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ADS/CFT and Cosmological Singularities

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In String Theory dynamical space-time is an emergent phenomenon. One manifestation of this is the holographic correspondence, where a field theory with no dynamical metric is dual to a theory containing gravity in higher number of dimensions. In these lectures I describe an attempt to understand space-like singularities and regions of large curvature where Einstein equations break down in terms of the dual gauge theory, and investigate whether the gauge theory can provide a continuation of time evolution beyond this region.

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1. Introduction

It has been always suspected that near singularities usual notions of space and time break down and a consistent quantization of gravity would provide a more abstract structure which replaces space-time. However we do not know as yet what this abstract structure could be in general. In some situations, String Theory has provided concrete ideas about the nature of this structure. These are situations where gravitational physics has a tractable holographic description [1] in terms of a non-gravitational theory in lower number of space-time dimensions. In view of the spectacular success of the holographic principle in black hole physics, it is natural to explore whether this can be used to understand conceptual issues posed by singularities.

In String Theory, holography is a special case of a more general duality between open and closed strings. This duality implies that the dynamics of open strings contains the dynamics of closed strings. Since closed strings contain gravity, space-time questions can be posed in an open string theory which does not contain gravity and therefore conceptually easier. Under special circumstances, the open string theory can be truncated to its low energy limit - which is a gauge theory on a *fixed* background. In these situations, open-closed duality becomes particularly useful. The best controlled example of this is the the celebrated AdS/CFT correspondence [2] which relates closed string theory in asymptotically anti-de-Sitter spacetimes to gauge theories living on their boundaries. The dynamical "bulk" spacetime (on which the closed string theory lives) is an approximation which holds in a specific regime of the gauge theory. In this regime, the closed string theory reduces to supergravity. Generically, there is no space-time interpretation, though the gauge theory may make perfect sense. This fact opens up the possibility that in regions where the bulk gravity description is singular, one may have a well formulated gauge theory description and one has an answer to the question : *What replaces space-time ?*

Treating time dependent backgrounds in string theory, particularly those with singularities, has been notoriously difficult. However, some progress has been made recently in holographic formulations of mentioned above. The basic idea is to find models where the space-time background on which the closed string theory is defined is singular, but the holographic gauge theory description is well formulated. Thus, the gauge theory hopefully provides a controlled description of the region which would appear singular if the gravity interpretation is extrapolated beyond its regime of validity.

In its simplest setting, the correspondence implies IIB string theory on asymptotically $AdS_5 \times S^5$ with a constant 5-form flux is dual to $3+1$ dimensional $N=4$ supersymmetric $SU(N)$ Yang-Mills theory with appropriate sources which lives on the boundary of AdS_5. If R_{AdS} denotes the radius of the S^5 as well as the curvature length scale of AdS_5 and g_s denotes the string coupling, the coupling constant g_{YM} and the rank of the gauge group N of the Yang Mills theory are related by

$$\frac{R_{AdS}^4}{l_s^4} = 4\pi g_{YM}^2 N \qquad g_s = g_{YM}^2 \tag{1.1}$$

This immediately implies that the gauge theory describes *classical* string theory in the 't Hooft limit

$$N \to \infty \qquad g_{YM} \to 0 \qquad g_{YM}^2 N = \text{finite} \tag{1.2}$$

The low energy limit of the closed string theory - supergravity - is a good approximation only in the strong coupling regime $g_{YM}^2 N \gg 1$. For small $g_{YM}^2 N$, supergravity, and hence conventional space-time, is not a good description of the gauge theory dynamics. Finite N corrections correspond to string loop effects.

I will review two approaches. The first approach involves deformations of $AdS_5 \times S^5$ in the Poincare patch which correspond to time-dependent [3, 4, 5] sources. For a suitable choice of the sources, the bulk solution develops a null or space-like singularity. In the second approach [6] one considers global AdS, so that the gauge theory is defined on a S^3 whose radius can be taken to be R_{AdS}. One then turns on a suitable time dependent source which varies slowly in time compared to R_{AdS}^{-1}. The bulk solution can be constructed in a systematic expansion in time derivatives, while the evolution of the state in the gauge theory language can be analyzed systematically in a modified form of adiabatic approximation involving coherent states.

The basic strategy of [3]-[5] is the following. Consider the $N = 4$ gauge theory defined on the Poincare boundary of an asymptotically $AdS_5 \times S^5$ space-time, deformed by a suitable time dependent source. The 't Hooft coupling is large. The source is chosen to be weak and slowly varying at early times and becomes strong at some intermediate time which may be chosen to be $t = 0$. The gauge theory is in its vacuum state in the far past. The AdS/CFT correspondence then ensures that at early times, the bulk space-time would be a non-normalizable deformation of $AdS_5 \times S^5$ by a supergravity mode dual to the source. Time evolution in the gauge theory is governed by the time-dependent hamiltonian. So long as the source is weak, the time evolution of the bulk theory is governed by the classical equations of motion of supergravity. At some later time, when the source becomes strong, curvature invariants and/or tidal forces in the bulk could become large and supergravity cannot be trusted any more. If we nevertheless continue to use supergravity we could encounter a singularity. The question is whether the gauge theory can be still used to ask whether a further time evolution is meaningful. Note that for this purpose a string scale curvature is physically equivalent to a mathematical singularity.

In the following we will choose a simple source - a time dependent coupling of the Yang-Mills theory of the form

$$g_{YM}^2(t) = \bar{g}_{YM}^2 \, F(t) \tag{1.3}$$

such that the function $F(t)$ becomes unity at $t = \pm\infty$ and dips down to a small value near $t = 0$. The quantity $\bar{g}_{YM}^2 N$ will be taken to be large. The non-dynamical space-time on which the gauge theory is defined remains flat.

In the rest of this paper we will choose $R_{AdS} = 1$. Then the string length is directly proportional to $(g_{YM}^2 N)^{-1/4}$.

2. Poincare Patch

In this section we consider Poincare patch cosmological solutions and their gauge theory duals

2.1 Supergravity Solution

Consider a general class of five dimensional metrics of the general form

$$ds^2 = \frac{1}{z^2}\left[dz^2 + \tilde{g}_{\mu\nu}(x)dx^\mu dx^\nu\right] \tag{2.1}$$

with a dilaton $\Phi(x)$ which depends on the 4 dimensional coordinates x^μ and the 5 form field is proportional to the volume form. This solves the supergravity equations of motion if

$$\tilde{R}_{\mu\nu} = \frac{1}{2}\partial_\mu\Phi\partial_\nu\Phi \qquad \tilde{\nabla}^2\Phi = 0 \tag{2.2}$$

where the tildes above mean that the quantities are evaluated with the 3+1 dimensional metric $\tilde{g}_{\mu\nu}$. $\tilde{g}_{\mu\nu}$ is in fact the metric on the boundary $z=0$ on which a dual field theory can be defined.

This means that we can lift any solution of 3+1 dimensional dilaton gravity to a solution in asymptotically AdS_5 spacetime. The simplest time-dependent solution with a space-like singularity is in fact the lifted Kasner metric with an acccompanying dilaton

$$ds^2 = \frac{1}{z^2}\left[dz^2 - dt^2 + \sum_{i=1}^{3} t^{2p_i}\,(dx^i)^2\right] \qquad \Phi(t) = \alpha\ \log(t) \tag{2.3}$$

where

$$\sum_{i=}^{3} p_i = 1 \qquad \sum_{i=1}^{3} p_i^2 = 1 - \frac{\alpha^2}{2} \tag{2.4}$$

A special case of this metric, which will be useful in what follows is the symmetric Kasner solution with $p_1 = p_2 = p_3 = \frac{1}{3}$ which may be written in the following form after a redefinition of the time :

$$ds^2 = \frac{1}{z^2}\left[dz^2 + |2t|(-dt^2 + (d\vec{x})^2)\right] \qquad \Phi(t) = \alpha\ \log(t) \qquad e^\Phi = |t|^{\sqrt{3}} \tag{2.5}$$

There is a curvature singularity at $t=0$. In (2.5) the boundary metric is conformal to flat space-time.

