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Bare vs. Effective Fixed Point Action in Asymptotic Safety: The Reconstruction Problem

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We propose a method for the (re)-construction of a regularized functional integral, well defined in the ultraviolet limit, from a solution of the functional renormalization group equation of the effective average action. The functional integral is required to reproduce this solution. The method is of particular interest for asymptotically safe theories. The bare action for the Einstein-Hilbert truncation of Quantum Einstein Gravity (QEG) is computed and its flow is analyzed. As a second example conformally reduced gravity is explored. Various conceptual issues related to the reconstruction problem are discussed.

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1. Introduction

One of the major challenges of contemporary theoretical physics is the search for a compelling quantum theory of gravity. Despite the great efforts made until now, the complete description of such a theory seems to be far from completion. However, there exists a variety of approaches that could be enlightening for understanding certain aspects of what might ultimately be the correct quantum theory of gravity $[1-4]$.

Certainly, one of the most interesting points of view one can adopt arises from the observation that the failure of perturbative approaches in gravity does not imply that such a quantum theory cannot exist. In principle there is the possibility of quantizing gravity non-perturbatively, with the aid of Exact Renormalization Group techniques, say. In fact within the so called asymptotic safety program [5]-[30], a lot of efforts were devoted to establishing the existence of an ultraviolet fixed point at which Quantum Einstein Gravity (QEG) can be renormalized. Detailed calculations revealed that the renormalization group (RG) flow of the theory does indeed possess an appropriate non-Gaussian fixed point (NGFP) in all approximations which were investigated.

Formulating QEG in terms of the gravitational average action as proposed in [6], the RG flow in question is that of the effective average action $\Gamma_k[g_{\mu\nu}, \cdots]$, henceforth abbreviated EAA [31],[36]. While similar in spirit to the idea of a Wilson-Kadanoff renormalization, it replaces the iterated coarse graining procedure by a direct mode cutoff at the infrared (IR) scale k. More importantly, the EAA is a scale dependent version of the ordinary *effective* action, while a "genuine" Wilsonian action S_Λ^W is a *bare* action, i.e. it is to be used under a regularized path integral. As a result, it depends on the ultraviolet (UV) cutoff Λ; its dependence on Λ is governed by a RG equation which is different from that for Γ_k. In fact the scale dependence of Γ_k is governed by a functional RG equation (FRGE) which is one of the most useful items in the EAA "tool box".

As a quantization method, the FRGE is in principle sufficient to fully define a quantum field theory: given a complete RG trajectory, well defined for all values of $k \in [0, \infty)$, we have complete knowledge of all properties of the QFT at hand. Its Green's functions are the derivatives of Γ_k and at $k = 0$ they coincide with those of the standard effective action $\Gamma \equiv \Gamma_{k=0}$ [31]. The RG trajectory chosen must be free from divergences in both the IR and the UV limit. To realize the asymptotic safe property, the trajectory should be arranged to hit the NGFP in the UV limit.

However, one should stress the difference between the EAA and the Wilsonian approach. In a sense, S_Λ^W for different values of Λ is a set of actions for the same system: the Green's functions have to be computed from S_Λ^W by a further functional integration over the low momentum modes, and this integration renders them independent of Λ. By contrast, the EAA can be thought of as the standard effective action for a family of different systems: for any value of k it equals the standard effective action of a model with the bare action $S_\Lambda + \Delta_k S$ where $\Delta_k S$ is the mode suppression term. The corresponding n-point functions, computed as functional derivatives of Γ_k without any further integration, are scale dependent and they provide an effective field theory description [37]-[49] of the physics at scale k.

Because of these differences between the EAA, Γ_k, and a genuine Wilson action S_Λ, this way of constructing an asymptotically safe field theory does not by itself yield a regularized path integral over metrics $\gamma_{\mu\nu}$ whose continuum limit would be related to the RG trajectory $\{\Gamma_k, 0 \le k < \infty\}$ in a straightforward way.

In general the relationship between Γ_k and S_Λ will depend on how we regularize the path integral measure $\mathscr{D}_\Lambda \gamma_{\mu\nu}$ when defining the generating functional. In the following we shall demonstrate that it is possible to reconstruct a regularized functional integral such that it describes a fixed, prescribed asymptotically safe theory in the infinite cutoff limit $\Lambda \to \infty$. Adopting a particularly convenient UV regularization scheme we shall see that the information contained in Γ_k is sufficient in order to determine the related bare action S_Λ in the limit $\Lambda \to \infty$. Given a complete RG trajectory $\{\Gamma_k, 0 \leq k < \infty\}$, computed from a FRGE *without any UV regulator*, we deduce from it how the bare coupling constants contained in S_Λ must behave in the UV limit when the path integral (with the measure $\mathscr{D}_\Lambda \gamma$ and action S_Λ defined according to the special scheme adopted) is required to reproduce the prescribed Γ_k trajectory.

There are various motivations for trying to construct a path integral representation of asymptotically safe QEG:

(a) Working with the EAA alone we have no access to the microscopic (or "classical") system whose standard quantization gives rise to this particular effective action. A functional integral representation of the asymptotically safe theory will allow the reconstruction of the microscopic degrees of freedom that we implicitly integrated out in solving the FRGE, as well as their fundamental interactions. The path integral provides us with their action, and from this action, by a kind of generalized Legendre transformation, we can reconstruct their Hamiltonian description. From this phase space formulation we can read off the classical system whose quantization (also by other methods, canonically say) leads to the given effective action. We expect this system to be rather complicated so that it cannot be guessed easily. This is why we start at the effective level where we know what to look for, namely a Γ whose functional derivatives (*S*-matrix elements) are such that observable quantities have no divergences on all momentum scales.

(b) Many general properties of a quantum field theory are most easily analyzed in a path integral setting, the implementation of symmetries, the derivation of Ward identities or the incorporation of constraints, to mention just a few.

(c) Many approximation schemes (perturbation theory, large-N expansion, etc.) are more naturally described in a path integral rather than a FRGE language. A standard way of doing perturbation theory is to compute, order by order, the counter terms to be included in S_Λ to get finite physical results in the limit $\Lambda \to \infty$. Now, QEG is not renormalizable in perturbation theory and hence new counter terms with free coefficients must be introduced at each order. If, on the other hand, QEG is asymptotically safe, defined by a complete trajectory $\{\Gamma_k, 0 \leq k < \infty\}$, this trajectory "knows" the correct UV completion of the perturbative calculation. But in order to extract this information from Γ_k and make contact with the perturbative language of S_Λ-counter terms we must convert the Γ_k-trajectory to a S_Λ-trajectory first.

(d) Ultimately we would like to understand how QEG relates to other approaches to quantum gravity, such as canonical quantization, loop quantum gravity [2–4] or Monte Carlo simulations [53]-[56], in which the bare action often plays a central role. In the Monte Carlo simulation of the Regge and dynamical triangulations formulation, for instance, the starting point is a regularized path integral involving some discrete version of S_Λ, and in order to take the continuum limit one must fine tune the bare parameters in S_Λ in a suitable way. If one is interested in the asymptotic scaling, for instance, and wants to compare the analytic QEG predictions to the way the continuum is approached in the simulations, one should convert the Γ_k-trajectory to a S_Λ-trajectory first. The

map from Γ_k to S_Λ depends explicitly on how the path integral is discretized; so each alternative formulation of QEG has its own S_Λ for one and the same Γ_k.

The remaining sections of this paper are organized as follows. In Section 2 we describe some features of the EAA when it has an additional UV cutoff built into it. Then, in Section 3, we explain the reconstruction of the bare from the running effective action. Thereafter the method is applied, in Section 4, to the Einstein-Hilbert truncation of QEG, and in Section 5, to conformally reduced gravity in the local potential approximation. A summary and outlook is given in Section 6. In an appendix we further elaborate on the relation between the bare and the effective average action by means of a simple example which is of physical interest in its own right: the cosmological constant induced by a scalar matter field.

2. Effective Average Action with UV cutoff

In this section we describe how the functional integral underlying the definition of the effective average action can be made well defined. We regularize it by introducing an UV cutoff Λ and then derive, in a completely well defined way, the corresponding EAA and its flow equation in presence of Λ. Many different regularization schemes are conceiveable here. For concreteness we use a kind of "finite mode regularization" which is ideally suited for implementing the "background independence" mandatory in QEG.

For simplicity, we consider a single scalar field on flat space. The generalization to more complicated theories can be achieved by obvious notational changes.

