PROCEEDINGS OF SCIENCE

New approximation methods in General Relativity

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We show how approximate solutions of the two-body problem in General Relativity, and the approximate solutions of Einstein's equations *in vacuo* can be constructed using small deformations of geodesics and of Einstein space-times embedded into a pseudo-Euclidean flat space of higher dimension. The method consists in using expansions of equations around a given simple solution (a circular orbit in the case of geodesics, and Minkowskian or Schwarzschild space in the case of Einstein's equations) in a series of powers of small deformation parameter, and then solving by iteration the corresponding linear systems of differential equations.

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1. Introduction

In these lectures we present an alternative method of calculus of trajectories and motions of test particles in the vicinity of a massive body, and a similar method for obtaining approximate solutions of Einstein's equations *in vacuo*. Both methods are based on successive approximations of solutions which are presented as infinite series in powers of some small parameter. Although we are still far from the results obtained by the usual methods of post-Newtonian approximations, the present approach is very well suited for recurrent numerical calculations.

Facing a highly non-linear and complex theory like Einstein's General Relativity one is forced to use approximation techniques in order to find realistic solutions. There are usually two alternative ways to treat the problem:

a) when a linearized version of theory exists, to find an **exact solution** of the simplified theory, and then add small perturbations transforming it into something that is closer to the exact theory solution;

b) when an exact (usually very symmetric) solution of the exact theory is at hand, try to deform it slightly and obtain, by successive approximations, a more general solution of the exact theory. This situation is be represented by the following diagram:

$$
\begin{array}{rcl}
\text{Full (exact) theory} & \longrightarrow & \text{Simplified (linearized) theory} \\
\downarrow & & \downarrow \\
\text{Approximate solutions} & \ldots\ldots & \text{Exact solutions} \\
\text{of exact theory} & & \text{of approximate theory}
\end{array}
$$

In General Relativity this diagram, which we would like to become commutative, represents two different approaches to the solution search in General Relativity: both cases are represented in the literature. The classical way to obtain the perihelion advance, as treated by Einstein, consists in solving in quadratures the geodesic equation describing the motion of a test particle in a Schwarzschild background, then in taking an approximate integral by expanding the integrand in powers of MG/r, supposed very small in realistic situations, and keeping only the linear term of the expansion. This approach is represented by the left side of the diagram.

The relativistic two-body problem including gravitational radiation was treated in an opposite manner: first, the exact solution of the equation of motion was found in the Newtonian theory, which can be considered as a limit of General Relativity when $c \to \infty$; then, starting from Minkowskian space-time and the Newtonian solution, corrections were added simultaneously to the trajectories and to the metric, with small parameter being v/c, its square also supposedly proportional to MG/r. This is represented on the right side of the diagram below:

$$
\begin{array}{rcl}
\text{General Relativity} & \xrightarrow{c\to\infty} & \text{Newtonian theory} \\
\downarrow & & \downarrow \\
\text{Special exact solution} & & \text{Post-Newtonian} \\
\text{(circular orbit,} & & \text{approximations} \sim v^2/c^2 \\
\text{Schwarzschild background)} & & \\
\downarrow & & \downarrow \\
\text{Successive deformations} & \ldots\ldots & \text{Post–post-Newtonian} \\
\text{"epicycles"} & & \text{approximations} \sim v^4/c^4 \\
\varepsilon \sim \text{eccentricity} & &
\end{array}
$$

In what concerns the two-body problem and the description of motion of bodies in the vicinity of a massive central body, the first approach has been successfully exploited by Ll. Bel, N. Deruelle, Th. Damour, G. Schaeffer, L. Blanchet, and others ([2], [3], [4], [5])

The second approach has been tested on the two-body problem in General Relativity quite recently, by one of the authors of this paper (RK), in collaboration with with A. Balakin, J.-W. van Holten, R. Colistete Jr. and C. Leygnac ([6], [7], [8], [9])..

The problem of motion of planets in General Relativity, considered as test particles moving along geodesic lines in the metric of Schwarzschild's solution, has been solved in an approximate way by Einstein ([1]) who found that the perihelion advance during one revolution is given by the formula

$$\Delta\phi = \frac{6\pi GM}{a(1-e^2)} \tag{1.1}$$

where G is Newton's gravitational constant, M the mass of the central body, a the greater half-axis of planet's orbit and e its eccentricity.

