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Can a nontrivial gravitational fixed point be identified in perturbation theory?

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The asymptotic safety scenario for quantum gravity hinges on the existence of a nontrivial fixed point for the dimensionless Newton constant g_N. We propose an affirmative answer to the question raised in the title provided a suitably improved perturbative framework is used. For any one loop renormalizable gravity theory one obtains flow equations which are uniquely determined by the coefficients of the powerlike and logarithmic divergences in the background covariant effective action. These flow equations exhibit nontrivial fixed points for g_N and the dimensionless cosmological constant λ with respect to which the flow is asymptotically safe. The gauge and scheme dependence can be discussed analytically. Results for higher derivative gravity in minimal gauge are presented. Remarkably, spectral positivity of the Hessians can be satisfied along the entire flow, evading the traditional positivity problem. Dependence on $O(10)$ initial data is erased to accuracy 10^{-7} after $O(10)$ units of the renormalization mass scale and the flow settles on a $\lambda(g_N)$ orbit.

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1. Nontrivial perturbative fixed points

Perturbation theory is as routinely applied by some as it is frowned upon by others, both not always thoughtfully. Before turning to gravity proper we offer some general remarks on what one can and cannot reasonably expect from perturbation theory (PT). The identification of a nontrivial fixed point in PT at first sight seems to be a contradiction in terms, alas it is not. By definition PT is a saddle point expansion in the loop counting parameter $\hbar$ introduced into the functional integral as the inverse prefactor of the bare action, $\hbar^{-1}S_0$. For $\hbar \to 0$ one obtains, depending on the signature, a generalized Laplace or stationary phase expansion which may or may not coincide with the expansion in some 'small' bare reference coupling. In the presence of massless degrees of freedom the orders in the loop counting parameter also do not coincide with the orders in Planck's constant. The 'bare' $\hbar$ expansion can typically be shown to provide an asymptotic expansion of the properly regularized functional integral. After renormalization such proofs are exceedingly difficult and are available only in superrenormalizable or purely fermionic field theories. Off hand therefore the renormalized PT expansion is only a formal power series in the loop counting parameter $\hbar$. An important advantage of PT however is that (in a perturbatively renormalizable field theory) the ultraviolet (UV) cutoff can termwise strictly be removed, independent of the nature of the coupling flow. The perturbatively defined coupling flow itself then provides a plausibility criterion [2] for the existence of an underlying 'exact' theory such that for physical quantities PT yields a valid asymptotic expansion, namely: All essential couplings g_j must be asymptotically safe with respect to a trivial or nontrivial fixed point,

$$\sum_{\mu_0 \leq \mu \leq \infty} \mathrm{g}_j(\mu) < \infty, \quad \lim_{\mu \to \infty} \mathrm{g}_j(\mu) = \mathrm{g}_j^* < \infty. \tag{1.1}$$

Here μ is the renormalization scale and the flow $\mu \to \mathrm{g}_j(\mu)$ is assumed to be non-constant. Clearly both properties in (1.1) are parameterization dependent and as stressed by S. Weinberg [5] one should ultimately define "the coupling constants as coefficients in a power series expansion of the reaction rates themselves around some physical renormalization point". With this understanding of the couplings the ultraviolet regime should in an asymptotically safe field theory be accessible to a suitably formulated perturbation theory. As emphasized in [2] the value of the fixed point coupling is of minor importance. One can take any one difference $\delta\mathrm{g}_1 := \mathrm{g}_1 - \mathrm{g}_1^*$ which is of order of $\hbar$ at $\mu = \mu_0$ as the reference coupling reinterpret the expansion as one in $\delta\mathrm{g}_1$ and proceed with the renormalization group improvement as usual. Being basis dependent the values of the fixed point couplings g_j^* are also not directly related to the 'Gaussian' or 'non-Gaussian' nature of the fixed point. A 'Gaussian' fixed point is one where there exists a preferred basis in the space of interaction monomials such that the 'exact' functional measure in the vicinity of the fixed point becomes Gaussian. For a non-Gaussian fixed point the existence of such a basis must be excluded, – a formidable task for a realistic field theory. We therefore distinguish here only between trivial fixed points with $\mathrm{g}_j^* = 0$ for all j in a 'natural' basis, and nontrivial ones where $\mathrm{g}_j^* \neq 0$ for at least one j.

We now apply these considerations to higher derivative (HD) gravity. HD gravity in four dimensions comes close to realizing a renormalizable quantum theory of gravity. Compared to the Einstein-Hilbert action two additional interaction monomials are added containing the independent curvature invariants with four derivatives of the metric tensor. The resulting gravity theory is perturbatively renormalizable to all loop orders [1] which is related to the strong, $1/p^4$ type, falloff of the free propagator at large momenta. With Euclidean signature the action reads

$$S = \int d^4x \sqrt{q} \Big[\tilde{\Lambda} - \frac{1}{\kappa^2} R + \frac{1}{2s} C^2 - \frac{\omega}{3s} R^2 \Big] . \qquad (1.2)$$

Here $q_{\alpha\beta}$ is the metric entering the functional integral, C^2 is the square of its Weyl tensor, and total derivative terms like $\nabla^2 R$ and the integrand of the Gauss-Bonnet term have been omitted. In terms of the cosmological constant Λ one has $\tilde{\Lambda} = 2\Lambda/\kappa^2$ and the parameterization of the other coefficients by couplings s, ω is chosen for later convenience. The two dimensionful couplings are replaced with dimensionless ones according to $\mathrm{g}_N = \mu^2 \kappa^2$ and $\lambda = \mu^{-4} \mathrm{g}_N \tilde{\Lambda}/2 = \mu^{-2}\Lambda$. Our curvature conventions are set by $(\nabla_\alpha \nabla_\beta - \nabla_\beta \nabla_\alpha) v^\gamma = R^\gamma{}_{\delta\alpha\beta} v^\delta$ and $R_{\alpha\beta} = R^\gamma{}_{\alpha\gamma\beta}$. Here $\nabla_\alpha v^\beta = \partial_\alpha v^\beta + \Gamma^\beta{}_{\alpha\gamma} v^\gamma$ with $\Gamma^\gamma{}_{\alpha\beta} = \frac{1}{2} g^{\gamma\delta}[\partial_\alpha g_{\beta\delta} + \partial_\beta g_{\alpha\delta} - \partial_\delta g_{\alpha\beta}]$, which gives $R^\delta{}_{\gamma\alpha\beta} = \partial_\alpha \Gamma^\delta{}_{\gamma\beta} - \partial_\beta \Gamma^\delta{}_{\gamma\alpha} + \Gamma^\delta{}_{\alpha\rho} \Gamma^\rho{}_{\gamma\beta} - \Gamma^\delta{}_{\beta\rho} \Gamma^\rho{}_{\gamma\alpha}$. In four dimensions the integrand E of the Gauss-Bonnet term and the square of the Weyl tensors are related by $C^2 = E + 2R^{\alpha\beta} R_{\alpha\beta} - \frac{2}{3} R^2$.