Let $\chi(x)$ be a real scalar field on a flat d-dimensional Euclidean spacetime. In order to discretize momentum space we compactify spacetime to a d-torus. As a result, the eigenfunctions of the Laplacian $\Box = \delta^{\mu\nu}\partial_\mu\partial_\nu \equiv -\hat{p}^2$ are plane waves $u(x) \propto \exp(ip \cdot x)$ with discrete momenta p_μ and eigenvalues $-p^2$. Given a UV cutoff scale Λ, there are only finitely many eigenfunctions with $|p| \equiv \sqrt{p^2} \leq \Lambda$. We regularize the path integral in the UV by restricting the integration to those modes. Therefore, the field χ and the source J have an expansion

$$\chi(x) = \sum_{|p|\in[0,\Lambda]} \chi_p \, u_p(x), \quad \text{and} \quad J(x) = \sum_{|p|\in[0,\Lambda]} J_p \, u_p(x) \tag{2.1}$$

Now we define a UV-regulated analogue of the standard functional $W_k[J]$:

$$\exp\left(W_{k,\Lambda}[J]\right) \equiv \int \mathscr{D}_\Lambda \chi \exp\left(-S_\Lambda[\chi] - \Delta_k S[\chi] + \int d^d x J(x)\chi(x)\right) \tag{2.2}$$

The notation in eq.(2.2) is symbolic. In fact, its RHS involves only finitely many integrations and is not a genuine functional integral. Here, the measure $\mathscr{D}_\Lambda \chi$ stands for an integration over the Fourier coefficients χ_p with p^2 below Λ^2:

$$\int \mathscr{D}_\Lambda \chi = \prod_{|p|\in[0,\Lambda]} \int_{-\infty}^{\infty} d\chi_p \, M^{-[\chi_p]} \tag{2.3}$$

The arbitrary mass parameter M was introduced in order to give the canonical dimension zero to (2.3). As always in the EAA construction [31, 35], the IR modes with $|p| < k$ are suppressed by a IR cutoff $\mathscr{R}_k(\hat{p}^2)$ which gives rise to a momentum dependent mass term:

$$\Delta_k S[\chi] = \frac{1}{2} \int d^d x \, \chi(x) \mathscr{R}_k(\hat{p}^2)\chi(x) \tag{2.4}$$

In (2.2) the bare action S_Λ is allowed to depend on the UV cutoff. Ultimately we would like to fix this Λ-dependence in such a way that, for every finite k and J, the path integral has a well defined limit for $\Lambda \to \infty$.

Following the standard construction [31], we define the EAA as

$$\Gamma_{k,\Lambda}[\phi] \equiv \widetilde{\Gamma}_{k,\Lambda}[\phi] - \frac{1}{2}\int \mathrm{d}^d x\, \phi(x)\mathscr{R}_k(\hat{p}^2)\phi(x) \tag{2.5}$$

where $\widetilde{\Gamma}_{k,\Lambda}[\phi]$ is the Legendre transform of $W_{k,\Lambda}[J]$ with respect to J and $\phi = \{\phi_p\}_{|p|\in[0,\Lambda]}$ is the expectation value field $\phi(x) \equiv \langle\chi(x)\rangle$ obtained by differentiating $W_{k,\Lambda}[J]$ with respect to the source $J(x)$.

It is then straightforward to show that the definition (2.5) implies the following exact FRGE for $\Gamma_{k,\Lambda}$:

$$k\partial_k\Gamma_{k,\Lambda}[\phi] = \frac{1}{2}\mathrm{Tr}_\Lambda\left[\left(\Gamma^{(2)}_{k,\Lambda}[\phi] + \mathscr{R}_k\right)^{-1} k\partial_k\mathscr{R}_k\right] \tag{2.6}$$

Here, Tr_Λ denotes the trace restricted to the subspace spanned by the eigenfunctions of p^2 with eigenvalues smaller than Λ^2:

$$\mathrm{Tr}_\Lambda[\cdots] = \mathrm{Tr}\left[\theta(\Lambda^2 - \hat{p}^2)[\cdots]\right] \tag{2.7}$$

It is worth mentioning that Γ_k satisfies the integro-differential equation

$$\exp\left(-\Gamma_{k,\Lambda}[\phi]\right) = \int \mathscr{D}_\Lambda f \exp\left(-S_\Lambda[\phi + f] + \int \mathrm{d}^d x\, f(x)\frac{\delta\Gamma_{k,\Lambda}[\phi]}{\delta\phi(x)}\right.$$
$$\left. -\frac{1}{2}\int \mathrm{d}^d x\, f(x)\mathscr{R}_k(\hat{p}^2)f(x)\right) \tag{2.8}$$

where we have introduced the fluctuation field $f(x) \equiv \chi(x) - \phi(x)$. Eq.(2.8) is the starting point for our investigations in the next section.

A natural question that arises immediately is whether the UV cutoff can be removed from the FRGE. Indeed, for this to be possible it is sufficient to assume that the cutoff is chosen such that $k\partial_k\mathscr{R}_k(p^2)$ decreases rapidly enough so that the trace of the flow equation (2.6) exists even in the limit $\Lambda \to \infty$. As a result, the "Λ-free" FRGE without UV cutoff, valid for all $k \geq 0$, has the familiar form:

$$k\partial_k\Gamma_k[\phi] = \frac{1}{2}\mathrm{Tr}\left[\left(\Gamma^{(2)}_k[\phi] + \mathscr{R}_k(p^2)\right)^{-1} k\partial_k\mathscr{R}_k(p^2)\right] \tag{2.9}$$

We denote the solutions of (2.9) as $\{\Gamma_k,\ 0 \leq k < \infty\}$, and those of the FRGE (2.6) with UV cutoff as $\{\Gamma_{k,\Lambda},\ 0 \leq k < \Lambda\}$.

It is easy to show [27] that the flow equations for Γ_k and $\Gamma_{k,\Lambda}$ are essentially the same as long as $k \ll \Lambda$. Generically, when k approaches Λ from below, there exist some small corrections due to the UV cutoff which affect $\Gamma_{k,\Lambda}$ and cause it to differ from Γ_k. However, it is always possible to chose a special IR cutoff $\mathscr{R}_k(p^2)$ such that those corrections vanish. In particular, this happens with the optimized cutoff [52] $\mathscr{R}_k(p^2) = (k^2 - p^2)\theta(k^2 - p^2)$. As a result the functional $\Gamma_{k,\Lambda}$ satisfies the same FRGE as Γ_k, but is defined in the interval $k \leq \Lambda$ only. For identical initial conditions, a simple relation between the solutions of the two flow equations exists therefore:

$$\Gamma_{k,\Lambda} = \Gamma_k \quad \text{when } 0 \leq k \leq \Lambda \tag{2.10}$$
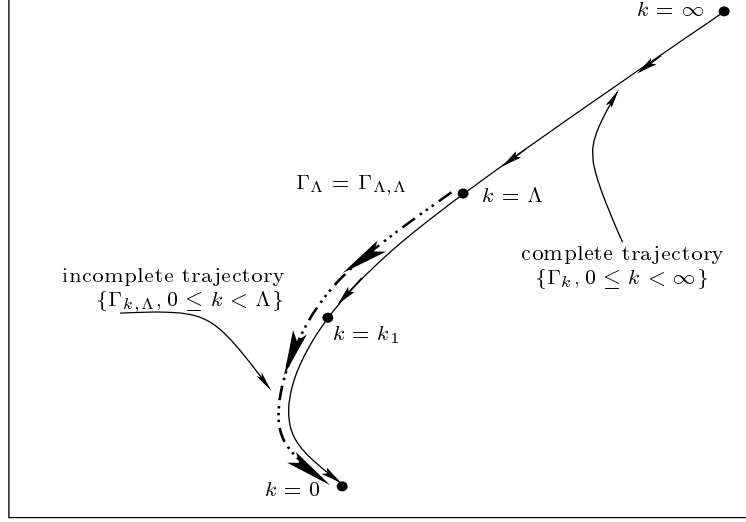
Figure 1: Employing the optimized cutoff every complete solution to the Λ-free FRGE gives rise to a solution of the FRGE with UV cutoff, valid up to any value of Λ.

Here Λ is a fixed, but arbitrary finite scale. In other words, $\{\Gamma_{k,\Lambda}, 0 \leq k < \Lambda\}$ is the restriction of the complete solution $\{\Gamma_k, 0 \leq k < \infty\}$ to the interval $k < \Lambda$. Thus, sending $\Lambda \to \infty$ in (2.10) is a trivial step. The situation is illustrated in Fig.1.

3. Reconstructing the bare action

The problem we want to address now is how one can determine the Λ-dependence of the bare action S_Λ, given some complete solution of the Λ-free flow equation, $\{\Gamma_k, k \in [0,\infty)\}$, According to (2.10), this complete solution implies a solution with a UV cutoff: $\{\Gamma_{k,\Lambda}, k \in [0,\Lambda]\}$. Setting $k = \Lambda$ we have in particular $\Gamma_{\Lambda,\Lambda} = \Gamma_\Lambda$ or, more explicitly,

$$\Gamma_{k=\Lambda,\Lambda} = \Gamma_{k=\Lambda} \tag{3.1}$$

Thus, knowing Γ_k for all k means that we know $\Gamma_{\Lambda,\Lambda}$ for all Λ.

