]).

The original brick wall model involved only the leading order WKB approximation [5, 18, 19]. A natural question that arises is whether working at the higher orders in the approximation will lead to any corrections to the Bekenstein-Hawking entropy. In this work, we extend the zeroth-order ($\hbar^0$) WKB analysis to the higher orders and show that (i) The contribution to the entropy from the higher-order WKB modes is of the same order as the leading order WKB modes. In other words, our analysis shows that it may be incomplete to calculate the contribution only from the leading order WKB modes. (ii) The brick-wall entropy $(S_{\rm BW})$ leads to generic corrections to area of the form:

$$S_{\rm BW} = S_{\rm BH} + \mathscr{G}(\mathscr{A}_{\rm H}) + \mathscr{F}(\mathscr{A}_{\rm H}) \log\left(\frac{\mathscr{A}_{\rm H}}{\ell_{\rm Pl}^2}\right) , \tag{1.1}$$

where $\mathscr{G}(\mathscr{A}_{\rm H})$, and $\mathscr{F}(\mathscr{A}_{\rm H})$ are polynomial functions of $\mathscr{A}_{\rm H}$. In the case of four-dimensions, the brick-wall entropy (upto sixth-order) [36] has the form given above with $\mathscr{G}(\mathscr{A}_{\rm H}) = 0$. In the case of six-dimensions, $\mathscr{G}(\mathscr{A}_{\rm H}) \neq 0$. (iii) We show that, only in the case of Schwarzschild, $\mathscr{F}(\mathscr{A}_{\rm H})$ is a constant.

The remainder of this article is organized as follows. In the next section, we shall outline some essential properties of static, spherically symmetric black holes in arbitrary spacetime dimensions. Then, in Section 3, we shall discuss the assumptions and approximations involved in evaluating the brick wall entropy, and describe the procedure for extending the calculation to the higher orders (in terms of $\hbar$) in the WKB approximation. In Section 4, in addition to the zeroth order, we shall evaluate the contributions to the brick wall entropy of four dimensional black holes at the second order (in terms of $\hbar$) in the WKB approximation. In Section 5, we explicitly write down the results for a few specific black hole solutions in four dimensions. Finally, in Section 6, after a rapid summary of the results we have obtained, we shall discuss as to how the sub-leading contributions we have evaluated compare with the results obtained from the other approaches.

Let us now briefly list the conventions and notations we shall adopt. We shall, in general, consider a $(D+2)$-dimensional, spherically symmetric, black hole spacetime. We shall work with the metric signature $(-,+,+,\cdots)$, and use the geometric units wherein $k_{\rm B} = c = G = 1$. We shall denote the derivative of any function with respect to the radial coordinate r of the black hole by an overprime. The quantum field Φ we shall consider will be a minimally coupled scalar field.

2. Key properties of static, spherically symmetric black holes:

Consider the following $(D+2)$-dimensional static and spherically symmetric line element

$$ds^2 = -f(r)\,dt^2 + \frac{dr^2}{g(r)} + r^2\,d\Omega_D^2\,, \tag{2.1}$$

$$= f(r)\left[-dt^2 + dx^2\right] + r^2\,d\Omega_D^2\,, \tag{2.2}$$

where $f(r)$ and $g(r)$ are arbitrary (but, continuous and differentiable) functions of the radial coordinate r, $d\Omega_D^2$ is the metric on a D-dimensional unit sphere, and

$$x = \int \frac{dr}{\sqrt{f(r)\,g(r)}} \tag{2.3}$$

denotes the tortoise coordinate. Throughout this work, we shall assume that the line-element (2.1) contains a singularity (say, at $r=0$) and *one*, non-degenerate, event horizon (located at, say, $r = r_{\rm H}$) But, we shall not assume any specific form of $f(r)$ or $g(r)$. In the rest of this section, we shall discuss some generic properties of the spacetime (2.1) near the horizon at $r = r_{\rm H}$.

In almost all approaches that evaluate the entropy of spherically symmetric black holes, their line-element close to the event horizon is approximated to be that of a Rindler spacetime (see, for instance, Ref. [32]). For the line-element (2.1), the Rindler behavior near the horizon can be arrived at by first carrying out the following transformation of the radial coordinate:

$$\gamma = \left(\frac{1}{\kappa}\right)\sqrt{f}, \tag{2.4}$$

where κ is a constant that denotes the surface gravity of the black hole and is defined as (see, for example, Ref. [40])

$$\kappa = \left[\sqrt{\frac{g(r)}{f(r)}}\,\left(\frac{f'(r)}{2}\right)\right]_{r=r_{\rm H}} . \tag{2.5}$$

In terms of the coordinate γ, the line-element (2.1) can be expressed as

$$ds^2 = -\kappa^2\,\gamma^2\,dt^2 + 4\,\left(\frac{f}{g}\right)\left(\frac{\kappa}{f'}\right)^2 d\gamma^2 + r^2\,d\Omega_D^2 . \tag{2.6}$$

Close to the horizon (i.e. near $r = r_{\rm H}$), this line-element reduces to

$$ds^2 \to -\kappa^2\,\gamma^2\,dt^2 + d\gamma^2 + r_{\rm H}^2\,d\Omega_D^2 \tag{2.7}$$

which describes the Rindler spacetime with a horizon that is located at $\gamma = 0$. It should be stressed here that such a behavior is exhibited by all non-degenerate black hole horizons in all dimensions.