Viewed as a running coupling Newton's constant has a special status. In some bounded kinematical regime it can be viewed as an inessential parameter whose value (like that of a wave function renormalization constant) can be changed at will. In principle G_N can be kept fixed, e.g. at $[G_N]^{1/2} = M_{\rm Pl} \approx 1.4 \times 10^{19}$GeV. This trivializes the g_N flow, $\mathrm{g}_N(\mu) = 16\pi(\mu/M_{\rm Pl})^2$, but may introduce spurious singularities in the flow of other dimensionless couplings; see [2] for a discussion. A better alternative is to define the running of g_N relative to that of a reference coupling. A convenient choice is a cosmological constant term $\tilde{\Lambda} \int dx \sqrt{g}$. Indeed in four dimensions $\tau := \mathrm{g}_N \lambda = 16\pi G_N \Lambda$ is dimensionless and invariant under constant rescalings of the metric [6]. As a consequence the associated flow equation must be independent of parameters entering a multiplicative metric renormalization. We shall continue to work with g_N and λ but will construct flow equations that have this interplay built in.

The existence of a nontrivial fixed point $\mathrm{g}_N^* > 0$ for g_N is a crucial ingredient for the asymptotic safety scenario [3, 2, 4]. The following heuristic argument suggests that if HD gravity indeed has a nontrivial fixed point for g_N it should be visible already in PT [2]: let $q_{\alpha\beta}$ be the 'quantum metric' entering the functional integral. Then any Wilsonian action of the form $S_\mu[q] = \int d^4x \sqrt{q} \sum_{i \geq 1} u_i(\mu) P_i(q)$ with asymptotically safe couplings and scalar interaction monomials $P_i(q)$ of mass dimension $-d_i$ will for $\mu \to \infty$ depend only on $\mu^2 q_{\alpha\beta}$, as $u_i(\mu) \sim \mu^{d_i} u_i^*$. For the coefficient of the Ricci scalar in S_μ this is 'as if' Newton's constant has picked up an integer anomalous dimension -2 along the trajectory connecting infrared to ultraviolet properties. A typical propagator would thus scale at low energies like $1/p^2$ and at high energies like $1/p^4$. But the latter is precisely the behavior which is in the realm of PT for HD gravity. On the other hand an anomalous dimension -2 goes hand in hand with a nontrivial fixed point for g_N. This can be seen by taking into account g_N's double role as an inessential parameter ("wave function renormalization constant") and a coupling. The yet-to-be-determined flow equation will thus naturally be parameterized by the anomalous dimension $\eta = \mu \frac{d}{d\mu} \ln \kappa^2$, in which case $\mu \frac{d}{d\mu} \mathrm{g}_N = (2+\eta) \mathrm{g}_N$, and $\mathrm{g}_N^* \neq 0$ if and only if $\eta = -2$. Finally we note that s and g_N are of degree 1 in the loop counting parameter $\hbar$ while the other couplings are of degree zero. Since s turns out to be asymptotically free in PT [11, 12] one can regard the perturbative expansion as an expansion in powers of s. Then

g_N may occur in degree zero ratios s/g_N (and in fact it does) and a putative nonzero fixed point value for g_N is well within the realm of the expansion.

Already the one loop order should contain the decisive piece of information provided a suitable framework is used. It is then a matter a computation to determine whether or not the nontrivial fixed point exists. In section 3 we present the result of such a computation [24] confirming the existence of a nontrivial fixed point. For definiteness we presented the argument here for HD gravity where the instrumental $1/p^4$ decay of the propagators is built into the kinematics. In section 4 we will put our results into the context of the asymptotic safety scenario which entails that a similar link between a nontrivial perturbative fixed point for g_N and a dynamically generated $1/p^4$ behavior should exist in any other gravity theory which renormalizable at one loop.

2. Refined gravitational PT

Perturbative computations in gravity theories have traditionally been been performed with dimensional regularization and minimal subtraction. The one loop flows of genuinely dimensionless couplings are in fact scheme independent, so any other regularization and scheme is bound to give the same result. A quick assessment shows that the situation is different for the originally dimensionful couplings g_N and λ. On general grounds scheme changes in PT should be in one-to-one correspondence to finite redefinitions of the couplings of the appropriate order in the loop counting parameter. For HD gravity at one loop one may take

$$g_N = g_N' + \frac{\hbar}{(4\pi)^2} {g_N'}^2 C_1' , \quad \lambda = \lambda' + \frac{\hbar}{(4\pi)^2} g_N' D_1' , \tag{2.1}$$

where C_1, D_1 are functions of $\lambda, \omega, s/g_N$ evaluated at the primed coupling values. If $\beta_{g'} = 2g_N' + O(\hbar)$, $\beta_{\lambda'} = -2\lambda' + O(\hbar)$, $\beta_s = O(\hbar)$, $\beta_\omega = O(\hbar)$ are the beta functions in the original scheme, then

$$\begin{aligned} \mu \frac{d}{d\mu} g_N &= \beta_g + \frac{\hbar}{(4\pi)^2} {g_N}^2 \left[2C_1 + 2g_N \frac{\partial C_1}{\partial g_N} - 2\lambda \frac{\partial C_1}{\partial \lambda} \right] , \\ \mu \frac{d}{d\mu} \lambda &= \beta_\lambda + \frac{\hbar}{(4\pi)^2} g_N \left[4D_1 + 2g_N \frac{\partial D_1}{\partial g_N} - 2\lambda \frac{\partial D_1}{\partial \lambda} \right] , \end{aligned} \tag{2.2}$$

are the beta functions in the new scheme. As usual most of the schemes corresponding to coupling redefinitions have no natural computational realization and are defined only implicitly. The question relevant here is: which of the schemes implicitly defined by (2.2) can be computationally realized and admit a 'Wilsonian' interpretation that in principle allows one to make contact to a nonperturbative formulation? Remarkably there is an essentially unique choice for C_1, D_1 which meets these requirements and which leads to beta functions unambiguously determined by the coefficients parameterizing the divergent part of the effective action. As a consequence the one loop beta functions of the originally dimensionful gravitational couplings acquire a status almost as robust as the beta functions of genuinely dimensionless couplings. For the rest of this section we now explain this fact in more detail and comment on the remaining non-universalities.