Next we shall explain how, given Γ_k, the bare action S_Λ can be found. In particular, setting $k = \Lambda$ we are given $\Gamma_{\Lambda,\Lambda} = \Gamma_\Lambda$. Since Γ_k is a solution for all values of k the action $\Gamma_{\Lambda,\Lambda}$ is known for all values of Λ.

Using equation (2.8) we can obtain the desired relation between Γ_k and S_Λ. For this purpose we evaluate the functional integral on the RHS of (2.8) by a saddle point expansion. Let the fluctuation field be $f(x) \equiv f_0(x) + h(x)$ where f_0 is the stationary point of the action

$$S_{\text{tot}}[f;\phi] \equiv S_\Lambda[\phi + f] - \int \mathrm{d}^d x\, f(x) \frac{\delta \Gamma_{k,\Lambda}[\phi]}{\delta \phi(x)} - \frac{1}{2} \int \mathrm{d}^d x\, f(x) \mathscr{R}_k(\hat{p}^2) f(x) \tag{3.2}$$

Now, expanding S_{tot} to second order in h and performing the Gaussian integral over h we obtain

the following relationship between the bare and the average action:

$$\Gamma_{k,\Lambda}[\phi] = S_\Lambda[\phi + f_0] - \int \mathrm{d}^d x\, f_0 \frac{\delta \Gamma_{k,\Lambda}[\phi]}{\delta\phi} + \frac{1}{2}\int \mathrm{d}^d x\, f_0 \mathscr{R}_k f_0 + $$
$$+ \frac{1}{2}\mathrm{Tr}_\Lambda \ln\left[\left(\frac{\delta^2 S_\Lambda[\phi + f_0]}{\delta\phi^2} + \mathscr{R}_k\right)M^{-2}\right] + \cdots \tag{3.3}$$

Recalling that the stationary point f_0 has an expansion in powers of $\hbar$ too, (3.3) yields, in a symbolic notation,

$$\Gamma_{k,\Lambda}[\phi] - S_\Lambda[\phi] = -\int f_0 \frac{\delta}{\delta\phi}\left(\Gamma_{k,\Lambda} - S_\Lambda\right)[\phi] + \frac{1}{2}\int f_0\left(S_\Lambda^{(2)}[\phi] + \mathscr{R}_k\right)f_0 + \mathscr{O}(f_0^3) + $$
$$+ \frac{\hbar}{2}\mathrm{Tr}_\Lambda \ln\left\{\left[S_\Lambda^{(2)}[\phi] + S_\Lambda^{(3)}[\phi]f_0 + S_\Lambda^{(4)}[\phi]f_0 f_0 + \cdots + \mathscr{R}_k\right]M^{-2}\right\} + \mathscr{O}(\hbar^2) \tag{3.4}$$

Together with the expansion of the stationary point condition $(\delta S_{\mathrm{tot}}/\delta f)[f_0] = 0$ the above equation is solved self-consistently if $f_0 = 0 + \mathscr{O}(\hbar)$ and $\Gamma_{k,\Lambda}[\phi] - S_\Lambda[\phi] = \mathscr{O}(\hbar)$, which leads to the following 1-loop formula for the difference between the average and the bare action:

$$\Gamma_{k,\Lambda}[\phi] - S_\Lambda[\phi] = \frac{1}{2}\mathrm{Tr}_\Lambda \ln\left\{\left[S_\Lambda^{(2)}[\phi] + \mathscr{R}_k\right]M^{-2}\right\} \tag{3.5}$$

Setting $k = \Lambda$ we arrive at the final result

$$\Gamma_{\Lambda,\Lambda}[\phi] - S_\Lambda[\phi] = \frac{1}{2}\mathrm{Tr}_\Lambda \ln\left\{\left[S_\Lambda^{(2)}[\phi] + \mathscr{R}_\Lambda\right]M^{-2}\right\} \tag{3.6}$$

This is an equation to be solved for S_Λ. It tells us how the bare action S_Λ must depend on Λ in order to give rise to the prescribed $\Gamma_{\Lambda,\Lambda}$. The relation (3.6), and its obvious generalizations to more complicated theories, is our main tool for (re)constructing the path integral that belongs to a known solution of the FRGE.

An important comment is in order here. Even though the parameter M was introduced in (2.3) only in order to make the measure dimensionless, it has a nontrivial impact on the solution of (3.6) for the bare action S_Λ. Indeed, different choices of M can lead to quite different actions , but all of them are physically equivalent. (See [27] for a detailed discussion.) In this sense, changing M simply amounts to shifting the contributions from the measure into the bare action. Therefore neither $\int \mathscr{D}_\Lambda \chi$ nor $\exp(-S_\Lambda)$ have a physical meaning separately, only the combination of them has.

4. QEG and the Einstein-Hilbert truncation

The results derived above can be generalized to the case of Quantum Einstein Gravity [27], following the strategy for constructing the EAA as in [6] and implementing an UV cutoff in addition. Indeed, we shall use the same notations and conventions as in [6]to which the reader is referred for further details.

4.1 The gravitational average action

The construction of the gravitational average actions starts out from a path integral $\int \mathscr{D}\gamma_{\mu\nu} \exp\left(-S[\gamma_{\mu\nu}]\right)$. First we introduce a background metric $\bar{g}_{\mu\nu}(x)$, decompose the integration variable as $\gamma_{\mu\nu} \equiv \bar{g}_{\mu\nu} + h_{\mu\nu}$, and gauge-fix the resulting path integral over $h_{\mu\nu}$. It is this integral that we make well defined by introducing an UV cutoff into the measure along with an IR-suppression term $\Delta_k S$ analogous to (2.4):

$$\int \mathscr{D}_\Lambda h \, \mathscr{D}_\Lambda C \, \mathscr{D}_\Lambda \bar{C} \, \exp\left(-\widetilde{S}_\Lambda[h,C,\bar{C};\bar{g}] - \Delta_k S[h,C,\bar{C};\bar{g}]\right) \tag{4.1}$$

Here C^μ and $\bar{C}_\mu$ are the Fadeev-Popov ghosts, and the total bare action, $\widetilde{S}_\Lambda \equiv S_\Lambda + S_{\text{gf},\Lambda} + S_{\text{gh},\Lambda}$, which is allowed to depend on Λ, includes the gauge fixing term $S_{\text{gf},\Lambda}$ and the ghost action $S_{\text{gh},\Lambda}$. The UV cutoff is implemented by restricting the expansion to eigenfunctions of the covariant Laplacian $\bar{D}^2 \equiv \bar{g}^{\mu\nu}\bar{D}_\mu\bar{D}_\nu$ with eigenvalues κ smaller than a given Λ^2. Hence the measure reads in analogy with (2.3)

$$\int \mathscr{D}_\Lambda h = \prod_{\kappa \in [0,\Lambda^2]} \prod_m \int_{-\infty}^{\infty} \mathrm{d}h_{\kappa m} \, M^{-[h_{\kappa m}]} \tag{4.2}$$

and likewise for the ghosts. Here, m is a degeneracy index. The remaining steps in the construction of the gravitational average action proceed exactly as in [6]. Note that in this construction the background metric $\bar{g}_{\mu\nu}(x)$ is crucial not only for the gauge fixing and the IR cutoff, but also for implementing the UV cutoff.

The key properties of the functional thus defined are the exact FRGE and the integro-differential equation which it satisfies. The flow equation reads

$$k\partial_k \Gamma_{k,\Lambda}[\bar{h},\xi,\bar{\xi};\bar{g}] = \frac{1}{2}\text{STr}_\Lambda\left[\left(\Gamma_{k,\Lambda}^{(2)} + \widehat{\mathscr{R}}_k\right)^{-1} k\partial_k\widehat{\mathscr{R}}_k\right] \tag{4.3}$$

Here the supertrace "STr" implies the extra minus sign in the ghost sector. In fact, the cutoff operator $\widehat{\mathscr{R}}_k$ and the Hessian $\Gamma_{k,\Lambda}^{(2)}$ are matrices in the space of dynamical fields $\bar{h},\xi$ and $\bar{\xi}$. The background covariant regularization of the measure entails the appearance of the restricted trace

$$\text{STr}_\Lambda[\cdots] \equiv \text{STr}\left[\theta(\Lambda^2 + \bar{D}^2)[\cdots]\right] \tag{4.4}$$

In parallel with Section 2, we denote the solutions of the Λ-free FRGE as $\Gamma_k[\bar{h},\xi,\bar{\xi};\bar{g}]$. According to equation (2.10) for $k = \Lambda$, we get the corresponding relation:

$$\Gamma_{\Lambda,\Lambda}[\bar{h},\xi,\bar{\xi};\bar{g}] = \Gamma_\Lambda[h,\xi,\bar{\xi};\bar{g}] \tag{4.5}$$