The above derivation of the Rindler line-element near the horizon is essentially equivalent to expanding the metric components $f(r)$ and $g(r)$ in (2.1) about $r_{\rm H}$ up to the linear order in the Taylor series. However, we find that, when evaluating the contributions to the brick wall entropy at the higher orders in the WKB approximation, we need to expand the quantities $f(r)$ and $g(r)$ to higher orders as follows:

$$f(r) \;=\; f'(r_{\rm H})\,(r - r_{\rm H}) + \left(\frac{f''(r_{\rm H})}{2}\right)\,(r - r_{\rm H})^2 + \dots , \tag{2.8}$$

$$g(r) \;=\; g'(r_{\rm H})\,(r - r_{\rm H}) + \left(\frac{g''(r_{\rm H})}{2}\right)\,(r - r_{\rm H})^2 + \dots . \tag{2.9}$$

As we shall see, in four dimensions, in addition to the surface gravity of the black hole, the corrections to the Bekenstein-Hawking entropy $S_{\rm BH}$ also depend on the second derivative of the metric evaluated at the horizon.

Another quantity which we shall require in our calculations is the proper or the coordinate invariant distance of the brick wall from the horizon. The proper radial distance to the brick wall, say, h_c, that is located at $r = h$ is given by

$$h_c = \int_{r_{\rm H}}^{r_{\rm H}+h} \frac{dr}{\sqrt{g(r)}} . \tag{2.10}$$

On using the expansion (2) for $g(r)$ up to the second order in this integral, we obtain the following relation between h and h_c:

$$h^{1/2} = \sqrt{\frac{2\,g'(r_{\rm H})}{g''(r_{\rm H})}}\;\sinh\left[\sqrt{\frac{g''(r_{\rm H})}{2}}\,\left(\frac{h_c}{2}\right)\right] . \tag{2.11}$$

For small h_c, this relation simplifies to

$$h_c = \sqrt{\frac{4h}{g'(r_{\rm H})}} , \tag{2.12}$$

and, for convenience, we shall use this expression for the proper distance to the brick wall.

3. Extending the brick wall model to higher WKB orders

In this section, after a rapid sketch of the assumptions and approximations that are involved in evaluating the black hole entropy using the brick wall model, we go on to outline the procedure for computing the brick wall entropy at the leading orders in the WKB approximation.

3.1 Basic assumptions

There are two crucial assumptions in the brick wall approach to black hole entropy. The first assumption concerns the modeling of the microscopic origin of the black hole entropy, and the second is regarding the handling of the divergences that arise close to the event horizon.

As we have mentioned before, the brick wall model is a semi-classical approach wherein the black hole is assumed to be described by a fixed classical geometry. It is further assumed that the black hole is in equilibrium with a thermal bath of quantum matter fields at the Hawking temperature. Moreover, it is the canonical entropy (actually, a specific component) of the quantum matter fields that are propagating outside the black hole horizon that is identified to be the entropy of the black hole.

In the process of calculating the canonical entropy of a matter field outside the black hole horizon, we need to evaluate the density of states of the field. However, one finds that, due to the infinite blue shifting of the modes in the vicinity of the event horizon, the density of states actually diverges. This divergence is regulated in the model by introducing a cut-off by hand above the horizon. The cut-off—popularly referred to as the brick-wall—is basically a static, spherical mirror at which the matter fields are assumed to satisfy, say, the Dirichlet boundary conditions. One finds that the leading component of the brick wall entropy diverges as h_c^{-2}, where h_c is the proper distance to the brick wall defined in Eq. (2.10). (The other component is essentially a volume dependent term that arises even in flat space.) It is this contribution that is identified to be the entropy of the black hole. Moreover, a specific choice for the cut-off h_c has to be made (this depends on the number of fields, the dimension of the spacetime, etc., but is generally of the order of the Planck length $\ell_{\rm Pl}$), in order to reproduce the Bekenstein-Hawking area law. As we mentioned, the area law has been recovered in this approach for a variety of black hole spacetimes and matter fields [20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 37].

3.2 Essential approximations

Two approximations turn out to be essential to make the computation of the brick wall entropy tractable. The first approximation is required in evaluating the density of states of matter fields around black holes, and the second involves expanding the metric near the event horizon.

As we pointed out above, in order to evaluate the brick wall entropy, one needs to evaluate the density of states of matter fields around black holes. However, apart from some lower dimensional cases, the density of states cannot be evaluated exactly. As a result, in the brick wall model, the density of states is usually evaluated at the leading order in $\hbar$ in the WKB approximation.

Moreover, barring a few special cases, one finds that, even after the WKB approximation, the brick wall entropy cannot be evaluated exactly. Recall that the dominant contribution to the entropy arises due to the modes close to the horizon. Motivated by this feature, one Taylor expands the metric functions $f(r)$ and $g(r)$ near the horizon in order to obtain a closed form expression for the brick wall entropy.

3.3 The methodology

Having discussed the assumptions and approximations involved in the brick wall approach, in the remainder of this sub-section, we shall outline the procedure for evaluating the brick wall entropy at the leading order in the WKB approximation.

The key assumption of the brick wall model, as we have pointed out above, is that the black hole is in equilibrium with a bath of thermal radiation at the Hawking temperature of the hole. The free energy F of a scalar field at the inverse temperature β is given by (see, for example, Ref. [5])

$$\begin{aligned} F &= \left(\frac{1}{\beta}\right) \int_0^\infty dE \left(\frac{d\Gamma(E)}{dE}\right) \ln\left[1 - \exp -(\beta E)\right] , \\ &= -\int_0^\infty dE \left(\frac{\Gamma(E)}{\exp(\beta E) - 1}\right) , \end{aligned} \tag{3.1}$$

where $\Gamma(E)$ denotes the total number of modes of the field with energy less than E. We have integrated the first of the above equation by parts to arrive at the second and have assumed that the boundary term vanishes. The canonical entropy associated with the free energy F is given by

$$S_{\rm C}(\beta) = \beta^2 \left(\frac{\partial F}{\partial \beta}\right) , \tag{3.2}$$

and, it is this entropy, evaluated at the Hawking temperature, that will be identified to be the entropy of the black hole.