This formula is deduced from the exact solution of the General Relativistic problem of motion of a test particle in the field of Schwarzschild metric, which leads to the expression of the angular variable φ as an elliptic integral, which is then evaluated after expansion of the integrand in terms of powers of the ratio $\frac{GM}{r}$ when it can be supposed to be very small.

It has been successfully confronted with observation, giving excellent fits not only for the orbits with small eccentricities (e.g., one of the highest values of e displayed by the orbit of Mercury, is $e = 0.2056$), but also in the case when e is very high, as for the asteroid Icarus ($e = 0.827$), and represents one of the best confirmations of Einstein's theory of gravitation. In the case of small eccentricities it can be developed into a power series:

$$\Delta\phi = \frac{6\pi GM}{a}\,(1+e^2+e^4+e^6+\ldots). \tag{1.2}$$

One can note at this point that even for the case of planet Mercury, the series truncated at the second term, i.e., taking into account only the factor $(1+e^2)$ will lead to the result that differs only by 0.18% from the result predicted by relation (1.1), which is below the actual error bar.

In what follows, we show how one can obtain the same results without taking integrals, but just by successive approximations around a circular orbit with constant angular velocity, leading to an iterative process of solving ordinary linear differential systems. The small parameter here will be not MG/r, which can take any value, but the excentricity e, controlling the maximal deviation from the initial circular orbit. It is amazing that with this method the effect of the perihelion advance is obtained already at the first order in the expansion in powers of e.

2. Geodesic deviations

In order to compare two close geodesics, one needs to see how a parallelly transported vector is modified when transported along the one or another neighbor geodesic. In fact, if an infinitesimal vector field is defined along a given geodesic curve, it may serve to produce another curve, infinitesimally close to the first one. A natural question can be asked then, is the new curve also a geodesic ? The answer is positive if the infinitesimal deformation vector satisfies the so-called geodesic deviation equation.

Given a (pseudo)-Riemannian manifold V_4 with the line element defined by metric tensor $g_{\mu\nu}(x^\lambda)$,

$$ds^2 = g_{\mu\nu}(x^\lambda)\, dx^\mu dx^\nu, \tag{2.1}$$

a smooth curve $x^\lambda(s)$ parametrized with its own length parameter (or proper time) s is a *geodesic* if its tangent vector $u^\mu = (dx^\mu/ds)$ satisfies the equation: $u^\mu = (dx^\mu/ds)$

$$u^\lambda \nabla_\lambda u^\mu = 0 \quad \Leftrightarrow \quad \frac{Du^\mu}{Ds} = \frac{du^\mu}{ds} + \Gamma^\mu_{\lambda\rho}\, u^\lambda\, u^\rho = 0. \tag{2.2}$$

where $\Gamma^\mu_{\rho\lambda}$ denote the Christoffel connection coefficients of the metric $g_{\mu\nu}$. Equivalently, the geodesic equation can be written in its more standard form:

$$\frac{d^2 x^\mu}{ds^2} + \Gamma^\mu_{\lambda\rho} \frac{dx^\lambda}{ds} \frac{dx^\rho}{ds} = 0. \tag{2.3}$$

Suppose now that a small deformation is produced along this geodesic line:

$$x^\mu(s) \to \tilde{x}^\mu(s) = x^\mu(s) + \delta x^\mu(s), \tag{2.4}$$

In order to check out to what extent the new curve $\tilde{x}^\mu(s)$ has conserved its geodesic character, we should veriify whether its satisfies its own geodesic equation:

$$\frac{d^2 \tilde{x}^\mu}{ds^2} + \tilde{\Gamma}^\mu_{\lambda\rho} \frac{d\tilde{x}^\lambda}{ds} \frac{d\tilde{x}^\rho}{ds} = 0. \tag{2.5}$$