In the overall setup described above, we want to keep the space-time of the gauge theory to be flat. This could be achieved by a Weyl transformation of the boundary metric. It is well known that such Weyl transformations are produced by a special class of coordinate transformations in the bulk - the Penrose-Brown-Henneaux (PBH) transformations [7]. For the symmetric Kasner metric, such transformations can be found explicitly,

$$z = \frac{32\rho T^{\frac{5}{2}}}{\sqrt{6}}\frac{1}{16T^2-\rho^2}, \qquad t = T\left(\frac{16T^2+5\rho^2}{16T^2-\rho^2}\right)^{\frac{2}{3}} \tag{2.6}$$

and the metric becomes

$$ds^2 = \frac{1}{\rho^2}\left[d\rho^2 - \frac{(16T^2-5\rho^2)^2}{256T^4}dT^2 + \frac{(16T^2-\rho^2)^{\frac{4}{3}}(16T^2+5\rho^2)^{\frac{2}{3}}}{256T^4}d\vec{x}^2\right] \tag{2.7}$$

The boundary $\rho=0$ is now explicitly flat.

Our setup also requires that e^Φ is bounded everywhere and asymptotes to a constant at early and late times. The Kasner solution is clearly not of this form. It turns out that there are solutions of this type whose near-singularity behavior is Kasner-like, but for which the dilaton is bounded. One such background is

$$ds^2 = \frac{dz^2}{z^2} + \frac{1}{z^2}(1-\frac{1}{\tau^4})\ \left[-d\tau^2 + \tau^2[dr^2 + \sinh^2 r\, d{\Omega_2}^2]\right] \tag{2.8}$$

with the dilaton

$$\Phi(\tau) = \sqrt{3}\ln\left[\frac{\tau^2-1}{\tau^2+1}\right] \tag{2.9}$$

The curvature singularity is now at $\tau = 1$. The boundary metric of (2.8) is conformal to a Milne wedge of flat space-time. Therefore there should be a PBH transformation which provides a new foliation in which the boundary metric is flat. In this case the PBH transformation cannot be found exactly. However since all we need is the form of the metric near the boundary, these may be determined as a power series expansion in z.

As promised, near $\tau \sim 1$ the metric and the dilaton become identical to the symmetric Kasner solution (after a trivial redefinition of time). In fact, this is a special case of a rather general fact. Since our solutions are lifts of 3+1 dimensional dilaton cosmologies, we can use the classic results of Belinski, Lifshitz and Khalatnikov (BKL) [8, 9]. The general analysis of BKL shows that for a large class of initial metrics, the geometry near a space-like singularity oscillates between suitable generalizations of Kasner-like metrics with the Kasner exponents p_i changing after every bounce in the oscillations. For dilaton driven cosmology, the number of oscillations are finite with the end-point corresponding to all the $p_i's$ being positive. The symmetric Kasner is a special case of this.

In the region where supergravity is valid we can compute the energy momentum tensor of the boundary gauge theory using standard techniques of Holographic Renormalization Group [10, 11, 12, 13].

The results are

$$< T_\mu^\nu >= \frac{N^2}{2\pi^2(\tau^4-1)^4}\,\mathrm{diag}\left(12-3\tau^4, 4+9\tau^4, 4+9\tau^4, 4+9\tau^4\right), \tag{2.10}$$

which clearly shows that $< T_\mu^\nu > \to 0$ as $\tau \to -\infty$, ensuring that we have indeed started with the vacuum state. This quantity diverges near the singularity at $\tau = 1$. However this is the region where this supergravity calculation cannot be trusted.

In what follows, it is useful to record the answer for the energy-momentum tensor for the symmetric Kasner solution,

$$< T_\mu^\nu >= \frac{N^2}{512\pi^2 t^4}\,\mathrm{diag}(9,\,13,\,13,\,13), \tag{2.11}$$

2.2 Properties of the dual gauge theory

We have chosen the gauge theory to have a bounded coupling which goes to zero, or becomes very small at some intermediate time. In the bulk, this leads to large curvatures and/or large tidal forces. Our aim is to determine whether this is a genuine sickness of the theory, or a breakdown of the supergravity approximation which can be cured by the gauge theory.

At first sight, it might appear that near the time of the bulk spacelike singularity, the theory is weakly coupled and there should be nothing wrong with it - indicating that one should be able to continue time evolution across this without trouble. However this is not correct. The lagrangian of the Yang-Mills theory with a general space-time dependent coupling is

$$L = \mathrm{Tr}\{-\frac{1}{4e^\Phi}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}(D_\mu X^a)^2 - \frac{1}{4}e^\Phi([X^a,X^b])^2 \tag{2.12}$$

$$+\bar{\Psi}\Gamma^\mu D_\mu\Psi + ie^{\Phi/2}\bar{\Psi}\Gamma^a[X^a,\Psi]\}. \tag{2.13}$$

where we have used the fact that $g^2_{YM}(x) = e^{\Phi(x)}$ where $\Phi(x)$ is the bulk dilaton. There are six scalars, $X^a, a = 1, \cdots 6$, and 4 two-component Weyl fermions of $SO(1,3)$, which have been grouped together as one Majorana Weyl Fermion of $SO(1,9)$ denoted by Ψ. The Gamma matrices $\Gamma^\mu, \mu = 0, 1, \cdots 3$, and $\Gamma^a, a = 1, \cdots 6$, together form the 10 Gamma matrices of $SO(1,9)$. The scalars and fermions transform as the adjoint of $SU(N)$. The covariant derivative of the scalars is,

$$D_\mu X^a = \partial_\mu - i[A_\mu, X^a], \tag{2.14}$$

and similarly for the fermionic fields. The field strength is,

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu]. \tag{2.15}$$

At the singularity e^Φ becomes small. As a result the prefactor in the gauge kinetic energy term in eq.(2.12) becomes large. As is usual in perturbation theory, we might want to absorb an appropriate power of the coupling into the gauge field by redefining

$$A_\mu \to e^{\Phi/2} A_\mu \tag{2.16}$$

Since Φ is not a constant, this would introduce terms which contain $\nabla\Phi$ in the lagrangian. We may arrange for ∇e^Φ to be finite and smooth everywhere, but $\nabla\Phi$ would be large, since e^Φ is itself small. It is easy to see that only terms which are quadratic in the fields will contain factors of $\nabla\Phi$ without accompanying factors of e^Φ. In the nonlinear terms, on the other hand, $\nabla\Phi$ factors are always accompanied by positive powers of e^Φ and these can be arranged to be small. Therefore, we need to concentrate on the quadratic terms in the action.

The only such terms we need to consider are those which involve the gauge fields. Under the above field redefinition, the quadratic part of the gauge field action becomes

$$-F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\left[(\frac{1}{2}(\nabla\Phi)^2 - \nabla^2\Phi)A_\nu A^\nu + \partial_\mu\Phi\partial_\nu\Phi A^\mu A^\nu + 2\nabla_\mu\Phi A^\nu\partial_\nu A^\mu\right] \tag{2.17}$$

When $\Phi(x^+)$ is a function of a null coordinate alone - as in the null cosmologies discussed above, we can choose a gauge $A_- = 0$. It is then easy to see that the extra terms in (2.17) vanish. Thus in this case the kinetic term becomes canonical and the interactions contain positive factors of e^Φ. Classically any initial condition with finite values of the transverse components of A_μ evolve into finite values near $t = 0$, so that the interaction terms are indeed small because of the smallness of e^Φ and its derivatives. This implies that nothing is truly pathological at $x^+ = 0$. The time evolution in the gauge theory continues even though from the supergravity time evolution appears to stop at a finite light front time.

In contrast, for a *time* dependent dilaton, $\Phi(t)$, the situation is completely different. In this case a convenient choice of gauge is $A_0 = 0$ together with $\partial_i A^i = 0, \quad i = 1, \cdots 3$. Then (2.17) gives rise to a time dependent mass term for the transverse components

$$m^2(t) = -(\frac{1}{2}(\nabla\Phi)^2 - \nabla^2\Phi) \tag{2.18}$$

This is the essential complication which needs to be dealt with.

2.2.1 The wavefunctional

The essential physics may be captured by a toy model of a single scalar field with the lagrangian

$$L = -\frac{1}{e^{\Phi}}[\frac{1}{2}(\partial X)^2 - V_{int}(X)]. \tag{2.19}$$

where $V_{int}(X)$ denotes some interaction term. The field X may be thought to represent one of the transverse gauge fields.