Consider a massive and minimally coupled scalar field Φ that is propagating in the line-element (2.1). Such a field satisfies the differential equation

$$\left(\Box - m^2\right) \Phi = 0, \tag{3.3}$$

where m denotes the mass of the field. The rotational symmetry of the line-element (2.1) allows us to decompose the normal modes $u_{E\ell m_i}$ of the field Φ as follows (see, for instance, Ref. [41]):

$$u_{E\ell m_i}(x^\mu) = \left(\frac{R(r)}{r^{D/2}\, G^{1/2}(r)}\right) Y_{\ell m_i}(\theta, \phi_i)\ e^{-(iEt/\hbar)} , \tag{3.4}$$

where E, ℓ and m_i (with $i \in [1, (D-1)]$) are the energy, angular momentum and the azimuthal angular momenta associated with the modes, respectively, the quantity $G(r)$ is given by

$$G(r) = \sqrt{f(r)\, g(r)}, \tag{3.5}$$

and $Y_{lm_i}(\theta, \phi_i)$ denote the hyper-spherical harmonics. On substituting the mode (3.4) in the equation of motion (3.3) and using the properties of the hyper-spherical harmonics, we find that the function $R(r)$ satisfies the differential equation

$$R''(r) + \left[\frac{V^2(r)}{\hbar^2} - \Delta(r)\right] R(r) = 0 , \tag{3.6}$$

where the quantities $V^2(r)$ and $\Delta(r)$ are given by

$$V^2(r) = \left(\frac{1}{G^2(r)}\right)\left(E^2 - f(r)\left[m^2 + \left(\frac{\ell(\ell+D-1)\hbar^2}{r^2}\right)\right]\right), \tag{3.7}$$

$$\Delta(r) = \left(\frac{G''(r)}{2G(r)}\right) - \left(\frac{G'^2(r)}{4G^2(r)}\right) + \left(\frac{D}{2r}\right)\left(\frac{G'(r)}{G(r)}\right) + \left(\frac{D(D-2)}{4r^2}\right). \tag{3.8}$$

The total number of modes $\Gamma(E)$ of the field Φ with energy less than E can be evaluated exactly if the solution to the differential equation (3.6) can be written down explicitly. However, apart from some simple $(1+1)$-dimensional example [28], it proves to be difficult to obtain an exact analytical solution for the function $R(r)$. As a result, the WKB approximation is almost always resorted to in the literature [20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35], and it is the leading order WKB solution for $R(r)$ that is utilized to evaluate the number of states $\Gamma(E)$, and the resulting free energy F and the entropy of the quantum field. Our goal here is to extend the analysis to the higher orders in the WKB approximation.

Let us begin by expressing the function $R(r)$ in the following WKB form:

$$R(r) = \left(\frac{c_0}{\sqrt{P(r)}}\right)\exp\left[\frac{i}{\hbar}\int^r d\tilde{r}\,P(\tilde{r})\right], \tag{3.9}$$

where c_0 is a constant. On substituting this expression in Eq. (3.6), we find that the function $P(r)$ satisfies the differential equation

$$\left(\frac{1}{\hbar^2}\right)\left[P^2(r) - V^2(r)\right] = \left(\frac{3}{4}\right)\left(\frac{P'(r)}{P(r)}\right)^2 - \left(\frac{1}{2}\right)\left(\frac{P''(r)}{P(r)}\right) - \Delta(r). \tag{3.10}$$

Let us now expand the function $P(r)$ in a power series in $\hbar^2$ as follows (see, for instance, Ref. [42]):

$$P(r) = \sum_{n=0}^{\infty} \hbar^{2n}\,P_{2n}(r). \tag{3.11}$$

On substituting this series in the differential equation (3.9) and collecting the terms of a given order in $\hbar^2$, we obtain following expressions for $P_{2n}(r)$ upto $n = 3$:

$$P_0(r) = \pm V(r) = \pm\left(\frac{1}{G(r)}\right)\left[E^2 - f(r)\left(m^2 + \left[\frac{\ell(\ell+D-1)\hbar^2}{r^2}\right]\right)\right]^{1/2}, \tag{3.12}$$

$$P_2(r) = \left(\frac{3}{8P_0(r)}\right)\left(\frac{P_0'(r)}{P_0(r)}\right)^2 - \left(\frac{4P_0''(r)}{P_0{}^2(r)}\right) - \left(\frac{\Delta(r)}{2P_0(r)}\right), \tag{3.13}$$

$$P_4(r) = -\left(\frac{5P_2{}^2(r)}{2V(r)}\right) - \left(\frac{4P_2(r)\Delta(r) + P_2''(r)}{4V^2(r)}\right) + \left(\frac{3P_2{}'(r)V'(r) - P_2(r)V''(r)}{4V^3(r)}\right), \tag{3.14}$$

$$\begin{aligned} P_6(r) = &-\left(\frac{5P_2(r)P_4(r)}{V(r)}\right) - \left(\frac{8P_2{}^3(r) + 4P_4(r)\Delta(r) + P_4''(r)}{4V^2(r)}\right) - \left(\frac{\Delta(r)P_2{}^2(r)}{2V^3(r)}\right) \\ &-\left(\frac{2P_2''(r)P_2(r) + 2P_4(r)V''(r) - 3P_2'^2(r) - 6P_4'(r)V'(r)}{8V^3(r)}\right). \end{aligned} \tag{3.15}$$

Note that the function $P_0(r)$ is related *algebraically* to the quantities $V(r)$ and $\Delta(r)$. It is evident that the higher order functions $P_{2n}(r)$ (with $n > 0$) can be expressed in terms of the functions at the lower orders and their derivatives and, eventually, in terms of the function $P_0(r)$.

