

## Thermodynamics of the pion gas using the $O(N)$ model in $1/N$ expansion

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We investigate the thermodynamics of a pion gas within the  $O(N)$  model in the  $1/N$  expansion. Using the auxiliary field technique, we compute the effective potential up to the next-to-leading order (NLO) and show that it can be renormalized in a temperature-independent manner. The crucial step for the consistency of the calculation turns out to be the elimination of the auxiliary field prior to renormalization. Subsequently, we solve the NLO gap equation for the chiral condensate as a function of temperature both in the chiral limit and with explicit symmetry breaking. We propose a simple semi-analytic estimate of the NLO correction to the condensate and compare it to the exact numerical solution. Finally, we show that in the chiral limit the chiral symmetry is restored at finite temperature by a second-order phase transition, and determine the critical scaling of the order parameter. We study the dependence of the critical temperature on the renormalized coupling and find that in contrast to the weak-coupling limit, at strong coupling the critical temperature increases at NLO.

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## 1. Introduction

The low-energy physics of Quantum Chromodynamics (QCD) is governed by chiral symmetry, associated with the presence of (almost) massless quarks. In the sector of the lightest  $u$  and  $d$  quarks, the spontaneous breaking of the  $SU(2)_L \times SU(2)_R$  chiral symmetry in the vacuum gives rise to an isospin triplet of pseudo-Nambu–Goldstone bosons, the pions. As the lowest excitations of the ordered ground state, these dominate the thermodynamics at low temperatures. Since the conventional perturbation theory breaks down at finite temperature, one needs a suitable nonperturbative resummation scheme to deal with the thermodynamics, in particular with the symmetry-restoring phase transition. In QCD with two quark flavors, one can make use of the isomorphism of the chiral group with the  $O(4)$  rotation group. The pions as well as the sigma meson are described using the scalar  $O(4)$  model, which is subsequently generalized to  $O(N)$ , allowing for a nonperturbative (one-particle-irreducible)  $1/N$  expansion [1].

In this contribution we further develop the calculation of [2, 3] at the next-to-leading order (NLO) in the  $1/N$  expansion. In particular we show that the NLO effective potential can be renormalized by temperature-independent counterterms. This allows us to renormalize and solve the NLO gap equation. From the methodical point of view, our main conclusion is that for a consistent renormalization one has to dispense with the auxiliary field introduced in [1] (see [4] for an alternative approach). From the physical point of view, we show that the NLO truncation predicts a second-order chirally-restoring phase transition and the critical behavior in accordance with general universality arguments. For details of the calculations as well as a more complete bibliographical account of related work we refer the reader to [5].

## 2. The auxiliary field technique

At zero temperature and density, the  $O(N)$  sigma model is defined by the Euclidean Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi_i)^2 + \frac{\lambda}{8N}(\phi_i \phi_i - N f_\pi^2)^2 - \sqrt{N} H \phi_N,$$

where  $i = 1, \dots, N$ , and the parameters  $f_\pi$  and  $\lambda$  denote the pion decay constant and coupling, respectively. The  $H$ -term accounts for explicit symmetry breaking by nonzero quark mass. The essence of the auxiliary-field trick [1] is to add a new field  $\alpha$  by means of a pure Gaussian action,

$$\Delta \mathcal{L}_\alpha = \frac{N}{2\lambda} \left[ \alpha - \frac{i\lambda}{2N}(\phi_i \phi_i - N f_\pi^2) \right]^2.$$

In case we wish to study pion gas at finite isospin density and pion condensation [6–8], we further have to introduce the isospin chemical potential  $\mu_I$  in terms of the covariant derivative of the scalar field. Choosing  $\phi_{1,2}$  to represent the charged pion  $\pi^\pm$ , this amounts to adding

$$\Delta \mathcal{L}_\mu = -i\mu_I(\phi_1 \partial_0 \phi_2 - \phi_2 \partial_0 \phi_1) - \frac{1}{2}\mu_I^2(\phi_1^2 + \phi_2^2).$$