We take the background effective action $\Gamma[h; g]$ as the basic object in the quantum theory. It is characterized by three properties: a functional integro differential equation, background covariance, and splitting Ward identities, see e.g. [8, 3]. The formal functional equation reads

$$\exp -\Gamma[h;g] = \int d\Omega_\chi(f) \exp\Big\{ - S_\chi[f;g] + \int d^4x \sqrt{g}\,(f-h)_{\mu\nu} \frac{\delta\Gamma[h;g]}{\delta h_{\mu\nu}}\Big\}. \tag{2.3}$$

Here $g_{\mu\nu}$ is a generic background metric and the integration is over the fluctuation part $f_{\mu\nu}$ of the original 'quantum metric' $q_{\mu\nu} = g_{\mu\nu} + f_{\mu\nu}$. The other argument of Γ is the average of the fluctuations, $h_{\mu\nu} = \langle f_{\mu\nu}\rangle_g$, so that $\langle q_{\mu\nu}\rangle_g = g_{\mu\nu} + h_{\mu\nu}$. The action $S_\chi[f;g]$ differs from the original one $S[g+f]$ by addition of a gauge fixing term $S_{\rm gf}[f;g]$ implementing $\chi_\mu(f;g) = 0$ and the dependence on f and g is no longer through the sum only. The kinematical measure $d\Omega_\chi(f)$ includes an integral over ghost fields $c^\mu, \bar{c}_\mu$ with action $S_{\rm gh}$. The gauge fixing action is obtained by Gaussian averaging of the (shifted) $\delta(\chi_\mu)$ constraint and contains a parameter a such that $\delta(\chi_\mu)$ is recovered in the limit $a \to 0$. The quantum action $S + S_{\rm gf} + S_{\rm gh}$ is BRST invariant and the measure is formally too. As usual $\Gamma[h;g]$ arises by Legendre transformation so that $\delta\Gamma[h;g]/\delta h_{\mu\nu} = \sqrt{g}\,J_{\mu\nu}$ is an extremizing source. The source-free condition $\delta\Gamma[h;g]/\delta h_{\mu\nu} = 0$ dynamically adjusts $h_{\mu\nu} = \langle f_{\mu\nu}\rangle_g$ to $g_{\mu\nu}$ in a selfconsistent way. In principle the background field formalism therefore is not tied to externally prescribed backgrounds.

Background covariance means that f and h transform as a symmetric tensors and that only covariant expressions enter built from f, h, g, the covariant derivative ∇ of g, and the curvature tensors of g. For a diffeomorphism φ with bull-back φ_* this means $S_\chi[\varphi_* f;\varphi_* g] = S_\chi[f;g]$ and $\Gamma[\varphi_* h;\varphi_* g] = \Gamma[h;g]$, assuming that the measure has the appropriate invariance. Note that this is a much weaker requirement than diffeomorphism invariance of a functional depending only on the sum $f+g$ or $h+g$. For the classical action $S[\varphi_*(g+f)] = S[g+f]$ entails upon expansion consistency conditions among the vertices. For the effective action the fact $S[g+f] = S[g'+f']$ with different decompositions into background and fluctuations gives rise to 'splitting Ward identities'.

In a perturbative construction one (re-)introduces a loop parameter $\hbar$ by rescaling $S_\chi \mapsto \frac{1}{\hbar} S_\chi$, $\Gamma \mapsto \frac{1}{\hbar}\Gamma$, and $f_{\mu\nu} \mapsto \sqrt{\hbar} f_{\mu\nu}$. The effective action is assumed to be of the form $\Gamma[h;g] = S[g] + \sum_{n\geq 1} \hbar^{n/2} \Gamma_{n/2}[h;g]$, where the non-integer coefficients vanish for $h=0$. Upon expansion of (2.3) one can compute the $\Gamma_{n/2}$'s recursively using only Gaussian averages. One finds $\Gamma_{1/2}[h;g] = S^{(1)}(g)\cdot h$ and $\Gamma_1[h;g] - \Gamma_1[0;g] = \frac{1}{2} S_\chi^{(2)}[g]\cdot h^2$, etc, where the superscripts here refer to functional derivatives with respect to the background metric and the dot indices integration and summation over tensor indices. The $\Gamma_1[0;g]$ term depends on the normalization of the underlying Gaussian measure. The normalization is important in the present context as it relates to 'vacuum energy contributions' which affect the renormalization of the gravitational couplings.

The normalization of the measure commonly adopted is that of the so-called geometric approach [7]. One decomposes f according to $f = f^\chi + Lv$, where f^χ satisfies the gauge condition, $\chi_\mu(f^\chi, g) = 0$, and $(L_g v)_{\mu\nu} := \nabla_\mu v_\nu + \nabla_\nu v_\mu$ is its gauge variation referring to a fixed g as the base point of the tangent space. The Jacobian $J_\chi(g)$ stemming from the change of variables $f_{\mu\nu} \mapsto (f^\chi_{\mu\nu}, v_\mu)$ is essentially the Faddeev-Popov factor and the measure in the perturbative expansion of (2.3) is interpreted as $d\Omega_\chi(f) = \mathscr{D}(f^\chi) J_\chi(g)$ with the formal product measure $\mathscr{D}(f^\chi)$ normalized by

$$\int \mathscr{D}(f^\chi) \exp\Big\{ - \frac{1}{2} \int d^4x \sqrt{g}\, f^\chi_{\mu\nu} [G(g)]^{\mu\nu,\alpha\beta} f^\chi_{\alpha\beta} \Big\} = 1\,, \tag{2.4}$$

where $G^{\mu\nu,\alpha\beta} = 1^{\mu\nu,\alpha\beta} + k_0 g^{\mu\nu} g^{\alpha\beta}$ is a ultralocal configuration space metric which is positive definite for $k_0 > -1/4$. The point relevant here is that the normalization of $\mathscr{D}(f^\chi)$ and hence also of the formal product measure $\mathscr{D}(f)$ is done with respect to a constant (rather than a differential) kernel on the tangent space at g. As a consequence *all* short distance singularities stemming from $S^{(2)}_\chi$ and the ghost Hessian are included in Γ_1. This is *not* what one would conventionally do in a Feynman diagram evaluation of a flat space quantum field theory, where the Gaussian measures occurring are normalized with respect to the inverse free propagator. The latter automatically removes divergences with field independent coefficients. In a gravitational context there is no sensible decomposition into a kinematical part with the background field g 'switched off' and a superimposed dynamics. By the above splitting principle, as far as the non-redundant degrees of freedom are concerned, switching off the background should be the same as switching off the full field in the Hessian. For a conventional field theory this indeed produces a free propagator, for a gravitational Hessian the notion is meaningless.

With the normalization fixed by (2.4) the problem of setting up a regularized perturbative expansion of (2.3) reduces to defining generic regularized background covariant Gaussian functional integrals. It is important that a systematic regularization is adopted from the beginning for the entire expression (2.3). After regularization the multiplicative identity $\mathrm{Det}\mathbf{AB} = \mathrm{Det}\mathbf{A}\,\mathrm{Det}\mathbf{B}$ is for non-trace-class operators *invalid* as a matter of principle – so formally equivalent expressions will differ after regularization and the correct interpretation is dictated by the initial definition of (2.3) via its regularized expansion. Once a suitable regularization has been fixed the divergent part Γ_1^{div} of the one loop background effective action can be computed and the beta functions extracted. For pure gravity theories the divergent part of the one loop background effective action is of the form

$$\Gamma_1^{\mathrm{div}}[0;g] = -\frac{1}{(4\pi)^2}\Big\{\Lambda_{\mathrm{UV}}^4 \int d^4x\sqrt{g}\,\Upsilon_1 + \Lambda_{\mathrm{UV}}^2 \int d^4x\sqrt{g}[\Upsilon_2 R + \mu^2\Upsilon_3] \\ + \ln(\Lambda_{\mathrm{UV}}/\mu)\int d^4x\sqrt{g}\Big[\zeta_1 C^2 + \zeta_2 R^2 + \zeta_3 E + \mu^2\zeta_4 R + \mu^4\zeta_5\Big]\Big\}. \tag{2.5}$$

Here Λ_{UV} is a momentum-type ultraviolet cutoff, μ is the renormalization scale, and the coefficients Υ_j, $i = 1,2,3$ and ζ_j, $j = 1,\dots,5$ are to be determined. Unlike dimensional regularization, where the Gauss-Bonnet term E is non-topological for $d \neq 4$ and in principle needs to be kept [14] our regularization operates in strictly four dimensions and the E can term be consistently omitted from the beginning. As indicated above some novel features occur in determining the beta functions for the originally dimensionful gravitational couplings which we address by a 'Wilsonian' reinterpretation of the perturbative flow.