Applying the Taylor expansion also to the Christoffel symbols,

$$\tilde{\Gamma}^\mu_{\lambda\rho} = \Gamma^\mu_{\lambda\rho}(\tilde{x}^\nu) = \Gamma^\mu_{\lambda\rho}(x^\nu) + \partial_\sigma \Gamma^\mu_{\lambda\rho}(x^\nu)\, \delta x^\sigma + \mathscr{O}((\delta x^\nu)^2),$$

and keeping only terms linear in δx^λ, we arrive at the following condition on the infinitesimal vector δx^λ:

$$\frac{d^2 \delta x^\mu}{ds^2} + 2\Gamma^\mu_{\lambda\rho}\, \delta x^\lambda \frac{d\,\delta x^\rho}{ds} + \partial_\sigma \Gamma^\mu_{\lambda\rho} \frac{dx^\lambda}{ds} \frac{dx^\rho}{ds}\, \delta x^\sigma = 0 + \mathscr{O}(\delta x^\nu)^2). \tag{2.6}$$

In this form of the geodesic deviation equation one easily identifies the relativistic generalizations of the Coriolis-type and centrifugal-type inertial forces, represented respectively by the second and third terms of Eq. (2.6). Although it does not look manifestly covariant, one can put forward its covariant character by replacing the ordinary second derivative by the covariant one, and adding compensating terms containing Christoffel symbols and their derivatives to obtain the well-known covariant form of the geodesic deviation equation:

$$\frac{D^2\delta x^\mu}{Ds^2} = R_{\lambda\rho\sigma}{}^{\mu}\,\frac{dx^\lambda}{ds}\,\frac{dx^\sigma}{ds}\,\delta x^\rho. \tag{2.7}$$

This first-order geodesic deviation equation is often called *the Jacobi equation*, and is manifestly covariant. Note that it represents only the *linear* approximation - one can say that if the deviation δx^μ satisfies the equation (2.7), then the curve $\tilde{x}^\mu(s) = x^\mu(s) + \delta x^\mu(s)$ close to the geodesic curve $x^\mu(s)$ is also a geodesic, but only up to the terms of quadratic and higher order in δx^μ which we deliberately neglected.

Now, one can become more ambitious and ask that the neighbor curve to the geodesic be a geodesic not only up to the linear approximation, but up to a given order in powers of the small deviation δx^μ. As in the usual differential calculus, we should expand the functions $\tilde{x}^\mu(s)$ into a Taylor series containing all the higher-order corrections, which we can stop at the desired order of approximation:

$$\tilde{x}^\mu(s) = x^\mu(s) + \delta x^\mu(s) + \frac{1}{2!}\,\delta^2 x^\mu(s) + \frac{1}{3!}\,\delta^3 x^\mu(s) + \ldots \tag{2.8}$$

It should be stressed now that although the first deviation δx^μ transforms as a vector, the second and higher-order deviations do not; for example, after a coordinate change $x^\mu \to y^{\lambda'}(x^\mu)$ we shall have

$$\delta y^{\lambda'} = \frac{\partial y^{\lambda'}}{\partial x^\mu}\,\delta x^\mu, \quad \text{but} \quad \delta^2 y^{\lambda'} = \frac{\partial y^{\lambda'}}{\partial x^\mu}\,\delta^2 x^\mu + \frac{\partial^2 y^{\lambda'}}{\partial x^\mu \partial x^\nu}\,\delta x^\mu \delta x^\nu, \tag{2.9}$$

including the terms quadratic in the first deviations. However, it is easy to introduce a covariant quantity of the same order; we shall denote it by b^μ; in order to make clear the infinitesimal character of this vectorial quantity, let us introduce the notation with small parameter ε in the following manner: let now

$$\frac{dx^\mu}{ds} = u^\mu(s), \quad \delta x^\mu(s) = \varepsilon\, n^\mu(s),$$

then it is easy to check that the vector defined as

$$\varepsilon\, b^\mu(s) = \delta^2 x^\mu(s)\,\Gamma^\mu_{\lambda\nu}\,\frac{dx^\lambda}{ds}\,\frac{dx^\nu}{ds} = \delta^2 x^\mu(s) + \Gamma^\mu_{\lambda\nu} n^\lambda n^\nu. \tag{2.10}$$