As a next step one defines the chiral condensate,  $\phi_0 \equiv \langle \phi_N \rangle / \sqrt{N}$ , the pion condensate,  $\rho_0 \equiv \langle \phi_1 \rangle / \sqrt{N}$ , and the condensate of the auxiliary field,  $iM^2 \equiv \langle \alpha \rangle$ . The fluctuations of the fields

are suppressed with respect to the condensates by the factor  $1/\sqrt{N}$ . This ensures that the NLO contribution to the effective action is given by a simple Gaussian integral over the fluctuations. The effective potential up to NLO thus becomes

$$V_{\text{eff}} = \frac{1}{2}NM^2(f_\pi^2 - \phi_0^2 - \rho_0^2) + \frac{NM^4}{2\lambda} + NH\phi_0 + \frac{1}{2}N\mu_I^2\rho_0^2 - \frac{1}{2}N\int_P \ln(P^2 + M^2) - \frac{1}{2}\int_P \ln[I(P, M)] - \frac{1}{2}\int_P \ln[(P^2 + m^2)^2 + 4\mu_I^2\omega_n^2] + \int_P \ln(P^2 + M^2), \quad (2.1)$$

where  $m^2 = M^2 - \mu_I^2$ ,  $\omega_n$  is the bosonic Matsubara frequency, and we used a compact notation for the sum-integral over the four-momentum  $P \equiv (i\omega_n, \vec{p})$ . Also, we denoted

$$I(P, M) = \frac{1}{\lambda} + \frac{\phi_0^2}{P^2 + M^2} + \frac{\rho_0^2(P^2 + m^2)}{(P^2 + m^2)^2 + 4\mu_I^2\omega_n^2} + \frac{1}{2}\int_Q \frac{1}{Q^2 + M^2} \frac{1}{(P+Q)^2 + M^2}.$$

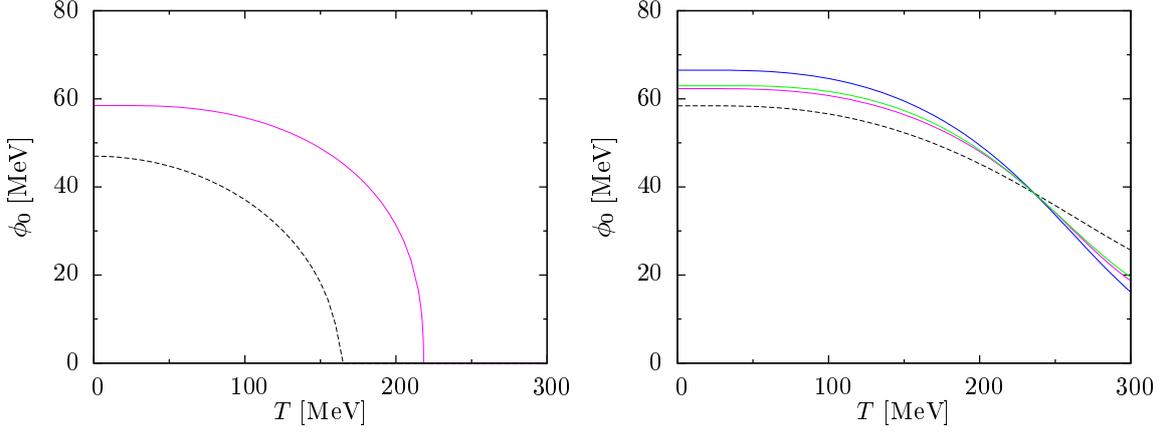
The leading-order (LO) effective potential is given by the first line of Eq. (2.1). The sum-integral involved is divergent, but the divergence can, after regularization using a four-dimensional Euclidean cutoff, be absorbed into renormalization of the parameters  $f_\pi^2$  and  $1/\lambda$ . The requirement of the stationarity of the effective potential leads to a set of gap equations, whose solution yields the standard phase diagram [6].

### 3. Next-to-leading order

The NLO effective potential is given by the second line of Eq. (2.1). The last two terms represent the correction to the pressure of the free pion gas induced by a finite chemical potential. With a four-dimensional cutoff (which is crucial for the successful renormalization at the next-to-leading order), they give rise to an unphysical  $\mu_I$ -dependent quadratic divergence. In the following, we therefore restrict ourselves to finite temperature, but zero chemical potential.

In general, the NLO effective potential has to be evaluated numerically. However, its divergent part can be calculated analytically [2, 3]. The coefficient of the quadratic divergence depends explicitly on temperature; this dependence disappears only when the LO gap equation for  $M$  is used. This suggests the necessity for a reinterpretation of the effective potential [5]. Since the auxiliary field  $\alpha$  was introduced as a composite operator of  $\phi_i$ , it may give rise to unphysical features of the effective potential. Once it is eliminated using its LO equation of motion, the resulting effective potential as a function of  $\phi_i$  alone can be renormalized in the usual manner. Using this elimination procedure, we renormalized the NLO effective potential and solved the gap equation for the chiral condensate. The results are shown in Fig. 1. The numerical values were obtained with  $f_\pi = 47 \text{ MeV}$  (differing by a factor 1/2 from the conventional value),  $\lambda = 30$ , renormalized at the scale  $100 \text{ MeV}$ , and  $H = (104 \text{ MeV})^3$ , adjusted in order to reproduce the physical pion mass in the vacuum.