One then finds that the Υ coefficients have a decisive influence on the flow. These coefficients are less universal than the coefficients of the logarithmic divergences; off hand the Υ's are both scheme and gauge dependent while the ζ coefficients are only gauge dependent. For both sets gauge independent combinations exist on general grounds and one can set up the flow equations (2.23) such that only Υ_1 and these gauge independent combinations enter. The scheme dependence of the Υ's is found to be be of a rather innocuous nature and to be parameterized by certain moments q_m, $m = 1/2,1,2$ of the cutoff function used as a regulator. These moments are by construction positive and $O(1)$, so using different schemes only leads to minor quantitative changes.

We now describe the main ingredients of the resulting improved perturbative framework [24] consecutively. In outline these are: (a) use of a background covariant operator cutoff in combination with the heat kernel that keeps track of powerlike divergences. (b) a nonminimal subtraction ansatz parameterizing generic finite coupling redefinitions. (c) a Wilsonian matching condition which identifies the bare couplings with the renormalized ones at the UV cutoff scale.

(a) Operator-heat-kernel regularization: In contrast to earlier perturbative computations we use a background covariant operator regularization [21] in combination with the heat kernel. Unlike dimensional regularization (which sees only logarithmic divergences) such a regulator in principle allows one to make contact with nonperturbative results. The operator regularization uses a function $z \mapsto F_{k,\Lambda_{\rm UV}}(z)$ that depends parametrically on an infrared cutoff k and an ultraviolet cutoff $\Lambda_{\rm UV}$. The infrared cutoff k is conceptually distinct from the renormalization scale μ; for the purposes here we may assume however $0 < c_s k < \mu \leq \Lambda_{\rm UV}$ with some fixed $c_s > 1$. For the regulator functions we take $F_{k,\Lambda_{\rm UV}}(z) = f(z/\Lambda_{\rm UV}^2) - f(z/k^2)$, for suitable $f : \mathbb{R}_+ \to \mathbb{R}_+$, interpolating between $f(y) = \ln y + {\rm const} + O(y)$ and zero. For a formally selfadjoint covariant differential operator $\mathbf{A}$ of order $2r$ on a d dimensional riemannian manifold our basic prescription is that $\ln \mathbf{A}$ is replaced with $F_{k^r,\Lambda_{\rm UV}^r}(A)$ acting as an integral operator obtained by averaging the heat kernel:

$$\ln \mathbf{A} \mapsto \ln \mathbf{A}_{k,\Lambda_{\rm UV}} := F_{k^r,\Lambda_{\rm UV}^r}(A)(x,y) = \int_0^\infty dt\, \widetilde{F}_{k^r,\Lambda_{\rm UV}^r}(t) A(x,y;t)\,, \tag{2.6}$$

where $A(x,y;t)$ is the heat kernel of $\mathbf{A}$ normalized according to

$$\Big(\frac{\partial}{\partial t} + \mathbf{A}\Big) A(x,y;t) = 0\,, \quad A(x,y;0) = \delta(x,y)\,, \tag{2.7}$$

with $\delta(x,y)$ normalized wrt $\sqrt{g}$; schematically $A(x,y;t) =: \langle x|\exp(-t\mathbf{A})|y\rangle$. Further $\widetilde{F}_{k,\Lambda_{\rm UV}}$ is the inverse Laplace transform of $F_{k,\Lambda_{\rm UV}}$ normalized such that $\widetilde{F}_{0,\infty}(t) = -1/t$ (where the limits of course cannot be taken under the integral). It follows that

$$\begin{aligned}
\mathbf{A}^{-1} &\mapsto \mathbf{A}^{-1}_{k,\Lambda_{\rm UV}} := \partial_z F_{\mu^r,\Lambda_{\rm UV}^r}(A)(x,y) = -\int_0^\infty dt\, t \widetilde{F}_{k^r,\Lambda_{\rm UV}^r}(t) A(x,y;t)\,,\\
{\rm Det}\mathbf{A} &\mapsto {\rm Det}_{k,\Lambda_{\rm UV}}\mathbf{A} := \exp\Big\{ \int_0^\infty dt\, \widetilde{F}_{k^r,\Lambda_{\rm UV}^r}(t) \int d^d x \sqrt{g} A(x,x;t)\Big\}\,,
\end{aligned} \tag{2.8}$$

and we define the regularized generic Gaussian functional integral by [24]

$$\int \mathscr{D} f\, \exp\Big\{ -\frac{1}{2} f\cdot \mathbf{A} \cdot f + J \cdot f\Big\} \;\;\mapsto\;\; ({\rm Det}_{k,\Lambda_{\rm UV}}\mathbf{A})^{-1/2} \exp\Big\{ \frac{1}{2} J \cdot \mathbf{A}^{-1}_{k,\Lambda_{\rm UV}} \cdot J\Big\}\,. \tag{2.9}$$

Note that the right hand side reduces to unity for $k = \Lambda_{\rm UV}$ in accordance with the picture that 'no modes' are being integrated out. Compatibility with 'completing the square' type manipulations requires that $\mathbf{A}$ itself is regularized according to $\mathbf{A} \mapsto (\mathbf{A}^{-1}_{k,\Lambda_{\rm UV}})^{-1}_{k,\Lambda_{\rm UV}}$. The regularization of non-trace-class operators inevitably violates naive multiplicative identities and the one adopted here is no exception: one may check that ${\rm Det}\,\mathbf{AB} \neq {\rm Det}\mathbf{A}{\rm Det}\mathbf{B}$ and $s(s\mathbf{A})^{-1} \neq \mathbf{A}^{-1}$, for the regularized versions.