The integro-differential equation analogous to (2.8) reads in QEG:

$$\begin{aligned}
\exp\left(-\Gamma_{k,\Lambda}[\bar{h},\xi,\bar{\xi};\bar{g}]\right) = \int \mathscr{D}_\Lambda h \mathscr{D}_\Lambda C \mathscr{D}_\Lambda \bar{C} \exp\Big[&-\widetilde{S}_\Lambda[h,C,\bar{C};\bar{g}] - \\
&-\Delta_k S[h-\bar{h},C-\xi,\bar{C}-\bar{\xi};\bar{g}] + \int \mathrm{d}^d x \, (h_{\mu\nu} - \bar{h}_{\mu\nu})\frac{\delta\Gamma_{k,\Lambda}}{\delta\bar{h}_{\mu\nu}} \\
&+ \int \mathrm{d}^d x \, (C^\mu - \xi^\mu)\frac{\delta\Gamma_{k,\Lambda}}{\delta\xi^\mu} + \int \mathrm{d}^d x \, (\bar{C}_\mu - \bar{\xi}_\mu)\frac{\delta\Gamma_{k,\Lambda}}{\delta\bar{\xi}_\mu} \Big]
\end{aligned} \tag{4.6}$$

4.2 The bare action at one loop

As in the scalar case above, we would like to use the information contained in a given solution $\Gamma_k[\bar{h}, \xi, \bar{\xi}; \bar{g}]$ of the Λ-free FRGE in order to find out which Λ-dependence must be given to the (total) bare action $\widetilde{S}_\Lambda$ if we want the path integral to possess a well defined limit $\Lambda \to \infty$ and to reproduce the prescribed Γ_k. Using eq.(4.5) and (4.6) and restricting ourselves to the 1-loop level, its derivation proceeds as in Section 3, with the result

$$\Gamma_{\Lambda,\Lambda}[\bar{h}, \xi, \bar{\xi}; \bar{g}] = \widetilde{S}_\Lambda[\bar{h}, \xi, \bar{\xi}; \bar{g}] + \frac{1}{2}\mathrm{STr}_\Lambda \ln\left[\left(\widetilde{S}_\Lambda^{(2)} + \widehat{\mathscr{R}}_\Lambda\right)[\bar{h}, \xi, \bar{\xi}; \bar{g}]\, \mathscr{N}^{-1}\right] \tag{4.7}$$

Here $\mathscr{N}$ is a block diagonal normalization matrix, equal to M^d and M^2 in the graviton and the ghost sector, respectively.

4.3 The twofold Einstein-Hilbert truncation

Solving the above equation for the bare action $\widetilde{S}_\Lambda[\bar{h}, \xi, \bar{\xi}; \bar{g}]$ is difficult, even at the one-loop level, since (4.7) is a complicated functional differential equation for the bare action. In practice one has to restrict the space of actions where Γ_k and $\widetilde{S}_\Lambda$ are defined by truncating them to a tractable number of terms. The simplest possibility, which we analyze here, is given by the Einstein-Hilbert truncation for both the effective and the bare action. As in [6] we make the ansatz

$$\begin{aligned}\Gamma_k[g, \bar{g}, \xi, \bar{\xi}] = &-(16\pi G_k)^{-1}\int \mathrm{d}^d x \sqrt{g}\left(R(g) - 2\bar{\lambda}_k\right) + S_{\mathrm{gh}}[g - \bar{g}, \xi, \bar{\xi}; \bar{g}]\\ &+ (32\pi G_k)^{-1}\int \mathrm{d}^d x \sqrt{\bar{g}}\bar{g}^{\mu\nu}(\mathscr{F}_\mu^{\alpha\beta} g_{\alpha\beta})(\mathscr{F}_\nu^{\rho\sigma} g_{\rho\sigma})\end{aligned} \tag{4.8}$$

The third term on the RHS of eq.(4.8) is the gauge fixing term[1] corresponding to the harmonic coordinate condition, involving $\mathscr{F}_\mu^{\alpha\beta} \equiv \delta_\mu^\beta \bar{g}^{\alpha\gamma}\bar{D}_\gamma - \frac{1}{2}\bar{g}^{\alpha\beta}\bar{D}_\mu$, and the second term is the associated ghost action. We make an analogous ansatz for the bare action:

$$\begin{aligned}\widetilde{S}_\Lambda[g, \bar{g}, \xi, \bar{\xi}] = &-(16\pi \check{G}_\Lambda)^{-1}\int \mathrm{d}^d x \sqrt{g}\left(R(g) - 2\check{\lambda}_\Lambda\right) + S_{\mathrm{gh}}[g - \bar{g}, \xi, \bar{\xi}; \bar{g}]\\ &+ (32\pi \check{G}_\Lambda)^{-1}\int \mathrm{d}^d x \sqrt{\bar{g}}\bar{g}^{\mu\nu}(\mathscr{F}_\mu^{\alpha\beta} g_{\alpha\beta})(\mathscr{F}_\nu^{\rho\sigma} g_{\rho\sigma})\end{aligned} \tag{4.9}$$

Eq.(4.8) contains the running dimensionful parameters G_k and $\bar{\lambda}_k$. The corresponding bare Newton and cosmological constant, respectively, are denoted $\check{G}_\Lambda$ and $\check{\lambda}_\Lambda$.

Setting the ghost terms to zero, $\xi = \bar{\xi} = 0$, and $\bar{g} = g$, the super trace has a derivative expansion of the form

$$\frac{1}{2}\mathrm{STr}_\Lambda \ln\left[\left(\widetilde{S}_\Lambda^{(2)} + \widehat{\mathscr{R}}_\Lambda\right)[0, 0, 0; \bar{g}]\mathscr{N}^{-1}\right] = B_0 \Lambda^d \int \mathrm{d}^d x \sqrt{g} + B_1 \Lambda^{d-2}\int \mathrm{d}^d x \sqrt{g}R(g) + \cdots \tag{4.10}$$

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[1]We employ a non-dynamical gauge fixing parameter $\alpha = 1$ here.

with dimensionless coefficients B_0 and B_1, respectively. In $d = 4$ and using the optimized cutoff shape function they are given by [27]

$$B_0 = \frac{1}{32\pi^2}\left[5\ln\left(1 - 2\check{\lambda}_\Lambda\right) - 5\ln\left(\check{g}_\Lambda\right) + Q_\Lambda\right] \tag{4.11a}$$

$$B_1 = \frac{1}{3}B_0 + \Delta B_1 \tag{4.11b}$$

$$\Delta B_1 \equiv \frac{1}{16\pi^2}\frac{2 - \check{\lambda}_\Lambda}{1 - 2\check{\lambda}_\Lambda} \tag{4.11c}$$

$$Q_\Lambda \equiv 12\ln\left(\Lambda/M\right) + b_0 \tag{4.11d}$$

with the constant $b_0 \equiv -5\ln\left(32\pi\right) - \ln 2$. Using (4.10) in (4.7) and equating the coefficients of the independent invariants we obtain two equations relating the effective to the bare parameters:

$$\frac{1}{G_\Lambda} - \frac{1}{\check{G}_\Lambda} = -16\pi\, B_1\, \Lambda^{d-2}, \qquad \frac{\bar{\lambda}_\Lambda}{G_\Lambda} - \frac{\check{\lambda}_\Lambda}{\check{G}_\Lambda} = 8\pi\, B_0\, \Lambda^d \tag{4.12}$$

In terms of the dimensionless quantities defined by $g_\Lambda \equiv \Lambda^{d-2}G_\Lambda$, $\check{g}_\Lambda \equiv \Lambda^{d-2}\check{G}_\Lambda$, and analogous relations for the bare couplings, we get:

$$\frac{1}{g_\Lambda} - \frac{1}{\check{g}_\Lambda} = -16\pi\, B_1 \tag{4.13a}$$

$$\frac{\lambda_\Lambda}{g_\Lambda} - \frac{\check{\lambda}}{\check{g}_\Lambda} = 8\pi\, B_0 \tag{4.13b}$$

The algebraic system of equations (4.13) should allow us to determine $\check{g}_\Lambda$ and $\check{\lambda}$ for given g_Λ and λ_Λ.

Unfortunately it is impossible to solve the system (4.13) analytically for the bare parameters. However, in [27] we solved those equations numerically, and found a well defined pair $(\check{g}, \check{\lambda})$ for all $g > 0$ and $\lambda < 1/2$, for a wide range of values of the constant $Q = 12\ln c + b_0$[2]. Different values of Q correspond to different normalizations of the measure.