On using the series expansion (3.11) in the standard semiclassical quantization procedure [5], we can express the total number of states $\Gamma(E)$ of the field with energy less than E as follows:

$$\Gamma(E) = \sum_{n=0}^{\infty} \Gamma_{2n}(E), \tag{3.16}$$

where we have defined $\Gamma_{2n}(E)$ as

$$\Gamma_{2n}(E) = \left(\frac{\hbar^{2n-1}}{\pi}\right) \int_{r_{\rm H}+h_c}^{L} dr \int_{0}^{\ell_{max}} d\ell\ (2\ell + D - 1) \\ \times \mathscr{W}(\ell)\, P_{2n}(r)\,, \tag{3.17}$$

with the quantity $\mathscr{W}(\ell)$ being given by

$$\mathscr{W}(\ell) = \left(\frac{(\ell + D - 2)!}{(D-1)!\ \ell!}\right). \tag{3.18}$$

It should be mentioned that, in the above expression for $\Gamma_{2n}(E)$, we have approximated the sum over the angular quantum numbers ℓ as an integral with a degeneracy factor $\mathscr{W}(\ell)$. Such an approximation is often made in the literature, and the approximation is considered to be valid since the separation between the states are expected to be small [26]. Moreover, the upper limit ℓ_{max} on the integral over ℓ is a function of energy E of the mode and the radial coordinate r, and it has to be chosen such that $P_0(r)$ is real[1]. Furthermore, the lower limit on the integral over radial coordinate, viz. h_c, is the invariant thickness of the 'brick-wall' defined in (2.10), and the upper limit L is the infra-red cutoff which we shall assume to be much larger than the horizon radius.

A few clarifying remarks are in order at this stage of our discussion. In the semi-classical quantization of, say, a one-dimensional non-relativistic quantum particle, the integral over the coordinate will be carried out over the range wherein P_0 is real [42]. In the case of bounded systems, these limits will prove to be the turning points of the potential, whereas in the case of potential barriers the limits will be between one of the turning points and infinity. In the context of black holes, the effective potential turns out to be a barrier and the integral over the radial coordinate is to carried out between the event horizon of the black hole and the first turning point that is located on the barrier. But, one finds that, most of the contribution to the density of states of the quantum field arises due to the modes close to the event horizon of the black hole, while the upper limit located on the barrier leads to a volume dependent contribution to the entropy. As a result, the contribution to the number of states and the free energy and the entropy of the quantum field due to the upper (infra-red) limit is usually ignored in the literature.

We should emphasize the point that, apart from replacing the sum over ℓ by an integral, we have not made any approximations until now. Hereafter, we shall make two approximations that

[1]Actually, the limits have to be chosen such that $P_{2n}(r)$ are real for all n. However, since, for $n > 0$, the functions $P_{2n}(r)$ can be expressed in terms of $P_0(r)$ and the real functions $V(r)$ and $\Delta(r)$, when $P_0(r)$ is real, $P_{2n}(r)$ are real as well. Therefore, the limits on ℓ proves to be the same for all n.

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we had discussed is some detail in the last subsection. Firstly, we shall approximate the line-element (2.1) near the event horizon of a spherically symmetric black hole to be that of Rindler spacetime, viz. Eq. (2.7). It should be pointed out that such an approximation is always made in the literature to arrive at closed form expressions for the free energy and the entropy of the quantum field. Secondly, we shall truncate the series (3.11) at a particular order, and evaluate the density of states and the associated free energy and the entropy of the quantum field around the black hole. It is important to note that, in the literature, it is only the leading term in the series (3.16) that has *always* been taken into account ignoring the higher orders when evaluating the brick-wall entropy.

3.4 The standard, leading order, result

Let us now reproduce the standard leading order result arriving at the Bekenstein-Hawking area law. In the case of massless scalar field, the leading order WKB modes are given by

$$P_0 = \pm\frac{1}{g(r)}\left[E^2 - g(r)\frac{L^2}{r^2}\right]^{1/2}. \tag{3.19}$$

Substituting the above expression in Eq. (3.17), we get

$$\Gamma_0(E) = \frac{2E^3}{3\hbar^3}\int_{r_{\rm H}+h}^{L}\frac{r^2\,dr}{g^2(r)}. \tag{3.20}$$

Substituting the above expression in (3.1) and integrating over E, the free energy F now reads

$$F_0 = -\frac{2\pi^3}{45\hbar^3}\frac{1}{\beta^4}\int_{r_{\rm H}+h}^{L}\frac{r^2}{g^2(r)}dr\,, \tag{3.21}$$

and the entropy is

$$S_0 = \frac{8\pi^3}{45\hbar^3}\frac{1}{\beta^3}\int_{r_{\rm H}+h}^{L}\frac{r^2}{g^2(r)}dr\,. \tag{3.22}$$

On expanding the metric near the horizon up to the first-order, we recover the following standard result [5],

$$S_0^{\rm (Std)} = \frac{r_{\rm H}^2}{90h_c^2}\,. \tag{3.23}$$

However, if we expand the metric to higher orders (2), we get

$$S_0 = \frac{r_{\rm H}^2}{90h_c^2} + \left[\frac{\kappa r_{\rm H}}{90} - \frac{g''(r_{\rm H})r_{\rm H}^2}{360}\right]\log\left(\frac{r_{\rm H}^2}{h_c^2}\right). \tag{3.24}$$