We will now analyze this model in detail for a dilaton profile given by

$$e^{\Phi(t)} = |t|^p \qquad p > 0 \tag{2.20}$$

as we approach $t \to 0^-$. This is motivated by the observation in the previous subsection that a large class of cosmological models become Kasner-like near the singularity. After a redefiniton of the field

$$Y(t,x) = e^{-\Phi/2}X(t,x) \tag{2.21}$$

the lagrangian becomes, upto a total derivative

$$L = -\frac{1}{2}(\partial Y)^2 - m^2(t)Y^2 - e^{-\Phi}V_{int}(e^{\Phi/2}Y). \tag{2.22}$$

For our choice of the dilaton profile,

$$m^2(t) = -\frac{p(p+2)}{4t^2} \tag{2.23}$$

This is a *tachyonic* mass term which diverges as $t \to 0^-$. The effect of this is quite significant. Ignoring the interaction term, a general solution of the classical equations of motion is

$$Y(t,\vec{x}) = \int d^3k\, e^{i\vec{k}\cdot\vec{x}}(-\omega t)^{1/2}\, [a_k\, H^{(1)}_\nu(-\omega t) + a^\star_k H^{(2)}_\nu(-\omega t)] \tag{2.24}$$

where

$$\nu = \frac{p+1}{2} \qquad \omega^2 = \vec{k}^2 \tag{2.25}$$

and $H^{(1)}, H^{(2)}$ denote the standard Hankel functions. For large arguments, the Hankel functions behave as

$$H^{(1)}_\nu(z) \sim \frac{1}{z^{1/2}}\, e^{iz} \tag{2.26}$$

so that the mode functions become standard plane waves as $t \to -\infty$. For small arguments, the Hankel functions behave as

$$H^{(1)}_\nu(z) \sim \frac{1}{z^\nu} \tag{2.27}$$

Using (2.25) it is now clear from (2.24) that for generic values of a and $a^\star$

$$Y(t,x) \to (-t)^{-p/2} \qquad t \to 0^- \tag{2.28}$$

and blows up. It is only for very special fine tuned initial conditions, $a = a^\star$ that Y remains finite.

This means that in the interaction, a term which is Y^n for some positive integer $n > 3$ behaves as

$$e^{\frac{n-2}{2}\Phi}(-t)^{-np/2} \sim (-t)^{-p} \tag{2.29}$$

and blows up as well. Clearly the interactions cannot be ignored even though the coupling is weak, simply because the fields are always generically driven to large values.

On the other hand, the original variables $X(t,x)$ have a finite limit as $t \to 0^-$, since

$$X(t,\vec{x}) = e^{\Phi/2} Y(t,\vec{x}) \sim (-t)^{p/2}\,(-t)^{-p/2} \tag{2.30}$$

This suggests that one should really think in terms of the original variables.

We will now examine the behavior of the wavefunctional of the theory. Let us first ignore the interactions. The fourier components of $X(t,x)$, which are denoted by $X_k(t)$,

$$X(t,\vec{x}) = \int \frac{d^3k}{(2\pi)^3} X_k(t)\, e^{-i\vec{k}\cdot\vec{x}} \tag{2.31}$$

and the mode expansion for the operator $X_k(t)$ is

$$X_k(t) = \frac{1}{\sqrt{2\omega}} e^{\Phi/2} (-\omega t)^{1/2}\,[\hat{a}_k\, f(t) + \hat{a}^\star_k f^\star(t)] \tag{2.32}$$

where

$$f(t) = \sqrt{\frac{\pi}{2}} (-\omega t)^{1/2}\, H^{(1)}_\nu(-\omega t) \tag{2.33}$$

$\hat{a}, \hat{a}^\dagger$ are now creation and annihilation operators. Consider first the state

$$\hat{a}_k |0> = 0 \tag{2.34}$$

The wavefunction of this state may be easily calculated

$$\Psi_0(X_k,t) = \prod_k \frac{A}{\sqrt{f^\star(t) e^{\Phi/2}}} \exp\left[\frac{i}{2}[\frac{\partial_t f^\star}{f^\star} + \frac{1}{2}\partial_t\Phi]\, e^{-\Phi}\, X_k X_{-k}\right] \tag{2.35}$$

where A is a normalization constant. For $t \to 0$, $f(t) \to e^{-i\omega t}$ and $\partial_t \Phi \sim \frac{p}{t}$, leading to the standard gaussian form of a harmonic oscillator wave function. For $t \to 0^-$ we have to use the small t behavior of the Hankel functions. The contribution to $\frac{\partial_t f^\star}{f^\star}$ from the leading term of the expansion, following from (2.27) cancel the term which comes from $\frac{1}{2}\partial_t\Phi$ and the first subleading correction leads to the following phase factor in the wavefunctional

$$\Psi_0 \sim \exp\,\left[i\, C X_k X_{-k}\, (-t)^{1-p}\right] \tag{2.36}$$

where C is a numerical constant. The probability density, however goes to a smooth gaussian

$$|\Psi_0|^2 \sim \prod_k \frac{|A|^2}{|f| e^{\Phi/2}} \exp\left[-\frac{\omega X_k X_{-k}}{|f|^2 e^{\Phi}}\right] \tag{2.37}$$

The behavior of the phase factor in (2.36) in the limit $t \to 0^-$ depends on the value of p. For $p < 1$ this has a smooth behavior, while for $p > 1$ the phase factor oscillates infinitely rapidly. These wild oscillations result in a diverging expectation value for the square of the conjugate momentum for X.

Significantly, the wavefunction for any coherent state exhibits precisely the same behavior. In fact, the same is true for a generic state of the system.

We have so far analyzed the behavior of the wavefunction in the free theory. However we have argued that interactions cannot be ignored near $t = 0$. With interactions, it is of course not possible to derive the exact wavefunctional. However it turns out that it is possible to deduce the behavior of the phase of the wavefunctional for an *arbitrary* interaction. The Schrodinger equation is

$$\int d^3k \left[-\frac{e^{\Phi}}{2} \frac{\partial^2}{\partial X_k \partial X_{-k}} + e^{-\Phi}\, V(X_k) \right] \Psi = i\partial_t \Psi \tag{2.38}$$

where the potential $V(X_k)$ includes terms which come from the space derivatives of the original field theory as well as interaction terms written in momentum space,

$$V(X_k) = \frac{1}{2}\omega^2 X_k X_{-k} + V_{int}(X_k) \tag{2.39}$$

is the potential term written in terms of the fourier modes X_k. Since $e^{\Phi} \sim (-t)^p$, the potential term dominates as $t \to 0^-$. To a first approximation we can then ignore the kinetic energy term and solve the Schrodinger equation easily

$$\Psi^{(0)}(X_k, t) = \prod_k \exp[-iG(t)V(X_k)]\ \xi(X_k) \tag{2.40}$$

where

$$G(t) = \int dt e^{-\Phi} = -\frac{(-t)^{1-p}}{1-p} \tag{2.41}$$

and $\xi(X_k)$ is a function of X_k only. The time dependence is seen to be exactly the same as in the quadratic approximation, and agrees precisely with (2.36)when only the quadratic term is retained in $V(X_k)$. We therefore see that the behavior of the phase factor is valid quite generally, independent of the quadratic approximation.

To check the self-consistency of the above procedure, we need to insert (2.40) into (2.38). A short calculation shows that the kinetic energy term is always subdominant, independent of the value of p. For $p > 1$ the kinetic energy term in fact diverges as $t \to 0^-$ - however slower than the potential energy term by a factor of t^2, while for $p < 1$ the kinetic energy term vanishes in this limit.

The form of the wavefunction (2.40) is quite general and does not depend on initial conditions. This is important since we have looked at the system with a e^{Φ} which behaves as some power of t. On the other hand the cosmological solutions we have discussed display such a behavior only near the singularity. If we start with an initial vacuum state for a system which has a dilaton profile which asymptotes to a constant value in the far past, this state will evolve into some nontrivial state when the time is close to $t = 0$ and where we can apply the considerations of this subsection. Since our conclusion in this subsection is valid for any general state, it directly applies to the cosmological solutions in question.

2.2.2 Energy Production

We have seen that because of wild oscillations the wave functional for $p > 1$ has no well defined limit as $t \to 0^-$ and therefore cannot be meaningfully continued beyond this time. On the other hand for $p < 1$ there is a finite limit, and a continuation is possible. This fact is independent of perturbation theory, which is not valid in any case in this region.