Indeed, for an effective RG trajectory, the fixed point behavior $\lim_{k\to\infty}(g_k, \lambda_k) = (g_*, \lambda_*)$ is mapped onto an analogous fixed point behavior at the bare level (after removing he explicit Λ dependence from the map by setting $M = c\Lambda$). The image of the GFP is always at $\check{g}_* = \check{\lambda}_* = 0$, while the coordinates of the "bare" NGFP, $\check{g}_*$ and $\check{\lambda}_*$, depend on the value of Q. This behavior is illustrated in Fig.2, where we present the result of applying the map $(g, \lambda) \mapsto (\check{g}, \check{\lambda})$ for different values of Q, to a set of representative effective RG trajectories on the half plane $g > 0$. However, we emphasize that all choices of Q are physically equivalent. Varying Q simply amounts to shifting contributions back and forth between the action and the measure.

It is instructive to determine the linearized flow near the two "bare" fixed points and to determine the corresponding critical exponents, if they can be defined.

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[2]The map $(g, \lambda) \mapsto (\check{g}, \check{\lambda})$ is explicitly Λ-dependent because of the parameter $Q_\Lambda \equiv 12\ln\left(\Lambda/M\right) + b_0$. This Λ-dependence can be removed by including appropriate factors of the UV cutoff into the measure. If we set $M = c\Lambda$ with an arbitrary $c > 0$ the quantity $Q = 12\ln c + b_0$ becomes a Λ-independent constant. As a result, the map $(g, \lambda) \mapsto (\check{g}, \check{\lambda})$ has no explicit dependence on any (UV or IR) cutoff.

(a)   **(b)**

(c)   **(d)**
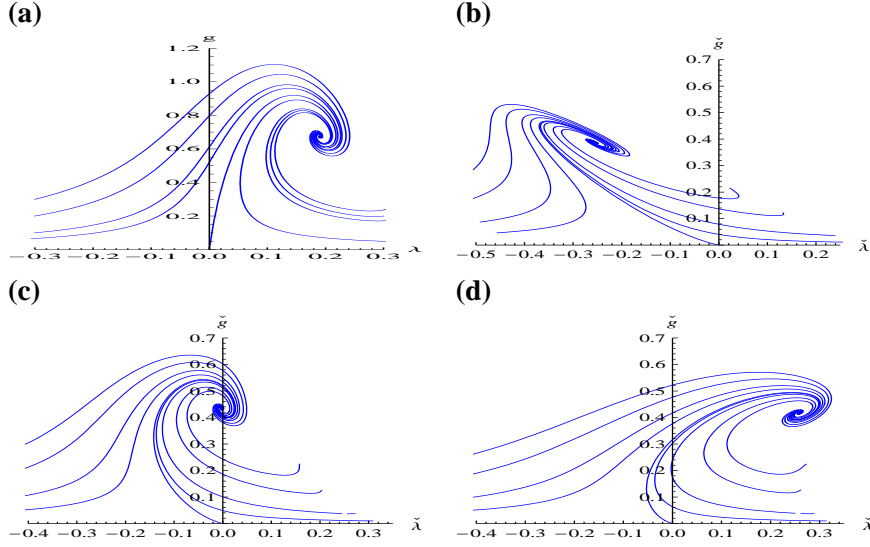
Figure 2: The diagram (a) shows the phase portrait of the effective RG flow on the (g, λ)-plane. The other diagrams are its image on the $(\check{g}, \check{\lambda})$-plane of bare parameters for three different values of Q, namely (b) $Q = +1$, (c) $Q = -0.1167$ where $\check{\lambda}_* = 0$, and (d) $Q = -1$, respectively.

Both the "effective" and the "bare" NGFP are inner points of the corresponding coupling constant space. The flow in the vicinity of one is the diffeomorphic image of the flow near the other. The RG running of the respective scaling fields is $\propto k^{-\theta}$ and $\propto \Lambda^{-\theta}$, respectively, with the same critical exponents θ. The "bare" GFP instead is located on the line $\check{g} = 0$, i.e. on the boundary of the domain on which the map from the effective to the bare couplings is defined. In its vicinity (on the half plane with $\check{g} > 0$) the "bare" running is characterized by logarithmically corrected power laws. The "effective" GFP, on the other hand, shows pure power law scaling. Near the GFP, we can expand the relations (4.13), obtaining, in leading order:

$$\check{g} = g + \mathcal{O}(g^2, \lambda^2) \tag{4.14}$$

$$\check{\lambda} = \lambda - \frac{g}{4\pi}\left(Q - 5\ln g\right) + \frac{g\lambda}{6\pi}\left[3 - Q + 5\ln g\right] + \mathcal{O}(g^2, \lambda^2) \tag{4.15}$$

These expansions are the first few terms of a power-log series. This implies that the bare running indeed follows logarithmically corrected power laws.

5. Conformally Reduced Gravity

As another example of the strategy described above, we next analyze conformally reduced gravity [24, 25] in which only the conformal factor of the metric is quantized. The simplicity of the model allows for the use of comparatively general truncations. We will use the method of the Local Potential Approximation (LPA) to deduce the general form of the bare potential contained in the reconstructed S_Λ of this model.

In conformally reduced gravity one considers only dynamical metrics $g_{\mu\nu} \equiv \phi^2 \hat{g}_{\mu\nu}$ and background metrics $\bar{g}_{\mu\nu} \equiv \chi_B^2 \hat{g}_{\mu\nu}$ which are conformal to a fixed reference metric $\hat{g}_{\mu\nu}$, usually taken to be $\hat{g}_{\mu\nu} = \delta_{\mu\nu}$. The background metric is used in order to construct a coarse graining operator

$\mathscr{R}_k[\chi_B]$ which cuts off the spectrum of $-\bar{\Box}$, the Laplace-Beltrami operator of $\bar{g}_{\mu\nu}$, at scale the k^2. In this way $1/k$ has the character of a proper length with respect to the background metric, exactly as in full QEG. Furthermore, we introduce a sharp UV cutoff by restricting the $-\bar{\Box}$ eigenvalues to be smaller than Λ^2. Following the same steps as in Section 2, one can construct a UV-regulated functional $W_{k,\Lambda}$ and with it the corresponding effective average action $\Gamma_{k,\Lambda}$. The reconstruction formula is a slight generalization of (3.6):

$$\Gamma_{\Lambda,\Lambda}[\phi,\chi_B] - S_\Lambda[\phi,\chi_B] = \frac{1}{2}\mathrm{Tr}\Big[\theta(\Lambda^2+\bar{\Box})\ln\Big\{\big[S_\Lambda^{(2)}[\phi,\chi_B]+\mathscr{R}_\Lambda[\chi_B]\big]M^{-2}\Big\}\Big] \tag{5.1}$$

Here, $S^{(2)}[\phi,\chi_B]_{xy} = \frac{1}{\sqrt{g(\hat{x})}\sqrt{g(\hat{y})}}\frac{\delta^2}{\delta\phi(x)\delta\phi(y)}S[\phi,\chi_B]$, and the explicit form of the coarse graining operator reads

$$\mathscr{R}_k[\chi_B] = -\frac{3}{4\pi G_k}\chi_B^2 k^2 R^{(0)}\Big(-\frac{\widehat{\Box}}{\chi_B^2 k^2}\Big) \tag{5.2}$$

In order to solve (5.1) we now make a local potential ansatz for both the effective and the bare action:

$$\Gamma_{k,\Lambda}[\phi,\chi_B] = -\frac{3}{4\pi G_{k,\Lambda}}\int \mathrm{d}^4 x\Big\{-\frac{1}{2}\phi\widehat{\Box}\phi + F_{k,\Lambda}(\phi,\chi_B)\Big\} \tag{5.3}$$

$$S_\Lambda[\phi,\chi_B] = -\frac{3}{4\pi\check{G}_{k,\Lambda}}\int \mathrm{d}^4 x\Big\{-\frac{1}{2}\phi\widehat{\Box}\phi + \check{F}_{k,\Lambda}(\phi,\chi_B)\Big\} \tag{5.4}$$

Inserting (5.3) and (5.4) into (5.1) we get:

$$\frac{F_\Lambda(\phi,\chi_B)}{G_\Lambda} - \frac{\check{F}_\Lambda(\phi,\chi_B)}{\check{G}_\Lambda} = -\frac{1}{48\pi}\chi_B^4\Lambda^4\ln\Big[\frac{1}{\check{G}_\Lambda\Lambda^2}\big(\Lambda^2\chi_B^2+\partial_\phi^2\check{F}_\Lambda(\phi,\chi_B)\big)\Big] \tag{5.5}$$

We have derived the last equation *without setting* $\phi = \chi_B$, that is, $g_{\mu\nu} = \bar{g}_{\mu\nu}$. Our motivation is simply to keep separated terms which are purely background dependent from those which are dynamical. The above truncations assume that the potentials have an extra, i.e. explicit dependence on χ_B (in addition to the one implicit in $\phi = \chi_B + \bar{f}$ where $\bar{f}$ is the fluctuation average). Extended truncations which have an explicit dependence on the background, were investigated in this setting in ref. [59].