Before we proceed with the calculations, there is yet another point concerning the higher order WKB terms that we need to discuss. As we mentioned above, the limits on the integral over ℓ has been chosen such that $P_0(r)$ is real. This condition essentially identifies the turning points of the potential. Notice that, in Eq. (3.3), all the higher order WKB terms—i.e. $P_{2n}(r)$ for $n>0$—contain $P_0(r)$ in the denominator. Obviously, these functions will diverge at the turning points, or equivalently, at the upper limits ℓ. Such a divergence is a well-known feature of the WKB approximation at the higher orders [42], and we shall devise a systematic procedure to isolate these divergences. We shall outline this procedure in the next section.

4. Higher order contributions to the brick wall entropy

In this section, we shall evaluate the brick wall entropy for spherically symmetric, four dimensional black holes by considering the contributions up to the $n = 1$ term in the series expansion (3.16) for the number of states of the quantum field. For simplicity, we shall consider here the case of $f(r) = g(r)$ in the line-element (2.1) and restrict ourselves to a massless scalar field (i.e. $m = 0$). For other general cases, we refer the readers to Ref. [36].

Let us now evaluate the contribution due to the $n = 1$ term in the series (3.16). For $f(r) = g(r)$, we find that the expression (3.13) for second order 'momentum' $P_2(r)$ can be written as

$$P_2(r) = \left(\frac{P_2^{(0)}(r)}{\mathscr{G}(\mathscr{E},r)}\right) + \lambda(r)\left(\frac{P_2^{(1)}(r)}{\mathscr{G}^3(\mathscr{E},r)}\right) + \lambda^2(r)\left(\frac{P_2^{(2)}(r)}{\mathscr{G}^5(\mathscr{E},r)}\right), \tag{4.1}$$

where the functions $P_2^{(0)}(r)$, $P_2^{(1)}(r)$ and $P_2^{(2)}(r)$ are given by

$$\begin{aligned}
P_2^{(0)}(r) &= -\left(\frac{g'}{2r}\right),\\
P_2^{(1)}(r) &= \left(\frac{g'^2(r)}{8g(r)}\right) - \left(\frac{3g'(r)}{4r}\right) + \left(\frac{g''(r)}{8}\right) + \left(\frac{3\,g(r)}{4r^2}\right),\\
P_2^{(2)}(r) &= \left(\frac{5}{32}\right)\left(\frac{g'(r)}{g(r)}\right)^2 - \left(\frac{5g'(r)}{8r}\right) + \left(\frac{5\,g(r)}{8r^2}\right),
\end{aligned} \tag{4.2}$$

and, for convenience, we have defined

$$\mathscr{G}(\mathscr{E},r) = [\mathscr{E} - \lambda(r)]^{1/2} \tag{4.3}$$

with $\mathscr{E} = E^2$ and $\lambda(r)$ being given by

$$\lambda(r) = \left[\ell(\ell+1)\hbar^2\right]\left(\frac{g(r)}{r^2}\right). \tag{4.4}$$

We now need to substitute the above expression for $P_2(r)$ in Eq. (3.17) and evaluate the number of modes Γ_2 with the upper limit $\ell_{\max}$ on the integral over ℓ being determined by the condition that the term $\mathscr{G}(\mathscr{E},r)$ vanishes. Clearly, the integral over ℓ will diverge in such a case. In order to isolate the finite contribution due to these higher order WKB modes, it is necessary that we follow a systematic procedure. The procedure we shall adopt is as follows. We shall first rewrite all the terms containing inverse powers of $\mathscr{G}(\mathscr{E},r)$ in terms of derivatives of $\mathscr{E}$ as follows:

$$\left(\frac{1}{\mathscr{G}(\mathscr{E},r)}\right) = 2\left(\frac{\partial\mathscr{G}(\mathscr{E},r)}{\partial\mathscr{E}}\right), \tag{4.5}$$

$$\left(\frac{1}{\mathscr{G}^3(\mathscr{E},r)}\right) = -4\left(\frac{\partial^2\mathscr{G}(\mathscr{E},r)}{\partial\mathscr{E}^2}\right), \tag{4.6}$$

$$\left(\frac{1}{\mathscr{G}^5(\mathscr{E},r)}\right) = \left(\frac{8}{3}\right)\left(\frac{\partial^3\mathscr{G}(\mathscr{E},r)}{\partial\mathscr{E}^3}\right). \tag{4.7}$$

Then, before evaluating the ℓ integral, we shall make use of the Leibnitz's rule, viz.

$$\frac{\partial}{\partial x}\int\limits_{a(x)}^{b(x)} dt\, f[x,t] = f[x,a(x)]\left(\frac{da(x)}{dx}\right) - f[x,b(x)]\left(\frac{db(x)}{dx}\right)\int\limits_{a(x)}^{b(x)} dt\left[\frac{\partial f(x,t)}{\partial x}\right]. \tag{4.8}$$

and interchange the order of differentiation and integration over the energy E and ℓ. When we do so, we find that the divergences occur at the turning point. But, we note that the origin of the divergent terms can be associated to the break-down of WKB approximation at the turning point. As a result, this is not a physical divergences and this is occurring due to the fact that the WKB approximation is not valid close to the turning points. Hence, it is perfectly consistence to separate out only the non divergent part and neglect the divergent contribution. Also, it can be shown that by introducing a cutoff close to the turning point that the results are independent of the cutoff. (For details, see Sec. (10.7) in Ref. [42].) We have checked the procedure up to the 6$^{\text{th}}$-order WKB modes and, indeed, it systematically separates the finite parts from the divergent ones (for details, see Ref. [36].