We will now show that regardless of the value of p, the energy produced in the $t \sim 0^-$ region is infinite for generic states. This follows from the observation that the energy is dominated by the potential term. Therefore,

$$< H >\sim e^{-\Phi} < V >= (-t)^{-p} < V > \tag{2.42}$$

which is infinite for ny state for which $< V >\neq 0$. This conclusion can be avoided for very special states for which $< V >= 0$.

2.2.3 The fate of the system

We have seen that for a gauge theory coupling which strictly vanishes as a power law at $t = 0$, an infinite amount of energy is pumped into the system. This suggests that such a theory is genuinely sick.

From a physical point of view, however, we are interested in the situation where e^{Φ} remains finite and becomes small enough at $t = 0$ so that the dual supergravity has a string scale curvature. This is for all practical purposes a singularity in supergravity. We need to examine whether the gauge theory admits a time evolution beyond this time in this case.

The above discussion shows that this is indeed possible. For example if we have

$$e^{\Phi} = (t^2 + \varepsilon^2)^{p/2} \tag{2.43}$$

the wave functional can always be continued, and the energy pumped into the system till $t = 0$ is finite, albeit large. As we proceed to positive values of t, energy will be *pumped out* of the system - leaving behind a finite amount of energy, unless the initial state is so finely tuned that as much energy would be extracted as put in initially.

Typically the remaining energy will thermalize, given enough time. Since we are considering the theory on $R^{(3,1)}$ the AdS/CFT correspondence implies that a thermal state will correspond to a black brane in the bulk.

It is important to check that thermalization does not occur at early enough times when supergravity is still valid. This is indeed true. This is the region where the bulk solution is still Kasner-like, but the curvatures are small. The energy density produced can be read off from (2.11)

$$\rho \sim \frac{N^2}{t^4} \tag{2.44}$$

so that the equivalent temperature T is given by

$$N^2\, T^4 \sim \rho \sim \frac{N^2}{t^4} \tag{2.45}$$

so that the thermalization time scale is

$$\tau \sim \frac{1}{T} \sim t \tag{2.46}$$

Thus the dimensionless quantity which determines the rate of change of temperature is

$$\frac{\partial_t T}{T^2} \sim 1 \tag{2.47}$$

On the other hand thermalization requires that this quantity should be much smaller than one.

Thermalization will, however, occur once one crosses the region $t = 0$. Since the coupling approaches a constant in the far future, for any net finite energy produced there will be sufficient time to thermalize. Therefore there will be a black brane in the bulk in the future. Our tools are not sufficient to derive detailed properties of this black hole. However, for any finite energy, the temperature of the black hole is finite and any such black hole in AdS space-time will have a curvature of the order of the AdS scale at the horizon. Therefore there will be a region outside the horizon which may be described by normal space-time geometry.

A black brane formation can be avoided only if the intial state is so finely tuned that *exactly* the same amount of energy is extracted from the system for $t > 0$ as pumped in during $t < 0$. For generic states this is not possible.

3. Global AdS : Slowly Varying Dilaton

I will now summarize our results for $N = 4$ gauge theory defined on S^3 with a time-dependent coupling so that the dual geometry is a cosmology in global AdS driven by a time-dependent dilaton. We will consider the case where the coupling is slowly varying in units of R_{AdS} and construct a systematic derivative expansion both in the gauge theory as well as in the dual gravity. Starting with a 't Hooft coupling which is large at early times, such a slowly varying dilaton can lead to a small value of the 't Hooft coupling at some intermediate time. When this happens, the bulk curvatures become large. This is not a singularity in the technical sense, but is like a space-like singularity for all physical purposes, since the spergravity equations break down. We will investigate if the gauge theory can be used to continue the time evolution beyond this point, and if so what is the outcome.

This section is entirely based on [6] which should be consulted for more details.

3.1 Supergravity solutions in a derivative expansion.

IIB supergravity in the presence of the RR five form flux is well known to have an $AdS_5 \times S^5$ solution. In global coordinates this takes the form,

$$ds^2 = -(1+\frac{r^2}{R_{AdS}^2})dt^2 + \frac{dr^2}{1+\frac{r^2}{R_{AdS}^2}} + r^2 d\Omega_3^2 + R_{AdS}^2 d\Omega_5^2. \tag{3.1}$$

Here R_{AdS} is given by,

$$R_{AdS} = (4\pi g_s N)^{1/4} l_s \sim N^{1/4} l_{pl} \tag{3.2}$$

where l_s is the string scale and $l_{pl} \sim g_s^{1/4} l_s$ is the ten dimensional Planck scale. g_s is the value of the dilaton, which is constant and does not vary with time or spatial position,

$$e^{\Phi} = g_s. \tag{3.3}$$

In the time dependent situations we consider below N will be held fixed. Let us discuss some of our conventions before proceeding. We will find it convenient to work in the 10-dim. Einstein frame. Usually one fixes l_{Pl} to be of order unity in this frame. Instead for our purposes it will be convenient to set

$$R_{AdS} = 1. \tag{3.4}$$

From eq.(3.2) this means setting $l_{Pl} \sim 1/N^{1/4}$. The $AdS_5 \times S^5$ solution then becomes,

$$ds^2 = -(1+r^2)dt^2 + \frac{1}{(1+r^2)}dr^2 + r^2 d\Omega_3^2 + d\Omega_5^2, \tag{3.5}$$

for any constant value of the dilaton, eq.(3.3). Let us also mention that when we turn to the boundary gauge theory we will set the radius R of the S^3 on which it lives to also be unity.

The essential idea in finding the solutions we describe is the following. Consider a situation where Φ varies with time slowly compared to R_{AdS}. Since the solution above exists for any value of g_s and the dilaton varies slowly one expects that the resulting metric at any time t is well approximated by the $AdS_5 \times S^5$ metric given in eq.(3.5). This zeroth order metric will be corrected due to the varying dilaton which provides an additional source of stress energy in the Einstein equations. However these changes should be small for a slowly varying dilaton and should therefore be calculable order by order in perturbation theory.

Let us make this more precise. Consider as the starting point of this perturbation theory the AdS_5 metric given in eq.(3.5) and a dilaton profile,

$$\Phi = \Phi_0(t) \tag{3.6}$$

which is a function of time alone. We take $\Phi_0(t)$ to be of the form,

$$\Phi_0 = f(\frac{\varepsilon t}{R_{AdS}}) \tag{3.7}$$

where $f(\frac{\varepsilon t}{R_{AdS}})$ is dimensionless function of time and ε is a small parameter,

$$\varepsilon \ll 1. \tag{3.8}$$

The function f satisfies the property that

$$f'(\frac{\varepsilon t}{R_{AdS}}) \sim O(1) \tag{3.9}$$

where prime indicates derivative with respect to the argument of f.

When $\varepsilon = 0$, the dilaton is a constant and the solution reduces to $AdS_5 \times S^5$. When ε is small,

$$\frac{d\Phi_0}{dt} = \frac{\varepsilon}{R_{AdS}} f'(\frac{\varepsilon t}{R_{AdS}}) \sim \frac{\varepsilon}{R_{AdS}} \tag{3.10}$$

so that the dilaton is varying slowly on the scale R_{AdS}, and the contribution that the dilaton makes to the stress tensor is parametrically suppressed. In such a situation the back reaction can be calculated order by order in ε. The time dependent solutions we consider will be of this type and ε will play the role of the small parameter in which we carry out the perturbation theory. A simple rule to count powers of ε is that every time derivative of Φ_0 comes with a factor of ε.

The profile for the dilaton we have considered in eq.(3.6) is S^5 symmetric. It is consistent to assume that the back reacted metric will also be S^5 symmetric with the radius of the S^5 being equal to R_{AdS}. The interesting time dependence will then unfold in the remaining five directions of AdS space and we will focus on them in the following analysis.

The zeroth order metric in these directions is given by,

$$ds^2 = -(1+r^2)dt^2 + \frac{1}{(1+r^2)}dr^2 + r^2 d\Omega_3^3. \tag{3.11}$$

And the zeroth order dilaton is given by eq.(3.6),

$$\Phi_0 = f(\varepsilon t). \tag{3.12}$$

We can now calculate the corrections to this solution order by order in ε.