Apart from a direct extremization of the effective potential, one may attempt to solve the gap equation by a systematic expansion in powers of  $1/N$ . Write generally the effective potential per degree of freedom as  $V(\phi) = V_0(\phi) + xV_1(\phi)$ , where  $V_{0,1}$  denote the LO and NLO contributions. Apparently, setting  $x = 0$  restricts the effective potential to its LO value, while  $x_0 = 1/4$  recovers the NLO result. Treating  $x$  as a continuous variable, i.e., demanding that the gap equation  $dV/d\phi = 0$



**Figure 1:** Chiral condensate as a function of temperature in the chiral limit (left panel) and at the physical point (right panel). LO (dashed line) and NLO (solid red line) values are compared. For explanation of the green and blue lines see the text.

be satisfied for all  $x \in [0, x_0]$ , one derives a formal integral equation for the condensate,

$$\phi(x_0) = \phi(0) - \int_0^{x_0} dx \frac{V_1'(\phi(x))}{V_0''(\phi(x)) + xV_1''(\phi(x))}.$$

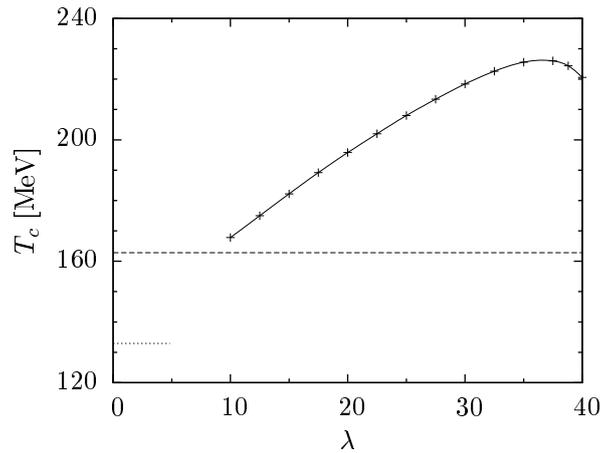
Assuming that the second derivative of the effective potential is constant (that is, approximating it at the LO point by a parabola), we find the explicit expression

$$\phi(x_0) = \phi(0) - \frac{x_0 V_1'(\phi(0))}{V_0''(\phi(0)) + x_0 V_1''(\phi(0))}.$$

This is the green line in Fig. 1. Note that the formal  $1/N$  expansion to the next-to-leading order would require neglecting  $V_1''$  in the denominator (blue line in Fig. 1). Obviously this is not a good approximation: It overshoots the NLO correction to the chiral condensate by nearly 100%.

In Fig. 2 we display the dependence of the NLO critical temperature on the renormalized coupling. While the weak-coupling analytic calculation predicts a negative correction,  $T_c = \sqrt{\frac{12}{1+\frac{2}{N}}} f_\pi$ , the numerical results show that at strong coupling, the critical temperature is increased by the NLO correction. Even though the auxiliary-field technique does not allow us to check the weak-coupling limit directly, the decreasing trend towards smaller values of the coupling is reassuring. Finally, from the solution of the gap equation near the critical temperature we were able to extract the critical exponent, which governs the scaling of the order parameter. The numerical value is in agreement with the analytic NLO expression,  $\nu = \frac{1}{2} - \frac{4}{N\pi^2}$ .

To conclude, we have shown that the NLO effective potential of the  $O(N)$  model can be consistently renormalized in a temperature-independent manner provided one uses the LO equation of motion to eliminate the auxiliary field before renormalization. We used this strategy to solve the NLO gap equation and proposed a simple semi-analytic approximation to the NLO chiral condensate which is very close to the exact value. We also investigated the coupling-dependence of the NLO critical temperature and found that the NLO correction is negative at weak coupling, but positive at strong coupling. In general, the NLO corrections to the observables turn out to be consistent with the  $1/N$  expansion, i.e., roughly of the order of ten percent.



**Figure 2:** NLO critical temperature as a function of the coupling. The dashed line denotes the LO (coupling-independent) value, while the dotted line indicates the NLO weak-coupling limit.

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