Having obtained the non-divergent part of the mode-functions as a function of E, our next step is to evaluate the contribution of these modes to the density of states $\Gamma_2(E)$. Using the general expression (3.17), we have

$$\Gamma_2(E) = \frac{\hbar}{\pi}\int\limits_{r_{\rm H}+h}^{L} dr \int\limits_{0}^{\ell_{max}} d\ell\,(2\ell+1)\,P_2(r)\,. \tag{4.9}$$

Substituting for $P_2(r)$ from Eq. (4.1) and using the relations (4), we get

$$\begin{aligned}
\hbar\Gamma_2(E) \;=\; & \frac{1}{\pi}\int\limits_{r_{\rm H}+h}^{L} dr\frac{r^2 P_0^{(2)}(r)}{2}\int\limits_{0}^{\mathscr{E}} d\lambda\,\frac{\partial\mathscr{G}(\mathscr{E},r)}{\partial\mathscr{E}} \\
& - \frac{1}{\pi}\int\limits_{r_{\rm H}+h}^{L} dr\, r^2 P_1^{(2)}(r)\int\limits_{0}^{\mathscr{E}} d\lambda\;\lambda\,\frac{\partial^2\mathscr{G}(\mathscr{E},r)}{\partial\mathscr{E}^2} \\
& + \frac{1}{\pi}\int\limits_{r_{\rm H}+h}^{L} dr\,\frac{3r^2 P_2^{(2)}(r)}{2}\int\limits_{0}^{\mathscr{E}} d\lambda\;\lambda^2\,\frac{\partial^3\mathscr{G}(\mathscr{E},r)}{\partial\mathscr{E}^3}\,.
\end{aligned} \tag{4.10}$$

Using the Leibniz rule (4.8) and following the steps discussed in [36], we get

$$\Gamma_2(E) = \frac{E}{\hbar\pi}\int\limits_{r_{\rm H}+h}^{L} dr\left[\frac{1}{3} - \frac{4rg'(r)}{3g(r)} + r^2\left\{\frac{g'(r)^2}{3g(r)^2} - \frac{g''(r)}{2g(r)}\right\}\right]. \tag{4.11}$$

Following points are worth noting regarding the above expression: (i) In the case of leading order WKB modes, the density of states goes as E^3 [see Eq. (3.20)]. However, for the second-order WKB modes the density of states scales as E. (ii) As in the leading-order, most of the contributions to the entropy come close to the horizon.

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Substituting the above expression in Eq. (3.1), and integrating over E, the free-energy is

$$F_2 = -\frac{\pi}{6\hbar\beta^2}\int_{r_{\rm H}+h}^{L} dr\left[\frac{1}{3}-\frac{4rg'(r)}{3g(r)}+r^2\left(\frac{g'(r)^2}{3g(r)^2}-\frac{g''(r)}{2g(r)}\right)\right]. \tag{4.12}$$

Using the relation (3.2), the entropy is given by

$$S_2 = \frac{\pi}{3\hbar\beta}\int_{r_{\rm H}+h}^{L} dr\left[\frac{1}{3}-\frac{4rg'(r)}{3g(r)}+r^2\left(\frac{g'(r)^2}{3g(r)^2}-\frac{g''(r)}{2g(r)}\right)\right]. \tag{4.13}$$

As mentioned above, maximum contribution to the entropy is from the modes close to the horizon. Hence, using the expansion (2) close to the horizon and the definition of surface gravity (2.5), we get,

$$S_2 = \frac{1}{9}\frac{r_{\rm H}^2}{h_c^2} - \left[\frac{g''(r_{\rm H})r_{\rm H}^2}{72}+\frac{\kappa}{9}r_{\rm H}\right]\log\left(\frac{r_{\rm H}^2}{h_c^2}\right) \tag{4.14}$$

where h_c is given by Eq. (2.12). We would like to stress the following points regarding the above result:

1. The dependence of the entropy on area (from the second-order WKB modes) is similar to that from the zeroth order WKB modes (3.24). Also the contribution to the entropy from the second order WKB modes contribute more as compared to the leading order WKB modes. This result has two immediate consequences:

   (a) To associate the brick-wall entropy to $S_{\rm BH}$ it is *necessary* to calculate all the higher order WKB mode contribution to the brick-wall entropy.

   (b) The sub-leading corrections (at the zeroth and second order WKB) depend only on the surface gravity and second derivative of the metric functions. They are of the form $\mathscr{F}(\mathscr{A}_{\rm H})\log(\mathscr{A}_{\rm H}/h_c^2)$.

2. If the surface gravity is inversely proportional to horizon radius and $g''(r_{\rm H})$ is inversely proportional to the square of the horizon radius, then second term in the RHS of (4.14) is a constant. In this case, the corrections to $S_{\rm BH}$ are purely logarithmic and does not contain any power-law dependence. This uniquely corresponds to Schwarzschild spacetime.