Let us make two more points at this stage. First, we will consider a dilaton profile Φ_0 which approaches a constant as $t \to -\infty$. This means that in the far past the corrections to the metric and the dilaton which arise as a response to the time variation of the dilaton must also vanish. Second, the perturbation theory we have described above is a derivative expansion. The solutions we find can only describe slowly varying situations. This still allows for a big change in the amplitude of the dilaton and the metric though, as long as such changes accrue gradually. It is this fact that makes the solutions non-trivial.

To the lowest order in ε the solution of the dilaton equations of motion turn out to be

$$\Phi_2(r,t) = \frac{1}{4}\ddot{\Phi}_0(t)\left[\frac{1}{r^2}\log(1+r^2) - \frac{1}{2}(\log(1+r^2))^2 - \mathrm{dilog}(1+r^2) - \frac{\pi^2}{6}\right]. \tag{3.13}$$

The solution is regular everywhere. Since $\mathrm{Lim}_{t\to-\infty}\dot{\Phi}_0(t), \ddot{\Phi}_0(t) = 0$, the correction vanishes in the far past, as required. The time varying dilaton provides an additional source of stress energy. The lowest order contribution due to this stress energy is $O(\varepsilon)^2$ as we will see below. It then follows, after a suitable coordinate transformation if necessary, that the $O(\varepsilon)$ corrections to the metric vanish and the first non-vanishing corrections to it arise at order ε^2. The essential point here is that any $O(\varepsilon)$ correction to the metric must be r dependent and thus would lead to a contribution to the Einstein tensor of order ε, which is not allowed. This is illustrated by the dilaton calculation above, where a similar argument lead to the $O(\varepsilon)$ contribution, Φ_1, vanishing. In this subsection we calculate the leading $O(\varepsilon^2)$ corrections to the metric.

Before we proceed it is worth discussing the boundary conditions which must be imposed on the metric. As was discussed in the previous subsection we consider a dilaton source, Φ_0, which approaches a constant value in the far past, $t \to -\infty$. The corrections to the metric that arise from such a source should also vanish in the far past. Thus we see that as $t \to -\infty$ the metric should approach that of AdS_5 space-time. Also the solutions we are interested in correspond to the gauge theory living on a time independent $S^3 \times R$ space-time in the presence of a time dependent Yang Mills coupling (dilaton). This means the leading behaviour of the metric for large r should be that of AdS_5 space. Changing this behaviour corresponds to turning on a non-normalisable component of the metric and is dual to changing the metric of the space-time on which the gauge theory lives.

We expect that these boundary conditions, which specify both the behaviour as $t \to -\infty$ and as $r \to \infty$ should lead to a unique solution to the super gravity equations. The former determine the normalisable modes and the latter the non-normalisable modes. This is dual to the fact that in the gauge theory the response should be uniquely determined once the time dependent Lagrangian is known (this corresponds to the fixing the non-normalisable modes) and the state of the system is known in the far past(this corresponds to fixing the normalisable modes).

Since Φ_0 is S^3 symmetric, we can consistently assume that the corrections to the metric will also preserve the S^3 symmetry. It turns that to $O(\varepsilon^2)$ we can choose the metric to beof the form

$$ds^2 = -e^{2A(t,r)}dt^2 + e^{2B(t,r)}dr^2 + r^2 d\Omega^2. \tag{3.14}$$

Then the solution to the supergravity equations with the boundary conditions detailed above is

$$e^{2A} = 1 + r^2 - \frac{1}{4}\dot{\Phi}_0^2 + \frac{1}{12}\dot{\Phi}_0^2 \frac{\ln(1+r^2)}{r^2}. \tag{3.15}$$

and

$$A = -B + \frac{1}{12}\dot{\Phi}_0^2[-\frac{1}{1+r^2}], \tag{3.16}$$

The solution above is regular and has no horizon. It has these properties due to the slowly varying nature of the boundary dilaton. The dual field theory in this case is in a non-dissipative phase. Once the dilaton begins to change sufficiently rapidly with time we expect that a black hole is formed, corresponding to the formation of a strongly dissipative phase in the dual field theory. In [18] the effect of a *small amplitude* time dependent dilaton with arbitrary time dependence was studied. Indeed it was found that when the time variation is fast enough there are no regular horizon-free solutions and a black hole is formed.

Finally, the analysis of this section holds when e^Φ is large enough to ensure applicability of supergravity. The fact that a black hole is not formed in this regime does not preclude formation of black holes from stringy effects when e^Φ becomes small enough. In fact we will argue in later sections that the latter is a distinct possibility.

An important feature of the lowest order calculation of this section is that the perturbations of the dilaton and the metric are essentially linear and do not couple to each other. To this order, the dilaton perturbation is simply a solution of the linear d'Alembertian equation in AdS_5. Similarly the metric perturbations also satisfy the linearized equations of motion in AdS, albeit in the presence of a source provided by the energy momentum tensor of the dilaton. This is a feature present only in the leading order calculation. As explained above, this arises because of the smallness of the parameter ε. We will use this feature to compare leading order supergravity results with gauge theory calculations in a later section.

Using usual hoplographic RG techniques the stress tensor for this solution turns out to be

$$< T^\mu_\nu >= r_0^4 T^\mu_\nu \tag{3.17}$$

Carrying out the calculation gives a finite answer,

$$\begin{aligned} < T^t_t > &= \frac{N^2}{4\pi^2}[-\frac{3}{8} - \frac{\dot{\Phi}_0^2}{16}] \\ < T^\theta_\theta >=< T^\psi_\psi >=< T^\phi_\phi > &= \frac{N^2}{4\pi^2}[\frac{1}{8} - \frac{\dot{\Phi}_0^2}{16}] \end{aligned} \tag{3.18}$$

We remind the reader that in our conventions the radius of the S^3 on which the boundary gauge theory lives has been set equal to unity. The first term on the right hand side of (3.18) arises due to the Casimir effect. The second term is the additional contribution due to the varying Yang Mills coupling.

From eq.(3.18) the total energy in the boundary theory can be calculated. We get,

$$E = - < T^t_t > V_{S^3} = \frac{3N^2}{16} + \frac{N^2 \dot{\Phi}_0^2}{32}. \tag{3.19}$$

where $V_{S^3} = 2\pi^2$ is the volume of a unit three-sphere. Note that the varying dilaton gives rise to a positive contribution to the mass, as one would expect. Moreover this additional contribution vanishes when the $\dot{\Phi}$ vanishes. In particular for a dilaton profile which in the far future, as $t \to \infty$, again approaches a constant value (which could be different from the starting value it had at $t \to -\infty$) the net energy produced due to the varying dilaton vanishes.

Finally the expectation value of the operator dual to the dilaton is

$$< \hat{\mathscr{O}}_{l=0} >= -\frac{N^2}{16} \ddot{\Phi}_0 \tag{3.20}$$

3.2 Quantum Adiabatic Expansion

It may appear natural to think that the gauge theory analog of the derivative expansion in supergarvity is the usual adiabatic approximation in supergravity.

More precisely, consider a time dependent Hamiltonian $H(\zeta(t))$, where $\zeta(t)$ is the time varying parameter. Now consider the one parameter family of time *independent* Hamiltonians given by $H(\zeta)$. To make our notation clear, a different value of ζ corresponds to a different Hamiltonian in this family, but each Hamiltonian is time independent. Let $|\phi_m(\zeta) >$ be a complete set of eigenstates of the Hamiltonian $H(\zeta)$ satisfying,

$$H(\zeta)|\phi_m(\zeta) >= E_m(\zeta)|\phi_m(\zeta) >, \tag{3.21}$$

in particular let the ground state of $H(\zeta)$ be given by $|\phi_0(\zeta) >$. We take $|\phi_m(\zeta) >$ to have unit norm. Then the adiabatic theorem states that if $\zeta \to \zeta_0$ in the far past, and we start with the state $|\phi_0 >$ which is the ground state of $H(\zeta_0)$ in the far past, the state at any time t is well approximated by,

$$|\psi^0(t) > \simeq |\phi_0(\zeta) > e^{-i\int_{-\infty}^t E_0(\zeta)dt}. \tag{3.22}$$

Here $|\phi_0(\zeta) >$ is the ground state of the time independent Hamiltonian corresponding to the value $\zeta = \zeta(t)$. Similarly in the phase factor $E_0(\zeta)$ is the value of the ground state energy for $\zeta = \zeta(t)$.