For a flat background $g_{\mu\nu} = \eta_{\mu\nu}$ and operators with constant coefficients one can switch to momentum space, insert

$$A(x,y;t) = \int \frac{d^d p}{(2\pi)^d}\, e^{ip\cdot(x-y)}\, e^{-tA(p)}\,, \tag{2.10}$$

and replace the above expressions by simpler ones involving the operator's symbol $A(p)$. Functions of the (matrix-valued) symbol are interpreted in terms of the spectral representation $A(p) = \sum_j \lambda_j(p)\Pi_j(p/\sqrt{p^2})$, where Π_j are (operator-dependent) mutually orthogonal projectors. In particular for the regularized $\mathbf{A}^{-1}$ and $\mathrm{Tr}\ln\mathbf{A}$ this gives

$$\begin{aligned} \mathbf{A}^{-1}_{k,\Lambda_{\rm UV}}(x,y) &= \sum_j \int \frac{d^d p}{(2\pi)^d} e^{ip\cdot(x-y)}\,\Pi_j\Big(\frac{p}{\sqrt{p^2}}\Big)\,\partial_z F_{k^r,\Lambda^r_{\rm UV}}(\lambda_j(p))\,, \\ \mathrm{Tr}F_{k^r,\Lambda^r_{\rm UV}}(A) &= \sum_j m_j \int \frac{d^d p}{(2\pi)^d} F_{k^r,\Lambda^r_{\rm UV}}(\lambda_j(p))\,, \end{aligned} \tag{2.11}$$

where $m_j = \mathrm{tr}\Pi_j$ is the multiplicity of λ_j.

Our cutoff functions will be of the symmetric form

$$\begin{aligned} F_{k,\Lambda_{\rm UV}}(z) &= f(z/\Lambda^2_{\rm UV}) - f(z/k^2)\,, \\ \widetilde{F}_{k,\Lambda_{\rm UV}}(t) &= \frac{1}{t}[\tilde{f}(t\Lambda^2_{\rm UV}) - \tilde{f}(tk^2)]\,, \end{aligned} \tag{2.12}$$

which ensures that for $k = \Lambda_{\rm UV}$ all regularized Gaussian integrals reduce to unity. It is convenient to specify only the functions $f(y)$ and $\tilde{f}(u)$ which are however related by $f(y) = \int_0^\infty \frac{du}{u}\tilde{f}(u)e^{-yu}$ only modulo (occasionally divergent) terms that cancel out in the differences. For definiteness we consider the following three choices:

$$\begin{aligned} f(y) &= -\ln(1+1/y) + \sum_{n=1}^{d/2}\frac{1}{n(1+y)^n}\,, && \text{`smooth'}\,, \\ f(y) &= \theta(1-y)\ln y\,, && \text{`optimal'}\,, \\ f(y) &= -\Gamma(0,y)\,, && \text{`sharp proper time'}\,. \end{aligned} \tag{2.13}$$

The 'optimal' cutoff [22] has no proper time counterpart, for the other two one has:

$$\begin{aligned} \tilde{f}(u) &= \frac{\Gamma(d/2+1,u)}{\Gamma(d/2+1)} = \sum_{n=0}^{d/2}\frac{1}{n!}u^n e^{-u}\,, && \text{`smooth'}\,, \\ \tilde{f}(u) &= -\theta(u-1)\,, && \text{`sharp proper time'}\,. \end{aligned} \tag{2.14}$$

Many applications require to isolate the divergences in $\ln\mathrm{Det}_{k,\Lambda_{\rm UV}}\mathbf{A}$ as $\Lambda_{\rm UV}\to\infty$. This can be achieved by inserting the small t asymptotic heat kernel expansion where UV singularities correspond to non-positive powers of t. For definiteness we take d even in the following. On general grounds the diagonal of the heat kernel $\langle x|\exp(-t\mathbf{A})|x\rangle$ will admit a small t asymptotic expansion for a large class of differential operators. For operators of order $2r$ which are products of Laplacian type operators $-\nabla^2 + U$ on a closed Riemannian manifold of dimension d the asymptotic expansion takes the form

$$\langle x|\exp(-t\mathbf{A})|x\rangle \sim \frac{1}{(4\pi)^{d/2}}\frac{\Gamma(\frac{d}{2r})}{r\Gamma(\frac{d}{2})}\sum_{n\geq 0} t^{\frac{2n-d}{2r}}E_{2n}(x|A)\,, \quad E_0(x|A) = 1\,. \tag{2.15}$$

The form (2.15) is then taken as an ansatz also for other classes of operators. Usually however only operators with trivial principal part are considered, $\mathbf{A} = 1(-\nabla^2)^r + \text{rest}$, and tabulated results are available only for this situation. Operators with nontrivial principal parts are needed in gravity. The results are considerably more complicated then, for example already $E_0(x|A)$ is a nontrivial matrix. For the divergent part of $\ln\det\mathbf{A}$ in (2.8) the generalized expansion (2.15) gives

$$\begin{aligned}\mathrm{Tr}F_{k^r,\Lambda_{\rm UV}^r}(A) &= -\frac{1}{(4\pi)^{d/2}}\frac{\Gamma(\frac{d}{2r})}{r\Gamma(\frac{d}{2})}\int d^dx\sqrt{g} \\ &\times\Big[\sum_{n=0}^{d/2-1}\Lambda_{\rm UV}^{(d-2n)}q_{(d/2-n)/r}E_{2n}(x|A)+2r\ln\Lambda_{\rm UV}E_d(x|A)\Big]+O(\Lambda_{\rm UV}^0)\,,\end{aligned}\tag{2.16}$$

where the q_n are moments of the cutoff function defined below. We use (2.16) to define the divergent part (2.5) of the effective action. From the scaling properties of the E_{2n} one sees that *only* the coefficient of the logarithmic divergence is invariant under re-scalings of the operator $\mathbf{A}$; overall normalizations thus matter. The moments q_n enter via

$$\begin{aligned}\int_0^\infty dt\,t^{-n}\widetilde{F}_{k,\Lambda_{\rm UV}}(t) &= \frac{1}{\Gamma(n)}\int_0^\infty dz\,z^{n-1}F_{k,\Lambda_{\rm UV}}(z)\\ &= \begin{cases}-2\ln\Lambda_{\rm UV}/k\,, & n=0\,,\\ -q_n(\Lambda_{\rm UV}^{2n}-k^{2n}) & 0<n\le 2\,.\end{cases}\end{aligned}\tag{2.17}$$

The case $n=0$ can also be interpreted as $\lim_{z\to 0}F_{k,\Lambda_{\rm UV}}(z)$. For the three cutoffs considered the q_n come out as

$$\begin{aligned}&\text{'smooth':} && q_n=\frac{\Gamma(d/2+1-n)}{n\Gamma(d/2+1)}\,,\\ &\text{'optimal':} && q_n=\frac{1}{n\Gamma(n+1)}\,,\\ &\text{'sharp proper time':} && q_n=\frac{1}{n}\,,\end{aligned}\tag{2.18}$$

for $0<n\le d/2$, where the actual values needed are $n=1/r,\dots,(d/2)/r$.