It is convenient for the analysis to rewrite the above equation in terms of dimensionless quantities. We use

$$g_\Lambda \equiv G_\Lambda\Lambda^2, \quad \varphi \equiv \Lambda\phi \quad \text{and} \quad b \equiv \Lambda\chi_B \tag{5.6}$$

$$Y_\Lambda(\varphi,b) \equiv \Lambda^2 F_\Lambda(\varphi/\Lambda, b/\Lambda) \tag{5.7}$$

and analogous relations for the bare quantities. The resulting equation is:

$$\frac{Y_\Lambda(\varphi,b)}{g_\Lambda} = \frac{\check{Y}_\Lambda(\varphi,b)}{\check{g}_\Lambda} - \frac{1}{48\pi}b^4\ln\Big[\frac{1}{\check{g}_\Lambda}\big(b^2+\partial_\varphi^2\check{Y}_\Lambda(\varphi,b)\big)\Big] \tag{5.8}$$

This equation strongly suggests that the bare potential $\check{Y}_\Lambda(\varphi,b)$ may depend explicitly on the background field.

Let us nonetheless start by exploring the "$b = \varphi$" truncation which is analogous to the $g_{\mu\nu} = \bar{g}_{\mu\nu}$-truncation used in QEG. Then eq.(5.8) reduces to

$$\frac{Y_\Lambda(\varphi)}{g_\Lambda} = \frac{\check{Y}_\Lambda(\varphi)}{\check{g}_\Lambda} - \frac{1}{48\pi}\varphi^4 \ln\left[\frac{1}{\check{g}_\Lambda}\left(\varphi^2 + \check{Y}_\Lambda{}''(\varphi)\right)\right] \tag{5.9}$$

According to ref.[25], the Λ-free effective potential exhibits the following NGFP on the infinite dimensional space of the Y's (for the R^4 topology):

$$Y_*(\varphi) = -\frac{1}{6}\frac{\lambda_*}{g_*}\varphi^4 \tag{5.10a}$$

$$\lambda_* \approx 0.279, \qquad g_* \approx 4.650 \tag{5.10b}$$

Therefore, one can insert this result on the LHS on eq.(5.9) and solve for the bare potential $\check{Y}_*(\varphi)$. It can be demonstrated that this indeed has a solution which can be found numerically. Asymptotically (for $\varphi \to \infty$) it behaves as

$$\check{Y}_*(\varphi) \approx \frac{\check{g}_*}{48\pi}\varphi^4 \ln \varphi^2 + \mathcal{O}(\varphi^4) \tag{5.11}$$

Remarkably, while this potential is of the familiar Coleman-Weinberg form, it is here part of the *bare* action; it corresponds to a simple φ^4 monomial in the effective one. Thus, as compared to a standard scalar matter field theory, the situation is exactly inverted.

It is not difficult to understand how this comes about: The difference $\Gamma_* - S_*$ is given by a trace $\mathrm{Tr}[\cdots]$ which is nothing but a differentiated one-loop determinant. As a consequence, Γ_* and S_* differ precisely by terms typical of a one loop effective action, and those include the potential term $\varphi^4 \ln \varphi$. Hence a φ^4 term in Γ_* unavoidably amounts to a Coleman-Weinberg term in S_*, at least *within the truncation considered*.

In fact, returning now to the more general truncations with an extra χ_B-dependence of $F_k(\phi, \chi_B)$ it can be shown that actually S_* and Γ_* do not differ by a "dynamical" term $\varphi^4 \ln \varphi$, nonanalytic in the quantum field, but rather merely by its background analog $b^4 \ln b$. It also can be shown [59] that the bare potential is analytic in φ if the effective one is so. This example nicely demonstrates that occasionally the oversimplifications caused by the class of "$b = \varphi$", or "$g_{\mu\nu} = \bar{g}_{\mu\nu}$" truncations can lead to a qualitatively wrong picture.

6. Discussion

Here we described some first steps towards solving the reconstruction problem for asymptotically safe quantum field theories. In particular we showed explicitly that, after specifying a UV regularization scheme and a measure, every solution of the flow equation for the effective average action without an UV cutoff gives rise to a regularized path integral with a well defined limit $\Lambda \to \infty$, and to a UV cutoff dependent bare action.

While the method we developed is completely general, this work was motivated by the Asymptotic Safety program in Quantum Einstein Gravity. As to yet the investigations based upon the EAA focused on computing RG trajectories of the Λ-free FRGE and establishing the existence of a non-Gaussian fixed point. The present work aims at completing the Asymptotic Safety program in the

sense of finding the, yet unknown, quantum system which we implicitly quantize by picking a solution of the flow equation. In fact, in our approach the primary definition of "QEG" is in terms of an RG trajectory of the EAA that emanates from the fixed point. The advantage of this strategy, defining the theory in terms of an effective rather than bare action, is that it automatically guarantees an "asymptotically safe" high energy behavior. The disadvantage is that in order to complete the Asymptotic Safety program, that is, to find the underlying microscopic theory, extra work is needed.

Once we know the microscopic, i.e. bare action we can attempt a kind of "Legendre transformation" to find appropriate phase space variables, a microscopic Hamiltonian, and thus a canonical description of the bare theory. Only at this level we can identify the degrees of freedom that got quantized, as well as their fundamental interactions. Since the Hamiltonian is unlikely to turn out quadratic in the momenta, the "Legendre tansformation" involved is to be understood as a generalized, i.e. quantum mechanical one. In the simplest case it consists in reformulating a given configuration space path integral $\int \mathscr{D}\Phi \exp\left(iS[\Phi]\right)$ as a phase space integral $\int \mathscr{D}\Phi \int \mathscr{D}\Pi \exp\left(i\int \Pi\dot{\Phi} - H[\Pi,\Phi]\right)$. With other words, we must undo the integrating out of the momenta.

However, given the complexity of Γ_* which most probably contains higher derivatives and non-local terms a generalized, Ostrogradski-type phase space formalism will emerge presumably.

Being interested in a canonical description of the "bare" NGFP action one might wonder if there exists an alternative formalism which deals directly with the RG flow of Hamiltonians rather than Lagrangians. It seems that there hardly can be a practicable approach of this kind which is similar in spirit to the EAA. The reason is as follows.

If we apply a coarse graining step to an action which contains only, say, first derivatives of the field, the result will contain higher derivatives in general. This poses no special problem in a Lagrangian setting, but for the Hamiltonian formalism it implies that new momentum variables must be introduced. As a result, the coarse grained Hamiltonian "lives" on a different phase space (in the sense of Ostrogradski's method) than the original one. Therefore, at least in a straightforward interpretation, there is no Hamiltonian analog of the flow on the space of actions. For this reason there is probably no simple way of getting around the "reconstruction problem".

However, the above discussion does not contradict other approaches where the renormalization procedure could be applied in a Hamiltonian description [57] since there the coarse graining is performed in space (rather than spacetime) only.

One should also emphasize that it is by no means clear from the outset what kind of fundamental degrees of freedom will be found in this Hamiltonian analysis. In our approach *the only nontrivial input is the theory space*, the space of functionals on which the renormalization group operates. Having fixed this space a FRGE can be written down, the resulting flow can be computed, its fixed point(s) Γ_* can be identified, and the associated asymptotically safe field theories can be defined *without any additional input*. As a consequence, the only statement about the degrees of freedom in these theories which we can make on general grounds is that they can be "carried" by precisely those fields on which Γ_k depends. (In the case at hand, theory space contains all functionals $\Gamma[g,\bar{g},\xi,\bar{\xi}]$ which are invariant under diffeomorphisms.) Clearly, just knowing the carrier field but not the action, here Γ_*, tells us comparatively little about the degrees of freedom. The action Γ_*, however, is a *prediction* of the theory, not an input. From this point of view it is quite nontrivial

that QEG was found to have RG trajectories which indeed describe classical General Relativity on macroscopic scales.

In this work we also investigated QEG in the Einstein-Hilbert truncation, constructing a map relating the effective to the bare Newton and cosmological constant, and we analyzed the properties of the "bare" RG flow. We saw in particular that the "effective" NGFP maps onto a corresponding "bare" one; in its vicinity the scaling fields show a power law running with the same critical exponents as at the effective level. The situation is different for the GFP which is a boundary point of parameter space. The pure power laws of the "effective" flow receive logarithmic corrections on the "bare" side. We also described the case of conformally reduced gravity within an (infine dimensional!) truncation of the LPA type. In this example we saw in particular that in order to get a qualitatively correct picture one must go beyond the class of "$g = \bar{g}$"-truncations.