Corrections can be calculated by expanding the state at time t in a basis of energy eigenstates at the instantaneous value of the parameter ζ. The first corrections take the form,

$$|\psi^1(t) >= \sum_{n \neq 0} a_n(t)|\phi_n(\zeta) > e^{-i\int_{-\infty}^t E_n dt} \tag{3.23}$$

where the coefficient $a_n(t)$ is,

$$a_n(t) = -\int_{-\infty}^t dt' \frac{< \phi_n(\zeta)|\frac{\partial H}{\partial \zeta}|\phi_0(\zeta) >}{E_0 - E_n} \dot{\zeta}\, e^{-i\int_{-\infty}^{t'}(E_0 - E_n)dt'} \tag{3.24}$$

In the formula above on the rhs $|\phi_n(\zeta) >, \frac{\partial H}{\partial \zeta}, E_n(\zeta)$, are all functions of time, through the time dependence of ζ.

For the adiabatic approximation to be good the first corrections must be small. To ensure this we impose the condition,

$$| < \phi_n | \frac{\partial H}{\partial \zeta} | \phi_0 > \dot{\zeta} | \ll (E_1 - E_0)^2 \tag{3.25}$$

where $(E_1 - E_0)$ is the energy gap between the ground state and the first excited state and $|\phi_n >$ is any excited state. (This would then imply that the lhs in eq.(3.25) is smaller than $(E_n - E_0)^2$ for all n.) This condition is imposed for all time for the adiabatic approximation to be valid [1] .

In our case the role of the parameter ζ is played by the dilaton Φ_0(with the gauge coupling $g_{YM}^2 = e^{\Phi_0}$). Thus eq.(3.25) takes the form,

$$| < \phi_n | \frac{\partial H}{\partial \Phi_0} | \phi_0 > \dot{\Phi}_0 | \ll (E_1 - E_0)^2. \tag{3.26}$$

Now, $\frac{\partial H}{\partial \Phi_0}$ is, up to a sign, exactly the operator $\hat{\mathcal{O}}_{l=0}$ which is dual to the modes of the dilaton which are spherically symmetric on the S^3. Therefore eq.(3.26) becomes

$$| < \phi_n | \hat{\mathcal{O}}_{l=0} | \phi_0 > \dot{\Phi}_0 | \ll (E_1 - E_0)^2. \tag{3.27}$$

We have argued above that the rhs is of order unity in our conventions due to the existence of a robust gap. On the lhs, $\dot{\Phi}_0 \sim O(\varepsilon)$, and as we will argue below the matrix element, $| < \phi_n | \hat{\mathcal{O}}_{l=0} | \phi_0 > \sim O(N)$. Thus eq.(3.27) becomes,

$$N\varepsilon \ll 1. \tag{3.28}$$

Eq.(3.28) is the required condition then for the applicability of quantum adiabatic approximation. When this condition is met, we can continue to trust the quatum adiabatic approximation in the gauge theory even when the 'tHooft coupling becomes of order unity or smaller at intermediate times. All the conditions which are required for the validity of this approximation continue to be hold in this case. First, as was discussed above the gap of order unity continues to exist. Second, the matrix elements which enter are in fact independent of λ since they correspond to the two-point function of dilaton which is a chiral operator. Thus the system continues to be well described in the quantum adiabatic approximation so long as eq.(3.28) is met. It follows then that in the far future the state of the system to good approximation is the ground state of the $\mathcal{N} = 4$ theory. This implies that the dual description in the far future is a smooth AdS_5 geometry.

3.3 Large N Adiabatic Approximation

The supergravity solution in the previous section describes *classical solutions* rather than states which contain a small number of bulk particles. The AdS/CFT correspondence implies that bulk classical solutions corresponds to coherent states in the boundary gauge theory with a large number of particles in which operators like $\hat{\mathcal{O}}$ have nontrivial expectation values. On the other hand, states obtained by the action of a few factors of $\hat{\mathcal{O}}$ on the vaccum are few-particle states in the bulk.

[1] The actual condition is that the corrections to $|\psi^0 >$ must be small. This means that at first order $< \psi^1 | \psi^1 >$ should be small. When eq.(3.25) is met $|a_n|$ is small, but in some cases that might not be enough and the requirement that the sum $\sum |a_n|^2$ is small imposes extra restrictions. There could also be additional conditions which arise at second order etc.

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The standard quantum adiabatic approximation described above attempts to determine the wave function in a basis formed out of such single particle states and does not apply to the supergravity solution.

We, therefore, need to formulate an adiabatic approximation in terms of coherent states of gauge invariant operators in the boundary theory to try and understand the supergravity solutions in a dual description. As is well known, these coherent states become classical in a smooth fashion in the $N \to \infty$ limit. (See e.g. [17]). Consider a complete (usually overcomplete) set of gauge invariant operators in the Schrodinger picture, $\hat{\mathscr{O}}^I$. A general coherent state is of the form

$$|\Psi(t)>= \exp\left[i\chi(t) + \sum_I \lambda^I(t)\hat{\mathscr{O}}^I_{(+)}\right]|0>_A . \tag{3.29}$$

Here $\hat{\mathscr{O}}^I_{(+)}$ denotes the creation part of the operator and $|0>_A$ denotes the adiabatic vacuum corresponding to some instantaneous value of the dilation Φ_0,

$$H[\Phi_0]|0>_A = E_{\Phi_0}|0>_A \tag{3.30}$$

with the ground state energy E_{Φ_0}.

The algebra of operators $\hat{\mathscr{O}}^I$, together with the Schrodinger equation then leads to a differential equation which determines the time evolution of the coherent state parameters $\lambda^I(t)$ in terms of the time dependent source $\Phi_0(t)$. The idea is then to solve this equation in an expansion in time derivatives of $\Phi_0(t)$. This is the coherent state adiabatic approximation we are seeking.

In general it is almost impossible to implement this program practically, since the operators $\hat{\mathscr{O}}^I$ have a non-trivial operator algebra which mixes all of them. The coherent state (3.29) is in the co-adjoint orbit of this algebra [17]. The resulting theory of fields conjugate to these operators would be in fact the full interacting string field theory in the bulk. In our case, however, the situation drastically simplifies for large 't Hooft coupling at the lowest order of an expansion in $\dot{\Phi}_0$. This is because these various operators decouple and their algebra essentially reduces to free oscillator algebras.

We have already found this decoupling in our supergravity calculation. The departure of the solution from $AdS_5 \times S^5$ is due to the time-dependence of the boundary value of the dilaton, and are small when the time variations are small, controlled by the parameter ε. To lowest order in ε (which is $O(\varepsilon^2)$) the deformation of the bulk dilaton in fact satisfied a linear equation in the AdS_5 background in the presence of a source provided by the boundary value $\Phi_0(t)$. This equation does not involve the deformation of the metric. Similarly, the equation for metric deformation does not involve the dilaton deformation to lowest order.

This allows us to treat each supergravity field and its dual operator separately. With this understanding we will now consider the coherent state (3.29) with only the operator dual to the dilaton, $\hat{\mathscr{O}}$. Since our source is spherically symmetric and higher point functions of the operators are not important in this lowest order calculation, we can restrict this operator to its spherically symmetric part.

The linearised approximation in the gravity theory means that only the two point function is non-trivial and all connected higher point functions vanish. The non-linear terms correspond to nontrivial higher order correlations. In this approximation the gauge theory simplifies a great deal. Each gauge invariant operator- which is dual to a bulk mode- gives rise to a tower of harmonic oscillators. The response of the gauge theory can be understood from the response of these oscillators.

In fact in the quadratic approximation the only oscillators which are excited are those which couple directly to the dilaton and so we only have to discuss their dynamics. The dilaton excitations we consider are S^3 symmetric and correspondingly the only modes of $\hat{\mathscr{O}}$ which are excited are S^3 symmetric. Here we denote these by $\hat{\mathscr{O}}_{l=0}$.