   In the case of Schwarzschild, we have

   $$f(r) = g(r) = 1-\frac{2M}{r} \tag{4.15}$$

   where M is the mass of the black hole. The horizon is at $r_{\rm H} = 2M$, $\kappa = 1/(4M)$ and $g''(r_{\rm H}) = -1/(2M^2)$. Substituting the above expressions in Eq. (4.14), we get

   $$S_2 = \frac{4}{9}\frac{M^2}{h_c^2} - \frac{1}{36}\log\left(\frac{r_{\rm H}^2}{h_c^2}\right). \tag{4.16}$$

   This result shows that, at least, in the zeroth and second order, there are no power-law corrections to $S_{\rm BH}$ for the four-dimensional Schwarzschild black hole, while, for all other black holes — since κ and $g''(r)$ has a more non-trivial structure – there are power-law corrections to the Bekenstein-Hawking entropy. This leads to the following conclusion: *The power-law corrections to the entropy occur for any non-vacuum solutions.* In Sec. (5) we obtain the entropy for some known black hole solutions.

5. Results for specific black holes

In this section, we shall explicitly write down the brick wall entropy (evaluated upto the second order in the WKB approximation) for a few well-known black hole solutions in four dimensions.

We find that, on combining the zeroth order (3.24) and the second order (4.14) terms, the total brick wall entropy can be expressed as

$$S_{\rm BW}^{(4D)} = S_{\rm BH} + \mathscr{F}^{(4D)}(\mathscr{A}_{\rm H})\, \log\left(\frac{\mathscr{A}_{\rm H}}{\ell_{\rm Pl}^2}\right), \tag{5.1}$$

where, in order for the leading term to match the Bekenstein-Hawking entropy, we have set the brick wall invariant cutoff h_c to be

$$h_c^2 = \left(\frac{11\,\ell_{\rm Pl}^2}{90\pi}\right). \tag{5.2}$$

and the quantity $\mathscr{F}^{(4D)}(\mathscr{A}_{\rm H})$ is given by

$$\mathscr{F}^{(4D)}(\mathscr{A}_{\rm H}) = -\left(\frac{1}{60}\right) g''(r_{\rm H})\, r_{\rm H}^2 - \left(\frac{1}{10}\right) \kappa\, r_{\rm H}\,. \tag{5.3}$$

5.0.1 Schwarzschild black hole

For the Schwarzschild black hole, the metric coefficients are given by Eq. (4.15) and the event-horizon of the black hole is located at $r_{\rm H} = (2M)$. The surface gravity κ and the second derivative of the metric at the horizon are given by

$$\kappa = \left(\frac{1}{4M}\right), \quad g''(r_{\rm H}) = -\left(\frac{1}{2M^2}\right). \tag{5.4}$$

On substituting these expressions in Eq. (5.1), we obtain that

$$S_{\rm Sch}^{(4D)} = S_{\rm BH} - \left(\frac{1}{60}\right) \log\left(\frac{\mathscr{A}_{\rm H}}{\ell_{\rm Pl}^2}\right). \tag{5.5}$$

5.0.2 Schwarzschild (anti-)de sitter spacetime

For the Schwarzschild (anti-)de sitter spacetime, the metric function $g(r)$ is given by

$$g(r) = \left(1 - \frac{2M}{\tilde{r}} \pm \frac{\tilde{r}^2}{l^2}\right) = \left(1 - \frac{2}{r} \pm \frac{r^2}{y}\right) \tag{5.6}$$

where $y = (l/M)^2$, $r = (\tilde{r}/M)$, M is the mass of the black hole, l is related to the positive (negative) cosmological constant and $-(+)$ corresponds to (anti-)de Sitter spacetime. Note that the coordinates y and r are dimensionless. While the Schwarzschild anti-de Sitter spacetime has only one horizon associated with the singularity at the origin, the Schwarzschild de Sitter has two—one event and one cosmological— horizons. Here, we shall focus on the entropy associated with the event-horizon.

Recall that the event horizon is identified by the condition $g(r) = 0$. On substituting the resulting $r_{\rm H}$ corresponding to the above $g(r)$ in Eq. (5.1), we find that the brick wall entropy upto the second order can be expressed as

$$S^{(4D)}_{\rm Sch-(a)dS} = S_{\rm BH} - \left(\frac{\pi^{1/2}}{15\,\mathscr{A}_{\rm H}^{1/2}} + \frac{\mathscr{A}_{\rm H}}{\pi y}\right) \log\left(\frac{M^2\,\mathscr{A}_{\rm H}}{\ell_{\rm Pl}^2}\right), \tag{5.7}$$

where $\mathscr{A}_{\rm H}$ defined in-terms of the coordinate r is also dimensionless. In contrast to the purely Schwarzschild case wherein the prefactor to the logarithmic correction was a constant, here the factor is a function of the horizon area.

5.0.3 Reissner-Nordström black hole

For the Reissner-Nordström black hole, we have

$$g(r) = \left(1 - \frac{2M}{\tilde{r}} + \frac{Q^2}{\tilde{r}^2}\right) = \left(\frac{(r-r_-)(r-r_+)}{r^2}\right), \tag{5.8}$$

where M and Q denotes the mass and the electric charge of the black hole. Also, $r = \tilde{r}/M$ and $r_\pm$ is the outer/inner horizon given by

$$r_\pm = \left(1 \pm \sqrt{1 - \frac{Q^2}{M^2}}\right), \tag{5.9}$$

where, again, r is a dimensionless variable. It is the outer horizon r_+ that is the event horizon of the black hole.

On substituting the above relations in Eq. (5.1), we obtain the brick wall entropy upto the second order to be

$$S^{(4D)}_{\rm RN} = S_{\rm BH} - \left(\frac{\pi^{1/2}}{15\,\mathscr{A}_{\rm H}^{1/2}}\right) \log\left(\frac{M^2\,\mathscr{A}_{\rm H}}{\ell_{\rm Pl}^2}\right), \tag{5.10}$$

where, again, $\mathscr{A}_{\rm H}$ defined in-terms of r is dimensionless. As in the previous example, the prefactor again turns out to be a function of the horizon area $\mathscr{A}_{\rm H}$.