Leaving aside gravity, in future work it will be interesting to analyze for instance also higher dimensional Yang-Mills theory along the same lines. In fact, in ref.[32] the effective average action of d-dimensional Yang-Mills theory was considered in a simple $\int (F_{\mu\nu}^a)^2$-truncation. According to this truncation[3], Γ_k has a NGFP in the UV if $4 < d < 24$. Inspired by the structure of the one-loop effective action in Yang-Mills theory one would expect that the "bare" counterpart of the $\int (F_{\mu\nu}^a)^2$-fixed point should contain terms like $\int (F_{\mu\nu}^a)^2 \ln (F_{\mu\nu}^a)^2$, and also nonlocal ones such as $\int F_{\mu\nu}^a f(-D^2) F_{\mu\nu}^a$. For the following reason it is of some practical importance to find out whether this is actually the case in a sufficiently general, reliable truncation. It seems comparatively easy to perform Monte-Carlo simulations in $d = 5$, say, so that one could possibly get an independent confirmation of the results obtained from the average action. However, the problem is that a priori we do not really know which bare theory should be simulated in order to arrive at the lattice version of the average action results. The present analysis suggests that if Yang-Mills theory is asymptotically safe in $d = 5$, the effective fixed point action Γ_* might be simple, but S_* could contain "exotic" nonlinear and nonlocal terms. If so, it is conceivable that S_* is sufficiently different from $\int (F_{\mu\nu}^a)^2$ to belong to a new universality class. In this case a Monte-Carlo simulation based upon the conventional Wilson gauge field action might not find a NGFP, while it should show up when a discretized version of S_* is used.

Completely analogous remarks apply to the nonlinear sigma model in $d > 2$ which, according to the lowest order truncation of the EAA, is asymptotically safe too [60].

A. The induced cosmological constant, and what we can learn from it

In this appendix we illustrate how the bare and the average action are related by means of a simple example: the cosmological constant induced by a scalar matter field quantized in a classical gravitational background. The example also serves as a toy model to highlight several issues arising in the complete formulation of QEG. For further details we refer to [27]

We start with an action of a scalar field which is minimally coupled with the classical metric $g_{\mu\nu}$. As we are interested only in the induced cosmological constant it will be sufficient to keep the

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[3]For a generalization see also [58].

$\int \mathrm{d}^d x \sqrt{g}$ gravitational invariant in the bare and average action, respectively:

$$S_\Lambda[\chi] = \frac{1}{2} \int \mathrm{d}^d x \sqrt{g} \Big[g^{\mu\nu} \partial_\mu \chi \partial_\nu \chi + \breve{m}^2 \chi^2 \Big] + \check{C}_\Lambda \int \mathrm{d}^d x \sqrt{g} \tag{A.1}$$

$$\Gamma_{k,\Lambda}[\phi] = \frac{1}{2} \int \mathrm{d}^d x \sqrt{g} \Big[g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + m^2 \phi^2 \Big] + C_{k,\Lambda} \int \mathrm{d}^d x \sqrt{g} \tag{A.2}$$

The solution $\Gamma_k[\phi]$ of the Λ-free FRGE has a structure similar to (A.2) involving a running parameter C_k. The three C-factors $\check{C}_\Lambda$, $C_{k,\Lambda}$ and C_k are related to the corresponding cosmological constants $\bar{\lambda}$ by $C \equiv (\bar{\lambda}/8\pi G)$ where Newton's constant G does not run in the approximation considered. Furthermore, for the purposes of this demonstration, the running of the masses is also neglected.

Notice that since S_Λ is quadratic in χ the functional integral (2.2) for $W_{k,\Lambda}[J]$, appropriately generalized to a curved background, can be solved exactly. With the restricted trace $\mathrm{Tr}_\Lambda[\cdots] \equiv \mathrm{Tr}[\theta(\Lambda^2 + \mathrm{D}^2)(\cdots)]$ one obtains

$$W_{k,\Lambda}[J] = \frac{1}{2} \int \mathrm{d}^d x \sqrt{g} \, J \left[-D^2 + \breve{m}^2 + \mathscr{R}_k(-D^2) \right]^{-1} J - \check{C}_\Lambda \int \mathrm{d}^d x \sqrt{g} -$$
$$- \frac{1}{2} \mathrm{Tr}_\Lambda \ln \left[\Big(-D^2 + \breve{m}^2 + \mathscr{R}_k(-D^2) \Big) M^{-2} \right] \tag{A.3}$$

In this simple case we can compute $\Gamma_{k,\Lambda}$ directly from the very definition of the EAA, eq.(2.5):

$$\Gamma_{k,\Lambda}[\phi] = \frac{1}{2} \int \mathrm{d}^d x \sqrt{g} \Big(g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + \breve{m}^2 \phi^2 \Big) + \check{C}_\Lambda \int \mathrm{d}^d x \sqrt{g} +$$
$$+ \frac{1}{2} \mathrm{Tr}_\Lambda \ln \left[\Big(-D^2 + \breve{m}^2 + \mathscr{R}_k(-D^2) \Big) M^{-2} \right] \tag{A.4}$$

The flow equation for $\Gamma_{k,\Lambda}[\phi]$ is a slight generalization of (2.6) with the flat metric replaced by $g_{\mu\nu}$ everywhere. In particular, the operator $\hat{p}^2 \equiv -D^2$ is now to be interpreted as the Laplace-Beltrami operator constructed with the metric $g_{\mu\nu}$. Upon inserting (A.2) the FRGE assumes the form

$$k\partial_k C_{k,\Lambda} \int \mathrm{d}^d x \sqrt{g} = \frac{1}{2} \mathrm{Tr} \Big[\theta(\Lambda^2 + D^2) \, \mathscr{K}(-D^2)^{-1} \, k\partial_k \mathscr{R}_k(-D^2) \Big] \tag{A.5}$$

with $\mathscr{K}(\hat{p}^2) \equiv \hat{p}^2 + m^2 + \mathscr{R}_k(\hat{p}^2)$. To make eq.(A.5) consistent we may retain only the volume term $\propto \int \mathrm{d}^d x \sqrt{g}$ in the derivative expansion of the trace on its RHS. It is easily found by inserting a flat metric. Using the optimized cutoff (A.5) it reduces to, with $v_d \equiv [2^{d+1} \pi^{d/2} \Gamma(d/2)]^{-1}$,

$$k\partial_k C_{k,\Lambda} = \frac{4v_d}{d} \left(\frac{k^2}{k^2 + m^2} \right) k^d \tag{A.6}$$

We observe that the RHS of (A.6) has become *independent of the cutoff* Λ.

Inserting the Γ_k-ansatz (involving C_k) into the Λ-free flow equation we find eq.(A.6), too, this time for C_k. Hence $k\partial_k C_k = k\partial_k C_{k,\Lambda}$ for all $k \le \Lambda$, and therefore $C_k = C_{k,\Lambda}$ for $k \le \Lambda$ if the same initial conditions are imposed on C_k and $C_{k,\Lambda}$.

If $k \gg m$, eq.(A.6) yields the familiar k^d-running of the cosmological constant; it is this scale dependence that would result from summing up the zero point energies of the (massless) field modes. If $k \ll m$ the running is much weaker since the RHS of (A.6) contains a suppression factor $(k/m)^2 \ll 1$. This is a typical decoupling phenomenon: In the regime $k \ll m$ the physical mass m is the active IR cutoff.

The RG equation (A.6) has the solution

$$C_{k,\Lambda} = C_{\text{ren}} + \frac{2v_d}{d} \int_0^{k^2} dy \frac{y^{\frac{d}{2}}}{y + m^2} \tag{A.7}$$

Here we fixed a specific RG trajectory by imposing the renormalization condition $C_{k=0,\Lambda\to\infty} = C_{\text{ren}}$ with $\bar{\lambda}_{\text{ren}} \equiv (8\pi G)C_{\text{ren}}$ the "renormalized cosmological constant", to be determined experimentally in principle. For $m = 0$ in particular, since $C_k = C_{k,\Lambda}$ for k below Λ,

$$C_k = C_{k,\Lambda} = C_{\text{ren}} + 4d^{-2} v_d k^d \tag{A.8}$$

If $d = 4$, say, in standard notation,

$$\bar{\lambda}_k = \bar{\lambda}_{\text{ren}} + \frac{1}{16\pi^2} G_0 k^4 \tag{A.9}$$

The scalar being massless, this running of the effective cosmological constant has the same structure as in pure quantum gravity [6].