In the Heisenberg picture $\hat{\mathscr{O}}_{l=0}$ can be expanded in terms of time dependent modes, this is dual to the fact that the S^3 symmetric dilaton can be expanded in terms of modes with different radial and related time dependence in the bulk. One finds that only even integer frequencies appear in the time dependence giving,

$$\hat{\mathscr{O}}_{l=0} = N\sum_{n=1}^{\infty} F(2n)[A_{2n}e^{-i2nt} + A_{2n}^\dagger e^{i2nt}]. \tag{3.31}$$

Here $A_{2n}, A_{2n}^\dagger$ are canonically normalised creation and destruction operators satisfying the relations,

$$[A_m, A_n] = [A_m^\dagger, A_n^\dagger] = 0 \qquad [A_m, A_n^\dagger] = \delta_{m,n}. \tag{3.32}$$

Their commutators with the gauge theory hamiltonian are

$$[H, A_{2n}^\dagger] = (2n)A_{2n}^\dagger \qquad\qquad [H, A_{2n}] = -(2n)A_{2n} \tag{3.33}$$

The normalization factor $F(2n)$ may be computed by comparing with the standard the 2-point function. The result is

$$|F(2n)|^2 = \frac{A\pi^4}{3}\, n^2(n^2-1) \tag{3.34}$$

for $n \geq 2$. $F(0)$ and $F(2)$ vanish, so this means that the sum in eq.(3.31) receives its first contribution at $n = 2$. It also means that the lowest energy state which can be created by acting with $\hat{\mathscr{O}}_{l=0}$ on the vacuum has energy equal to 4. This is what we expect on general grounds, since the energies of states created by an operator with conformal dimension Δ are given by

$$\omega(n,l) = \Delta + 2n + l(l+2) \qquad n = 0, 1, 2\cdots \tag{3.35}$$

The constant A in eq.(3.34) is the normalization of the 2-point function which may be determined e.g. from a bulk calculation. Before proceeding let us also note that $F(2n)$ grows like $F(2n) \sim n^2$, eq.(3.34), for large mode number n. This enhances the coupling of the higher frequency modes to the dilaton and will be important in our discussion of renormalisation below.

From now onwards we will find it convenient to work in the Schrodinger representation, in which operators are time independent. The operator $\hat{\mathscr{O}}_{l=0}$ in this representation is given by,

$$\hat{\mathscr{O}}_{l=0} = N\sum_n F(2n)[A_{2n} + A_{2n}^\dagger]. \tag{3.36}$$

From eq.(3.33) it follows that the Hamiltonian for $A_{2n}, A_{2n}^\dagger$ modes can be written as,

$$H = \sum_n 2nA_{2n}^\dagger A_{2n}. \tag{3.37}$$

Note this Hamiltonian measures the energy above that of the ground state.

The operators, $A^\dagger_{2n}, A_{2n}$ create and destroy a single quantum of excitation when acting on the vaccum of the $\mathcal{N}=4$ theory with the instantaneous value of $g^2_{YM} = e^{\Phi_0}$.

The time dependence of the Hamiltonian due to the varying dilaton can be expressed as follows,

$$\frac{\partial H}{\partial t} = \frac{\partial H}{\partial \Phi}\dot{\Phi}_0 = -\hat{\mathcal{O}}_{l=0}\dot{\Phi}_0 \tag{3.38}$$

leading to,

$$\frac{\partial H}{\partial t} = -\hat{\mathcal{O}}_{l=0}\dot{\Phi}_0 = -N\sum_n F(2n)[A_{2n} + A^\dagger_{2n}]\dot{\Phi}_0, \tag{3.39}$$

where we have used eq.(3.36). It is useful to write this as

$$\frac{\partial H}{\partial t} = -N\sum F(2n)\sqrt{4n}\dot{\Phi}_0[\frac{A_{2n}+A^\dagger_{2n}}{\sqrt{4n}}], \tag{3.40}$$

So we see that the gauge theory, in the quadratic approximation maps to a tower of oscillators, with frequencies, $\omega_n = 2n$.

Consider therefore the coherent state

$$|\psi> = \hat{N}(t) e^{(\sum_n \lambda_n(t) A^\dagger_{2n})}|\phi_0> . \tag{3.41}$$

Here $|\phi_0>$ is the adiabatic vacuum, which in is the ground state of the $\mathcal{N}=4$ theory with coupling $g^2_{YM} = e^{\Phi_0}$. $\hat{N}(t)$ is a normalisation constant. Using the Schrodinger equation for this state and using the properties of the oscillators one can now solve for the coherent state parameter $\lambda_n(t)$ with the condition that it vanishes in the far past. The result is

$$\begin{aligned}
\lambda_n(t) &= -N\frac{F(2n)\, e^{-2int}}{2n}\int_{-\infty}^t \dot{\Phi}_0(t')\, e^{2int'}\, dt' \\
&= N\frac{F(2n)}{2n}\left[\frac{\dot{\Phi}_0}{(2in)} - \frac{e^{-2int}}{(2in)}\int_{-\infty}^t \ddot{\Phi}_0(t')e^{2int'}\right] \\
&= N\frac{F(2n)}{2n}\left[\frac{\dot{\Phi}_0}{(2in)} + \frac{\ddot{\Phi}_0}{4n^2} + \cdots\right]
\end{aligned} \tag{3.42}$$

The second and the third lines in (3.42) are obtained by performing successive integration by parts. This is an expansion in time derivatives - the adiabatic expansion which is the analog of the derivative expansion of the solution of Einstein's equations in the bulk.

The condition that the source is varying slowly is

$$|\frac{\ddot{\Phi}_0}{n\dot{\Phi}_0}| \ll 1 \qquad \forall n. \tag{3.43}$$

It is clearly sufficient to satisfy this condition for $n=1$,

$$|\frac{\ddot{\Phi}_0}{\dot{\Phi}_0}| \sim \varepsilon \ll 1. \tag{3.44}$$

This condition is met for the dilaton profile we have under consideration. When this condition is true λ_n can be evaluated by keeping the first term in eq.(3.42). The condition that the state is classical, is that $\lambda_n \gg 1$, this gives

$$|NF(2n)\sqrt{4n}\dot{\Phi}_0| \gg (2n)^{5/2}. \tag{3.45}$$

Noting from eq.(3.34) that $F(2n) \sim n^2$ for large n we see that the factors of n cancell out on both sides, leading to the conclusion that when,

$$|N\dot{\Phi}_0| \sim N\varepsilon \gg 1 \tag{3.46}$$

all the oscillators are in a classical state.

The summary is that when the two conditions,

$$\varepsilon \ll 1, N\varepsilon \gg 1 \tag{3.47}$$

are both valid, the gauge theory is described to leading order in ε as a system of harmonic oscillators. The oscillators which couple to the dilaton are excited by it and are in a classical state. This description can be used to calculate the resulting expectation value of operators. To leading order in ε we get ,

$$< \frac{A_{2n} + A_{2n}^\dagger}{\sqrt{4n}} >= -N\frac{F(2n)\sqrt{4n}}{(2n)^4}\ddot{\Phi}_0. \tag{3.48}$$

Substituting in eq.(3.36) next gives,

$$< \hat{\mathscr{O}}_{l=0} >= -CN^2\ddot{\Phi}_0 \tag{3.49}$$

where C is

$$C = \sum \frac{F(2n)^2}{4n^3}. \tag{3.50}$$

The functional dependence on Φ_0 and N in eq.(3.48) agrees with what we found in the supergravity calculation, eq.(3.20). The constant of proportionality C is in fact quadratically divergent. This follows from noting that for large n, $F(2n) \sim n^2$.

A little thought tells us that the divergence should in fact have been expected. The supergravity calculation also had a divergence and the finite answer in eq.(3.20) was obtained only after regulating this divergence and renormalising. Therefore it is only to be expected that a similar divergence will also appear in the description in terms of the oscillators. In the subsection which follows we will discuss the issue of renormalisation in more detail. The bottom line is that counter terms can be chosen so that the coefficient in eq.(3.20) agrees with that in the supergravity calculation.

It is also important to discuss how the energy behaves. The energy above the ground state is easily seen to be

$$< E > -E_{gnd} = \frac{1}{2}CN^2\dot{\Phi}_0^2 \tag{3.51}$$

We note that the functional dependence on $\dot{\Phi}_0, N$ match with those obtained in the supergravity calculations, eq.(3.19). The constant of proportionality which is obtained by summing over the oscillator modes in the case of the energy is the same as C defined above, eq.(3.50). It is also therefore quadratically divergent.