It turns out that for $n = 2$, there is no contribution to brick wall entropy. We find that, at the sixth order, i.e. for $n = 3$, all the conclusions we have reached for the $n = 1$ case remain valid except the total entropy is dependent on the third derivative of the metric function evaluated on the horizon. We have also repeated these calculations in six dimensions (for further details, see Ref. [36]).

6. Discussion

6.1 Summary

As we have pointed out repeatedly, the brick wall model has been a very popular approach that has been utilized to recover the Bekenstein-Hawking entropy $S_{\rm BH}$ in a multitude of situations [20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35]. In all these efforts, it is only the leading term in the WKB expansion (3.16) that has been taken into account in evaluating the density of states and the associated free energy and entropy of quantum fields around black holes. Also, the metric has almost always been assumed to be of the Rindler form near the event horizon.

In this work, we have extended the brick wall approach to the higher orders in the WKB approximation. Moreover, by expanding the metric functions $f(r)$ and $g(r)$ beyond the leading order near the event horizon, we have been able to evaluate the corrections to the Bekenstein-Hawking entropy for spherically symmetric black holes in four and six dimensions. To begin with, we have illustrated that, even the often considered zeroth order term in the WKB approximation leads to corrections to the Bekenstein-Hawking entropy, provided the metric functions are expanded beyond the linear order near the horizon. Secondly, we have shown that all the higher order terms in the WKB approximation have the same form as the zeroth order term. Lastly, we find that, the higher order WKB terms actually contribute *more* to the entropy than the lower order terms.

Specifically, we have shown [36] that, upto the second order in the WKB approximation, the brick wall entropy of four dimensional black holes can be expressed as

$$S_{\rm BW}^{(4\rm D)} = S_{\rm BH} + \mathscr{F}^{(4\rm D)}(\mathscr{A}_{\rm H})\, \log\left(\frac{\mathscr{A}_{\rm H}}{\ell_{\rm Pl}^2}\right),$$

where $\mathscr{F}^{(4\rm D)}(\mathscr{A}_{\rm H}) \propto \mathscr{A}_{\rm H}^n$ with $n < 1$. Whereas, in six dimensions, we find that the brick wall entropy up to the second order has the form

$$S_{\rm BW}^{(6\rm D)} = S_{\rm BH} + \mathscr{G}(\mathscr{A}_{\rm H}) + \mathscr{F}^{(6\rm D)}(\mathscr{A}_{\rm H})\, \log\left(\frac{\mathscr{A}_{\rm H}}{\ell_{\rm Pl}^2}\right),$$

where $\mathscr{G}(\mathscr{A}_{\rm H}) \propto \mathscr{A}_{\rm H}^n$ and $\mathscr{F}^{(6\rm D)}(\mathscr{A}_{\rm H}) \propto \mathscr{A}_{\rm H}^m$ with $(n,m) < 1$. Note that, while the brick wall entropy in four dimensions depends only on the first and the second derivatives of the metric at the horizon, in six dimensions, it depends on the third derivative as well. It is tempting to propose that, at least in even dimensions, the brick wall entropy will depend on as many as derivatives of the metric as half the number of spacetime dimensions! However, the black hole entropy is a coordinate invariant concept. If the brick wall entropy depends on arbitrary derivatives of the metric functions at the horizon, then it is not a priori evident that the resulting entropy will be coordinate invariant. We believe that this is an issue that needs to be addressed satisfactorily.

6.2 Comparison with results from other approaches

Power law and logarithmic corrections to the Bekenstein-Hawking entropy $S_{\rm BH}$ that we have obtained in the brick wall approach has been encountered earlier in a few other approaches to black hole entropy. For instance, the Noether charge approach predicts a generic power law correction to the Bekenstein-Hawking entropy [8]. However, unlike our approach wherein the brick wall entropy can be completely expressed in terms of the metric and its first few derivatives at the event horizon, the Noether charge entropy can not be mapped to the horizon properties. It is also interesting to note that, in the case of the four dimensional Reissner-Nordström black hole, for large horizon area, i. e. when $M \gg \ell_{\rm Pl}$, the brick wall entropy $S_{\rm RN}^{(4\rm D)}$ [cf. Eq. (5.10)] reduces to

$$S_{\rm RN}^{(4\rm D)} \simeq S_{\rm BH} - \left(\frac{2\pi^{1/2}}{15}\right)\left(\frac{1}{\mathscr{A}_{\rm H}^{1/2}} - \frac{\ell_{\rm Pl}^2\, \mathscr{A}_{\rm H}^{3/2}}{M^2}\right). \tag{6.1}$$

Similar power law corrections arise on evaluating the entanglement entropy of such black holes [17]. This behavior seem to suggest a possible relationship between the brick wall model and the approach due to entanglement entropy. Another interesting feature is the absence of power law corrections in case of four dimensional Schwarzschild black hole. It seems to indicate that power law corrections to the Bekenstein-Hawking entropy are related with the presence of matter. The logarithmic corrections that we have obtained as in Eq. (5.5) for the case of the four dimensional Schwarzschild black hole has also been arrived at in other methods such as the approach through conformal field theory [12], statistical fluctuations around thermal equilibrium [16] and spin foam models [11]. However, it should be pointed out that the prefactor to the logarithmic term that we obtain turns out to be different from the one that arises in the other approaches.

Our analysis unambiguously indicates that corrections to the Bekenstein-Hawking entropy can arise even in a semi-classical approach. Clearly, it will be worthwhile to extend our analysis to the case of rotating black holes. We intend to carry out such an exercise in the near future.

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