(b) Nonminimal subtraction: The divergences (2.5) are absorbed as usual by coupling and field renormalizations. For the gravitational couplings we use the nonminimal subtraction ansatz

$$\begin{aligned}\tilde{\Lambda}_0 &= \mu^4\frac{2\lambda}{g_N}\Big\{1+\frac{\hbar}{(4\pi)^2}\Big[a_{10}+a_{11}\ln(\Lambda_{\rm UV}/\mu)+a_{12}\Big(\frac{\Lambda_{\rm UV}}{\mu}\Big)^2+a_{13}\Big(\frac{\Lambda_{\rm UV}}{\mu}\Big)^4\Big]+O(\hbar^2)\Big\}\,,\\ \kappa_0^2 &= \mu^{-2}g_N\Big\{1+\frac{\hbar}{(4\pi)^2}\Big[b_{10}+b_{11}\ln(\Lambda_{\rm UV}/\mu)+b_{12}\Big(\frac{\Lambda_{\rm UV}}{\mu}\Big)^2\Big]+O(\hbar^2)\Big\}\,,\end{aligned}\tag{2.19}$$

while for originally dimensionless couplings minimal subtraction with only log terms is used. Inserting the coupling and field redefinitions into the bare action S_0 and expanding to $O(\hbar)$ gives $S_0=S+\Delta S$, where S is the renormalized action and ΔS is the counterterm. The cancellation condition $\Gamma_1^{\rm div}=-\Delta S$ fixes all minimal subtraction parameters as well as a_{11},a_{12},a_{13} and b_{11},b_{12} in (2.19) but leaves a_{10} and b_{10} undetermined. The flow equations for the couplings follow as usual from the fact that the bare couplings $\tilde{\Lambda}_0$, κ_0^2, etc are μ-independent. For the gravitational couplings one finds flow equations of the form (2.2) where β_g, β_λ are the beta functions in minimal subtraction and C_1, D_1 are related to a_{10}, b_{10} by

$$a_{10}=g_NC_1-\frac{g_N}{\lambda}D_1\,,\quad b_{10}=-g_NC_1\,.\tag{2.20}$$

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(c) 'Wilsonian' matching condition: So far the bare couplings $\kappa_0^2, \tilde{\Lambda}_0$ were only assumed to be μ-independent. In a Wilsonian interpretation they should coincide with the running ('renormalized') couplings at scale $\mu = \Lambda_{\rm UV}$. This additional requirement fixes the subtraction point (2.19) uniquely:

$$\begin{aligned} \kappa_0^2 &\stackrel{!}{=} \Lambda_{\rm UV}^{-2}\, g_N(\mu = \Lambda_{\rm UV}) \qquad \text{iff} \quad b_{10} + b_{12} = 0\,, \\ \tilde{\Lambda}_0 &\stackrel{!}{=} \Lambda_{\rm UV}^4 \left(\frac{2\lambda}{g_N}\right)(\mu = \Lambda_{\rm UV}) \quad \text{iff} \quad a_{10} + a_{12} + a_{13} = 0\,. \end{aligned} \tag{2.21}$$

The matching condition can be viewed as a (considerable) shortcut to imposing renormalization conditions proper, which for large μ gives identical results. Explicitly (2.21) amounts to

$$C_1 = \Upsilon_2\,, \quad D_1 = \frac{1}{2}(\Upsilon_1 + \Upsilon_3) + \lambda \Upsilon_2\,, \tag{2.22}$$

in terms of the coefficients in (2.5). As a consequence the flow equations for g_N and λ are now uniquely determined by the counterterm coefficients in (2.5) and possibly the parameters ξ entering through the field renormalizations:

$$\mu \frac{d}{\partial \mu} g_N = \beta_{\rm g}(\Upsilon, \zeta, \xi)\,, \qquad \mu \frac{d}{\partial \mu} \lambda = \beta_\lambda(\Upsilon, \zeta, \xi)\,. \tag{2.23}$$

The dependence on ξ must be such that $g_N \beta_\lambda + \lambda \beta_g$ is ξ independent.

3. Results for HD gravity

We now outline the application of the previous framework to HD gravity with classical action (1.2) [24]. [1] This requires a choice of gauge fixing and gauge averaging. We use a three parameter harmonic gauge

$$\begin{aligned} S_{\rm gf} &= \frac{1}{2s} \int d^4x \sqrt{g}\, \chi_\mu Y^{\mu\nu} \chi_\nu\,, \\ \chi_\mu &= \nabla^\nu f_{\mu\nu} + b_1 \nabla_\mu f\,, \\ Y^{\mu\nu} &= -\frac{1}{a}\Big[g^{\mu\nu}\nabla^2 + (b_2 - 1)\nabla^\mu \nabla^\nu - R^{\mu\nu}\Big]\,, \\ b_1 &= -\frac{1}{4c_1}\frac{1+4\omega}{1+\omega}\,, \quad b_2 = \frac{2c_2}{3}(1+\omega)\,, \end{aligned} \tag{3.1}$$

where the gauge condition $\delta(\chi_\mu - \theta_\mu)$ has been averaged with a normalized Gaussian of covariance $Y^{\mu\nu}$. The reparameterization of b_1, b_2 in terms of c_1, c_2 is such that $a = c_1 = c_2 = 1$ corresponds to the so-called minimal gauge where in the gauge fixed Hessian all terms quartic in ∇_μ except $(\nabla^2)^2$ drop out. The ghost action associated with (3.1) has kernel $\Delta^{\mu\nu} := -g^{\mu\nu}\nabla^2 - (1+2b_1)\nabla^\mu \nabla_\nu - R^{\mu\nu}$. The parameter in (2.4) is $k_0 = b_1$.

Consistent with the regularized Gaussians of section 2 we define the one-loop effective action by

$$\Gamma_1 = \frac{1}{2}{\rm Tr} F_{k^2, \Lambda_{\rm UV}^2}(\mathscr{H}) - \frac{1}{2}{\rm Tr} F_{k, \Lambda_{\rm UV}}(Y) - {\rm Tr} F_{k, \Lambda_{\rm UV}}(\Delta)\,, \tag{3.2}$$

[1] The presentation has been revised Nov. 2009 to match the publication.

where $\mathscr{H}$ is the Hessian of $2s(S+S_{\rm gf})$. Its divergent part will be of the form (2.5) with the coefficients Υ and ζ to be determined. In a non-gravitational context one usually subtracts from (3.2) a corresponding contribution from a reference operator. The reference operator is chosen so as to represent the non-interacting system and in particular removes quartic divergences. In gravity such a reference system bears on a definition of selfenergy and it is unlikely that a preferred choice exists. As discussed before the Gaussian normalization condition (2.4) amounts to having no subtractions in (3.2). Another modification of (3.2) would be to add to $\mathscr{H}$ its Vilkovisky-de-Witt (VdW) correction [9]. We verified that the setting used here correctly reproduces the VdW form of ζ_5 [10] upon adding the correction, but that it leaves Υ_1, Υ_3 unaffected.