By performing a derivative expansion of $\text{Tr}_\Lambda \ln[\cdots]$ in (A.4) we can obtain the scalar's contribution to the induced cosmological constant ($\int \sqrt{g}$ term), the induced Newton constant ($\int \sqrt{g}R$ term), and similarly to the higher derivative terms. Here we are interested in the cosmological constant only, and comparing (A.4) to (A.2) yields

$$C_{k,\Lambda} - \check{C}_\Lambda = \frac{1}{2} \Big[\int d^d x \sqrt{g} \Big]^{-1} \text{Tr}_\Lambda \ln \Big[\cdots \Big] \Big|_{\int \sqrt{g} \text{ term}}$$
$$= \frac{1}{2} \int \frac{d^d p}{(2\pi)^d} \, \theta(\Lambda^2 - p^2) \, \ln\Big([p^2 + m^2 + \mathscr{R}_k(p^2)]M^{-2} \Big) \tag{A.10}$$

Employing the optimized cutoff again, (A.10) evaluates to

$$C_{k,\Lambda} = \check{C}_\Lambda + \frac{2v_d}{d} k^d \, \ln\Big(\frac{k^2 + m^2}{M^2} \Big) + v_d \int_{k^2}^{\Lambda^2} dy \, y^{d/2-1} \ln\Big(\frac{y^2 + m^2}{M^2} \Big) \tag{A.11}$$

Note that in (A.10) and (A.11) we replaced $\check{m}$ with m since comparing the ϕ^2-terms in (A.4) and (A.2), respectively, implies that $\check{m} = m$ within the simple truncation used.

For $m = 0$ and $d = 4$, say, eq.(A.11) implies the following explicit result for the running effective cosmological constant in terms of the bare one:

$$C_{k,\Lambda} = \check{C}_\Lambda + v_4 \Big[\Lambda^4 \ln(\Lambda/M) - \frac{1}{4}(\Lambda^4 - k^4) \Big] \tag{A.12}$$

For arbitrary d and m, the limit $k \to \Lambda$ of eq.(A.11) reads

$$\check{C}_\Lambda = C_{\Lambda,\Lambda} - \frac{2v_d}{d} \Lambda^d \, \ln\Big(\frac{\Lambda^2 + m^2}{M^2} \Big) \tag{A.13}$$

This equation tells us how, for a given effective cosmological constant $C_{\Lambda,\Lambda}$, the bare one, $\check{C}_\Lambda$, must be adjusted in order to give rise to the prescribed effective one. The value of $C_{\Lambda,\Lambda}$ in turn depends on the RG trajectory chosen, i.e., in this simple situation, on the value of C_{ren}. In fact, from the explicit solution (A.7) we get

$$C_{\Lambda,\Lambda} = C_{\text{ren}} + \frac{2v_d}{d} \int_0^{\Lambda^2} dy \frac{y^{\frac{d}{2}}}{y + m^2} \tag{A.14}$$

The above simple formulae illustrate various conceptual lessons of general significance. The first lesson we illustrate with this model is the non-uniqueness of the bare action. For the massless case $m = 0$ in eq.(A.13), the cosmological constant in the bare action is

$$\check{C}_\Lambda = C_{\Lambda,\Lambda} - 4d^{-1}v_d\,\Lambda^d\,\ln\left(\Lambda/M\right) \tag{A.15}$$

while the one in $\Gamma_{k,\Lambda}$ and Γ_k at $k = \Lambda$ reads

$$C_{\Lambda,\Lambda} = C_{\text{ren}} + 4d^{-2}v_d\,\Lambda^d = C_{k=\Lambda} \tag{A.16}$$

It is clear from here that choosing different values of the free parameter M will affect the bare cosmological constant (A.15) but not the effective one, eq.(A.16). The *effective* cosmological constant $C_{k=\Lambda}$ will always be proportional to Λ^d for $\Lambda \to \infty$ and approach *plus* infinity.

As a first choice consider $M = $ const, i.e. M is a positive constant independent of Λ. Then, according to (A.15), the *bare* cosmological constant $\check{C}_\Lambda$ is proportional to $-\Lambda^d\ln\Lambda$ for $\Lambda \gg M$ and it approaches *minus* infinity in the limit $\Lambda \to \infty$.

As a second choice assume M is proportional to the UV cutoff, $M = c\Lambda$, with some constant $c > 0$. Then $\check{C}_\Lambda = C_{\text{ren}} + 4d^{-2}v_d\Lambda^d\{1 - d\ln c\}$ diverges proportional to Λ^d if $c \neq \exp\left(1/d\right)$, and depending on the value of c it might approach $-\infty$ or $+\infty$. In the special case $c = \exp\left(1/d\right)$ the bare cosmological constant $\check{C}_\Lambda$ equals C_{ren} for all Λ, i.e. it is *finite* even in the limit $\Lambda \to \infty$. Also $c = 1$ is special: in this case, accidentally, the bare and the effective average action contain *the same* cosmological constant: $\check{C}_\Lambda = C_{\Lambda,\Lambda}$.

Even though they can lead to dramatically different bare actions, the various choices for M are all physically equivalent. The ordinary effective action and the EAA are independent of M. Changing M simply amounts to shifting contributions from the measure into the bare action or vice versa.

This illustrates a general lesson which, while true everywhere in quantum field theory, is particularly important in the asymptotic safety context: It makes no sense to talk about a bare action unless one has specified a measure before; neither $\mathscr{D}_\Lambda\chi$ nor $\exp\left[-S_\Lambda\right]$ have a physical meaning separately, only the combination $\int \mathscr{D}_\Lambda\chi \exp\left[-S_\Lambda\right]$ has. Here we illustrated this phenomenon by a simple rescaling of the integration variable but clearly it extends to more general transformations of χ whose Jacobian is interpreted as changing the action S_Λ to a new one, S_Λ'.

The concrete lesson for the asymptotic safety program is that one should not expect a fixed point solution of the FRGE, Γ_*, to correspond to a unique bare action.

Also, a natural question to ask is if there is a flow equation that governs the Λ-dependence of the bare actions defined with our strategy. For the present toy model, this flow equation can be easily derived using (A.13):

$$\Lambda\partial_\Lambda\check{C}_\Lambda = -\frac{4v_d}{d}\,\Lambda^d\,\left[\frac{d}{2}\ln\left(\frac{\Lambda^2 + m^2}{M^2}\right) - \frac{\Lambda\partial_\Lambda M}{M}\right] \tag{A.17}$$

This equation tells how the bare action must change when Λ is sent to infinity, given the requirement that the parameter $C_{k=0}$ in the ordinary effective action assumes the prescribed value C_{ren}.

Obviously *the RG equation for the bare cosmological constant is quite different from the corresponding equation at the level of the effective average action*, eq.(A.6).

So, for constructing a path integral describing an asymptotically safe theory, why not use a full fledged functional flow equation for the bare action? Why is the RG flow of Γ_k crucial for the QEG program, while S_Λ plays only a secondary role? There are at least two answers to these questions:

The first answer is that *the property of asymptotic safety is decided about at the effective rather than bare level.* By its very definition, asymptotic safety requires observable quantities such as scattering cross sections to be free from divergences. Since the *S*-matrix elements are essentially functional derivatives of $\Gamma \equiv \Gamma_{k=0}$ this requires the ordinary effective action to be free from such divergences. This is indeed the case if Γ is connected to a UV fixed point Γ_* by a regular RG trajectory. So, in order to test wether this condition is satisfied we need to know the Γ_k-flow. The concomitant S_Λ-flow is of no direct physical relevance. In principle is is even conceivable that, while Γ_k approaches to a fixed point in the UV, the bare action does not; the resulting theory could nevertheless have completely acceptable physical properties.

For these reasons the basic tool in searching for asymptotic safety is the flow equation for the EAA and not its analog for the bare action.

A second answer to the above question is that we would like the scale dependent functional obtained by solving the flow equation to have a chance of defining an effective field theory in the sense that its tree level evaluation at some scale approximately describes all quantum effects with this typical scale. For Γ_k this is indeed the case[4], but not for S_Λ. The reason is that, given S_Λ, there is still a functional integration to be performed in order to go over to the effective level; using Γ_k instead, it has been performed already.

The above toy model illustrates this point: From eq.(A.8) or eq.(A.9) we conclude that for every finite $\bar{\lambda}_{\mathrm{ren}} \equiv (8\pi G)C_{\mathrm{ren}}$ the *running effective* cosmological constant $\bar{\lambda}_k \equiv (8\pi G)C_k$ becomes large and positive for growing k and finally approaches *plus* infinity for $k \to \infty$. Applying the effective field theory interpretation we would insert this $\bar{\lambda}_k$ into the effective Einstein equation. It then predicts that, at high momentum scales, spacetime is strongly curved and has *positive* curvature.

From the above remarks it is clear that the *running bare* action does not contain this information. Depending on our choice for M the bare cosmological constant $\check{C}_\Lambda$ approaches to $+\infty, -\infty$ or a finite value where $\Lambda \to \infty$. So clearly it would not make any sense to insert it into Einstein's equation in order to "RG improve" it.

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[4]Of course we are not saying here that Γ_k necessarily provides a numerically precise description. To what degree this is actually possible (fluctuations are small, etc.) depends on the details of the physical situation.

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