The fact that the two constants of proportionality in eq.(3.51) and eq.(3.49) are the same follows on general grounds. Noether's argument in the presence of the time dependence means

$$< \frac{dE}{dt} >= -\dot{\Phi}_0 < \hat{\mathscr{O}}_{l=0} > \tag{3.52}$$

leading to the equality of the two constants. Earlier we had also seen that the supergravity calculation satisfies this relation. It follows from these observations that if after renormalisation the answer for $< \hat{\mathscr{O}}_{l=0} >$ agrees between the supergravity theory and the oscillator description developed here, then the expectation value for E will also agree in the two cases.

Here we have analysed the gauge theory to leading order in ε. Going to higher orders introduces anharmonic couplings between the different oscillators. These couplings arise because of connected three-point and higher point correlations in the gauge theory. The three point function for example is suppressed by $1/N$, the four point function by $1/N^2$ and so on. For computations in the ground state these would therefore be suppressed in the large N limit. However as we have seen here the time dependence results in a coherent state which contains $O(N\varepsilon)^2$ quanta being produced. The 3- pt function in such a state is suppressed by $O(\varepsilon)$ and not by $O(1/N)$. Since $\varepsilon \ll 1$, this is still enough though to justify our neglect of the cubic terms to leading order in ε. Similarly the effect of 4-pt correlators in the coherent state are suppressed by $O(\varepsilon)^2$ etc. This is in agreement with the supergravity calculation, where the cubic terms in the equations of motion are suppressed by $O(\varepsilon)$ etc.

To go to higher orders in ε using the oscillator description the effect of the anharmonic couplings induced by the higher order correlations would have to be introduced. In addition one would have to keep the contributions from the quadratic approximation to the required order in ε. As long as the 'tHooft coupling stays big for all times and the supergravity approximation is valid, there is no reason to believe that these effects will be significant and the behaviour of the system should be well described by the leading harmonic oscillator description, in agreement with what we saw in supergravity. When the 'tHooft coupling begins to get small though the anharmonic couplings could potentially significantly change the behaviour of the system.

3.4 The regime of large curvatures

So far we have considered what happens in the parametric regime, eq.(3.47), when the 'tHooft coupling stays big all times. In this case the supergravity description is always valid. We saw above that the gauge theory can be described in this regime in terms of approximately decoupled classical harmonic oscillators and this reproduces the supergravity results.

Now let us consider what happens when the dilaton takes a larger excursion so that the 'tHooft coupling at intermediate times becomes of order unity or even smaller. Some of the resulting discussion is already contained in the introduction above.

A natural expectation is that description in terms of classical adiabatic system of weakly coupled oscillators should continue to apply even when the 'tHooft coupling becomes small. There are several reasons to believe this. First, anharmonic terms continue to be of order ε and thus are small. The leading anharmonic terms arise from three -point correlations, $< \hat{O}_1\hat{O}_2\hat{O}_3 >$. In the vaccum these go like $1/N$. In the coherent state produced by the time dependence these go like ε. The enhancement by $N\varepsilon$ arises because the coherent state contains $O((N\varepsilon)^2)$ quanta, so that the probability goes as $(N\varepsilon)^2/N^2 \sim \varepsilon^2$. Four-point functions give rise to terms going like $O(\varepsilon^2)$ and so on, these are even smaller. In the absence of anharmonic terms the theory should reduce to a system of oscillators. Second, the existence of a gap of order $1/R$ means that for each oscillator the time dependence is slow compared to its frequency. Therefore the system continues to be very far from resonance and should evolve adiabatically. Finally, in the parametric regime, eq.(3.47) the analysis of the previous subsections should then apply leading to the conclusion that an $O(N\varepsilon) \gg 1$ quanta are produced making the coherent state a good classical state.

If this expectation is borne out the system should settle back into the ground state of the final $\mathcal{N} = 4$ theory in the far future and should have a good description in terms of smooth AdS space then.

However, as discussed in the introduction, there are reasons to worry that this expectation is not borne out. New features could enter the dynamics when the 'tHooft coupling becomes small at intermediate times, and these could change the qualitative behaviour of the system. These new features have to do with the fact that string modes can start getting excited in the bulk when the curvature becomes of order the string scale. These modes correspond to non-chiral operators in the gauge theory and the corresponding oscillators have a time dependent frequency. When the 'tHooft coupling is big these frequencies are much bigger than those of the supergravity modes and as a result the string mode oscillators are not excited. But when the 'tHooft coupling becomes of order unity some of the frequencies of these string modes become of order the supergravity modes and hence these oscillators can begin to get excited [2]. In fact the string modes are many more in number than the supergravity modes, since there are an order unity worth of chiral operators in the gauge theory and an $O(N^2)$ worth of non-chiral ones.

The worry then is that if a significant fraction of these string oscillators get excited the correct picture which could describe the ensuing dynamics is one of thermalisation rather than classical adiabatic evolution. In this case the energy pumped into the system initially would get equipartitioned among all the different degrees of freedom. Subsequent evolution would then be dissipative, and the energy would increases in a monotonic manner, as it does for a large black hole.

Due to the dissipative behaviour the energy which is initially pumped in would not be recovered in the future. Rather one would expect that when the 'thooft coupling becomes large again, the energy, which is of order $N^2\varepsilon^2$ remains in the system. The gravity description of the resulting thermalized state depends on the value of ε relative to $\lambda \equiv g_{YM}^2 N$ and N. In this late time regime of large 't Hooft coupling, the various possibilities can be figured out from entropic considerations in supergravity (see e.g. section 3.4 of [16]). The result in our case is the following. For $\varepsilon \ll (g_{YM}^2 N)^{5/4}/N$ a gas of supergravity modes is favored. For $(g_{YM}^2 N)^{5/4}/N < \varepsilon \ll (g_{YM}^2 N)^{-7/8}$ one would have a gas of massive string modes. For $(g_{YM}^2 N)^{-7/8} < \varepsilon \ll 1$ one gets a small black hole, i.e. a black hole whose size is much smaller than R_{AdS}. A big black hole requires $O(N^2)$ energy which is parametrically much larger. Thus, the strongest departure from AdS space-time in the far future would be presence of small black holes. Such black holes would eventually evaporate by emitting Hawking radiation. However this takes an $O(N^2 R_{AdS})$ amount of time which is much longer than the time scale $O(R_{AdS}/\varepsilon)$ on which the 'tHooft coupling evolves. As a result for a long

[2] The primary reason for them getting excited are the anharmonic terms which couple them to the modes dual to the dilaton.

time after the 'tHooft coupling has become big again the gravity description would be that of a small black hole in AdS space.

An important complication in deciding between these two possibilities is that the rate of time variation is ε which is also the strength of the anharmonic couplings between the supergravity oscillators and string oscillators. If the rate of time variation could have been made much smaller, thermodynamics would become a good guide for how the system evolves. In the microcanonical ensemble, which is the correct one to use for our purpose, with energy $N^2\varepsilon^2$ the entropically dominant configurations are as discussed in the previous paragraph, and this would suggest that dissipation would indeed set in. However, as emphasised above this conclusion is far from obvious here since the time variation is parametrically identical to the strength of the anharmonic couplings.

In fact we know that the guidance from thermodynamics is misleading in the supergravity regime, where the 'tHooft coupling stays large for all times. In this case we have explicitly found the solution in §2. It does not contain a black hole. Moreover, it does not suffer from any tachyonic instability - since it is a small correction from AdS space which does not have any tachyonic instability [3]. The only way a black hole could form is due to a tunneling process but this would be highly suppressed in the supergravity regime.

One reason for this suppression is that the energy in the supergravity solution discussed in §2 is carried by supergravity quanta which have a size of order R_{AdS}. This energy would have to be concentrated in much smaller region of order the small black hole's horizon to form the black hole and this is difficult to do. In contrast, away from the supergravity regime this could happen more easily. When the 'tHooft coupling becomes small at intermediate times, strings become large and floppy, of order R_{AdS}, at intermediate times. If a significant fraction of the energy gets transferred to these strings at intermediate times it could find itself concentrated within a small black hole horizon once the 'tHooft coupling becomes large again.

In summary we do not have a clean conclusion for the future fate of the system in the parametric regime, eq.(3.47). Note however that in both possibilities discussed above most of space-time in the far future is smooth AdS space, with the possible presence of a small black hole. Hopefully, the framework developed here will be useful to think about this issue further.

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