The functional form of the coefficients in (2.5) can be constrained without genuine dynamical input. Keeping track of the grading by the loop counting parameter $\zeta_1, \zeta_2, \zeta_4, \zeta_5$ and $\Upsilon_1, \Upsilon_2, \Upsilon_3$ must be real valued functions of $s/g_N, \lambda, \omega$. Further

$$\Upsilon_1, \quad \Upsilon_2, \quad \frac{\Upsilon_3}{\lambda}, \quad \frac{\zeta_4}{\lambda}, \quad \frac{\zeta_5}{\lambda^2}, \quad \frac{\xi}{\lambda}, \tag{3.3}$$

must be polynomials in $s/(g_N\lambda)$ and that the last three quantities cannot have terms. This can be seen by deriving the flow equations as described in section 2 and requiring that the explicit dependence on the UV cutoff cancels. Finally, using the field equations of (1.2) one sees that ζ_1, ζ_2, $\zeta_5/\lambda^2 + 4\zeta_4/\lambda$ and Υ_1, $4\Upsilon_2 + \Upsilon_3/\lambda$, contain only on-shell information and thus should be independent of the choice of gauge and field reparameterization constant.

The evaluation of the divergent part of (3.2) now amounts to the determination of the short time asymptotics for the heat kernels of the operators $\mathscr{H}$, Y, and Δ. Both Y and Δ are second order operators with trivial principal part, for which tabulated heat kernel coefficients are available [20]. In a curved background and in a generic gauge (3.1) $\mathscr{H}$ is a very complicated operator for which no tabulated results are available; moreover there is no choice of gauge parameters for which its principal part is trivial. We thus resorted to an evaluation on a flat background in a generic gauge which allows one to determine Υ_1, Υ_3 (and as a check ζ_5) in a generic gauge. Finally Υ_2 can be obtained by transversal-traceless decomposition of the Hessian on maximally symmetric backgrounds. As a check we also evaluated Υ_2 directly on a generic background in minimal gauge, where the principal part is a nontrivial but constant matrix.

The evaluation of (3.2) on a flat background reveals that – in contrast to the common wisdom about the system and in contrast to the situation in Einstein gravity – there is *no* problem with positivity. The Hessian on a flat background can be diagonalized exactly and the positivity of the spectrum can be investigated. There are four spectral values $\lambda_1(p)$, $\lambda_2(p)$, $\lambda_3(p)$, $\lambda_4(p)$, with multiplicities $5,3,1,1$, respectively. The last two are non-rational functions of the momenta with a large p expansion of the form $p^{-4}\lambda_i(p) = \mu_i + O(sp^{-2})$ (which also applies to λ_1, λ_2, where the expansion terminates). Spectral positivity is decided by the signs of the μ_i and one can show

$$\mu_i > 0 \quad \text{for} \quad -1 < \omega < 0\,,\ c_1 > 1/4\,,\ c_2/a > 0\,. \tag{3.4}$$

The interval $-1 < \omega < 0$ is invariant under the renormalizition flow (3.6) and contains the known UV fixed point $\omega_* \approx -0.0228$ [12, 11, 13]. Hence for μ sufficiently large no problem with positivity of the propagator (i.e. the inverse Hessian) ever arises.

The divergent part of Γ_1 is absorbed by the coupling renormalizations described before and field renormalizations. By inspection of the equations of motion for (1.2) one sees that the only useful field renormalizations are of the form

$$q^0_{\alpha\beta} = q_{\alpha\beta} + \frac{\hbar}{(4\pi)^2}\ln(\Lambda_{\rm UV}/\mu)\, g_N \xi\, q_{\alpha\beta} + O(\hbar^2)\,, \tag{3.5}$$

where ξ can be a function of $s/g_N, \lambda, \omega$. Inserting the coupling renormalizations and (3.5) into the bare action S_0, expanding, and requiring that the divergent terms equals $-\Gamma^{(1)}_{\rm div}$ yields cancellation conditions fixing all parameters in terms of the ζ_j, Υ_j, except for ξ and a_{10}, b_{10}. Imposing the Wilsonian matching condition also fixes a_{10}, b_{10} and one arrives at flow equations uniquely determined by ζ_j, Υ_j and ξ. The result is:

$$\begin{aligned} \mu\frac{ds}{d\mu} &= -\frac{\hbar}{(4\pi)^2}\,2\zeta_1 s^2\,, \\ \mu\frac{d\omega}{d\mu} &= -\frac{\hbar}{(4\pi)^2}\,s(3\zeta_2 + 2\omega\zeta_1)\,, \end{aligned} \tag{3.6}$$

$$\begin{aligned} \mu\frac{d}{d\mu}g_N &= 2g_N + \frac{\hbar}{(4\pi)^2}g_N^2[\zeta_4 + \xi + 2\Upsilon_2]\,, \\ \mu\frac{d}{d\mu}\lambda &= -2\lambda + \frac{\hbar}{(4\pi)^2}\frac{g_N}{2}\Big[\zeta_5 + 4\lambda\zeta_4 + \Upsilon_3 + 4\lambda\Upsilon_2 + 4\Upsilon_1 - (2\lambda\xi + 2\lambda\zeta_4 - \Upsilon_3)\Big]\,. \end{aligned} \tag{3.7}$$

The $(s,\,\omega)$ flow equations are those of [12, 11, 14] while the $(g_N,\,\lambda)$ equations are new. The parameters in (3.6) are gauge independent and equal $\zeta_1 = 133/20$, $3\zeta_2 + 2\omega\zeta_1 = (25 + 1098\omega + 200\omega^2)/60$ [12, 14]. As a consequence the coupling s is asymptotically free with UV fixed point $s_* = 0$. The ω flow can likewise be integrated analytically and has the unique UV fixed point $\omega_* = (-549 + 7\sqrt{6049})/200 \approx -0.0228$ mentioned earlier. The other ζ coefficients are known in several gauges [11, 12, 13, 14, 10] while the Υ coefficients have not previously been computed. The (g_N, λ) flow equations manifestly depend on the coefficients of the powerlike divergences and warrants a detailed discussion.

The flow equations (3.7) admit a nontrivial fixed point which is solely determined by the Υ_1 and Υ_2 coefficents. Indeed by (3.3) ζ_4/λ, ζ_5/λ^2 and ξ/λ are at least $O(s)$ and thus vanish at the UV fixed point $s_* = 0$ of the s flow. Anticipating that also Υ_3 is linear in $s/(g_N\lambda)$ one sees that (3.7) has a nontrivial fixed point at

$$\frac{g_N^*}{(4\pi)^2} = -\frac{1}{\Upsilon_2^*}\,, \quad \lambda_* = -\frac{\Upsilon_1^*}{2\Upsilon_2^*}\,, \tag{3.8}$$

where $\Upsilon_2^* := \Upsilon_2|_{\omega_*, s=0}$, $\Upsilon_1^* := \Upsilon_1|_{\omega_*, s=0}$. Importantly the fixed point is gauge-independent whenever a definition of Γ_1 is used that renders it gauge independent on-shell. This is *not* the case for the definition (3.2) and improved variant will be presented elsewhere. The scheme dependence always enters only through the q_n of Eq. (2.17).

The results for Υ_1, Υ_2, Υ_3 based on (3.2) in a generic gauge are too bulky to be reported here. For simplicity we present them here in minimal gauge. First the fixed point values

$$\begin{aligned} \Upsilon_2^* &= -1.9867\,q_1 - 0.09836\,q_{1/2}\,, \\ \Upsilon_1^* &= 5.8114\,q_1 - 6.1026\,q_2\,, \end{aligned} \tag{3.9}$$

where for all cutoffs usually considered $q_1/q_2 \geq 2$.

To study the flow (3.6) itself we now make the choice

$$\xi = -\zeta_4 + \Upsilon_3/(2\lambda)\,, \tag{3.10}$$

which gives rise to (g_N, λ) flow equations depending only on the gauge independent ζ combination, $\Upsilon_2 + \Upsilon_3/(4\lambda)$ and Υ_1 without affecting the fixed point (3.8). One has $\Upsilon_1 = u_1(\omega)$, $\Upsilon_2 + \Upsilon_3/(4\lambda) = u_2(\omega) - \frac{s}{g_N \lambda} u_3(\omega)$, with

$$\begin{aligned}
u_1(\omega) &= q_1 \frac{26\omega - 1}{12\omega} - q_2 \Big[\frac{9}{2} + \frac{9}{8}\frac{1}{(1+\omega)^2} + \frac{4}{9}(1+\omega)^2\Big] \\
u_2(\omega) &= -\frac{\sqrt{\pi}}{8} q_{1/2} \Big[3(1+\omega) - \frac{\omega+2}{3}\sqrt{-\frac{1+\omega}{3\omega}}\Big] - q_1 \frac{87 + 118\omega + 56\omega^2 + 16\omega^3}{72(1+\omega)}\,. \\
u_3(\omega) &= \frac{3\sqrt{\pi}}{64} q_{1/2} \Big[3 - \Big(-\frac{1+\omega}{3\omega}\Big)^{3/2}\Big]\,,
\end{aligned} \tag{3.11}$$

In combination with the known $\zeta_1, \zeta_2, \zeta_5 + 4\lambda\zeta_4$ [12, 11, 14, 13, 10] this defines the flow (3.6).

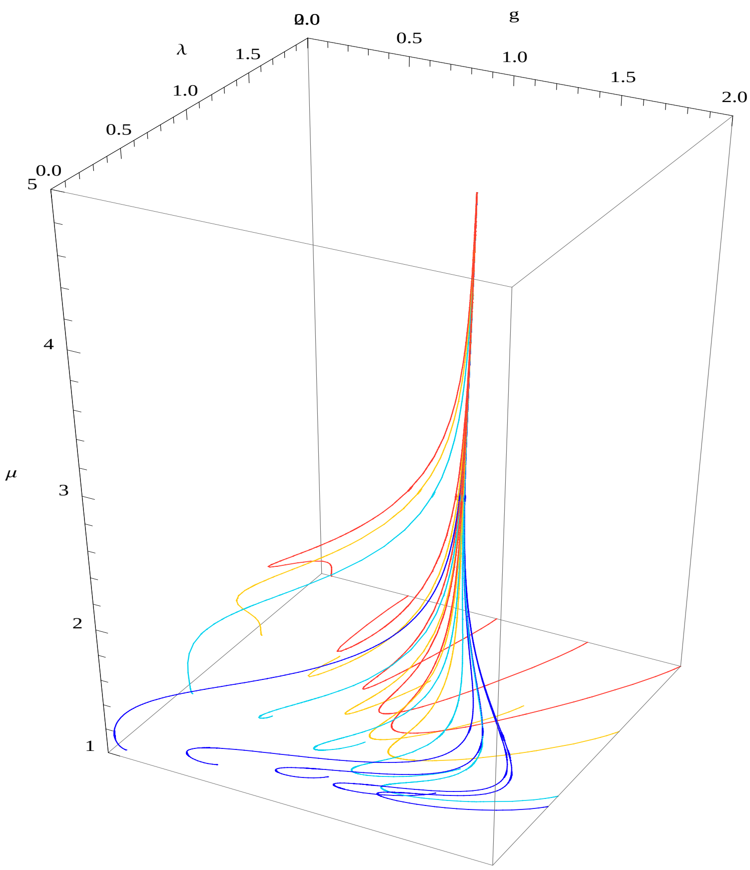

Figure 1: Wilsonian 1-loop flow in HD gravity in minimal geuge.

Fig. 1 shows the result of a numerical integration after rescaling $g_N \mapsto (4\pi)^2 g_N$, $s \mapsto (4\pi)^2 s$, with $s(1) = 1$, $\omega(1) = -1/2$, and the smooth cutoff. The initial data for g_N, λ were varied in the range $[0, 2]$. One sees that g_N, λ are initially non-monotonous functions of μ, monotonous behavior sets in quickly but non-uniformly in the initial data. At $\mu = 10$ the memory of the initial data is erased to accuracy 10^{-7} and the merged trajectory eventually hits the fixed point located at $g_N^* \approx 1.3697, \lambda_* \approx 0.9451$, however with 1% deviations even at $\mu = 10^9$.

4. Implications for the asymptotic safety scenario

Two main issues need to be addressed in order to promote HD gravity to a viable field theory of quantum gravity. First, since perturbation theory (PT) presumably captures only a small part of the physics content of the theory a formulation that is renormalizable in the Kadanoff-Wilson sense needs to be found. This requires a genuinely nonperturbative regularization yet-to-be-found in which one would expect the associated Wilsonian actions to contain higher derivative terms in addition to those in (1.2) needed for powercounting renormalizability. Second, higher derivative interactions are potentially problematic from the viewpoint of unitarity. Although the problem as originally construed is absent in HD gravity proper, see Eq. (3.4), little is known, conceptually and computationally, about what physical quantities ought to obey which physically relevant notion of unitarity. The asymptotic safety scenario [5, 2, 3, 4] purports the optimistic view that both problems can be overcome. In brief: *S-matrix-like quantities in higher derivative type gravity can be constructed via a massive scaling limit based on a nontrivial fixed point beyond asymptotic expansions and are compatible with the physically relevant notion of unitarity.* We regard the perturbative construction of the nontrivial fixed point described here as very compelling. It complements earlier results obtained via the truncated average effective action [15, 16, 17, 18, 19]; some comments on the relation will be offered elsewhere. An important implication is that the interplay between perturbative and nonperturbative quantum gravitational physics may be similar as in Yang-Mills theories, with the nonperturbative dynamics important mostly in the infrared. A second consequence is that the dimensional reduction phenomenon [3, 2] for the residual interactions in the extreme UV can be investigated perturbatively. The conjectured picture is [2]: The functional averages of physical quantities can in an asymptotically safe theory of quantum gravity based on a nontrivial fixed point in the extreme ultraviolet be asymptotically reproduced by a two-dimensional statistical field theory which is: selfinteracting, not a conformal field theory, and asymptotically safe itself. Gravitationally motivated field theories with the correct qualitative properties can be obtained through Killing vector reductions [23] but the proper dynamical reduction phenomenon remains to be understood.

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