Developing the geodesic equation up to the second order in ε, after some algebra using the the Bianchi and Ricci identities for the Riemann tensor, we get the following condition for the vanishing of the second order terms:

$$\frac{D^2 b^\mu}{Ds^2} + R_{\rho\lambda\sigma}{}^{\mu} u^\lambda u^\sigma b^\rho$$
$$= [\nabla_\nu R_{\lambda\rho\sigma}{}^{\mu} - \nabla_\lambda R_{\nu\sigma\rho}{}^{\mu}] u^\lambda u^\sigma n^\rho n^\nu + 4 R_{\lambda\rho\sigma}{}^{\mu} u^\lambda n^\rho \left(\frac{Dn^\sigma}{Ds}\right). \tag{2.11}$$

Note that the second-order deviation vector $b^\mu(s)$ satisfies an inhomogeneous extension of the first-order geodesic deviation equation, with the same left-hand side as for the first deviation n^μ, but with the extra terms on the right-hand side, containing various expressions quadratic in the first deviation and its first derivatives.

The procedure can be extended to an arbitrarily high order geodesic deviations $\delta^n x^\mu(s)$, allowing us to *construct* a desired set of geodesics in the neighborhood of the reference $x_0^\mu(s)$, when the congruence of geodesics is not given a priori in closed form. Indeed, all that is needed is the set of deviation vectors $(n_0^\mu(s), b_0^\mu(s), \ldots)$ *on the reference geodesic*; these vectors are completely specified as functions of s by solving the geodesic deviation equations (2.7), (2.11) and their extensions to higher order, for given $x_0^\mu(s)$. As in the case of the first-order deviation, it is sometimes convenient to write equation (2.11) in the equivalent but non-manifest covariant form

$$\frac{d^2 b^\mu}{ds^2} + \partial_\rho \Gamma^\mu_{\lambda\sigma} u^\lambda u^\sigma b^\rho + 2\Gamma^\mu_{\lambda\sigma} u^\lambda \frac{db^\sigma}{ds}$$

$$= 4\left(\partial_\lambda \Gamma^\mu_{\sigma\rho} + \Gamma^\nu_{\sigma\rho}\Gamma^\mu_{\lambda\nu}\right)\frac{dn^\sigma}{ds}(u^\lambda n^\rho - u^\rho n^\lambda) \tag{2.12}$$

$$+\left(\Gamma^\tau_{\sigma\nu}\partial_\tau\Gamma^\mu_{\lambda\rho} + 2\Gamma^\mu_{\lambda\tau}\partial_\rho\Gamma^\tau_{\sigma\nu} - \partial_\nu\partial_\sigma\Gamma^\mu_{\lambda\rho}\right)(u^\lambda u^\rho n^\sigma n^\nu - u^\sigma u^\nu n^\lambda n^\rho).$$

Again, the above formula is valid up to the third-order terms; one can continue *ad infinitum* producing better approximations each time. The third-order deviation vector and its equation can be found in ([8]).

Now we can proceed to the simple application of these formulae to the two-body problem treated as motion of a test particle with negligible mass m when compared to the mass M of the central body.

3. Deviations from circular orbits

Let us consider the geodesic deviation equation starting with a circular orbit in the field of a spherically-symmetric massive body, i.e. in the Schwarzschild metric. The geodesic deviation analysis in General Relativity has been performed by S.L. Bazanski and P. Jaranowski ([10], [11], [12]), but only for straight geodesics; a non-geodesic circular motion in the gravitational field of a massive body was also studied by Aliev and Gal'tsov ([13]). The circular orbits and their stability have been analyzed and studied in several papers [14, 16, 17] and books, e.g. the well-known monograph by Chandrasekhar [18].

The gravitational field is described by the line-element (we have put $c = 1$; G the gravitational constant):

$$g_{\mu\nu}dx^\mu dx^\nu = -ds^2 = -B(r)dt^2 + \frac{1}{B(r)}dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right), \tag{3.1}$$

with

$$B(r) = 1 - \frac{2MG}{r}. \tag{3.2}$$

Let us recall the essential features of the solution of the geodesic equations for a test particle of mass $m << M$.

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As the spherical symmetry guarantees conservation of angular momentum, the particle orbits are always confined to an equatorial plane, which we choose to be the plane $\theta = \pi/2$.

The angular momentum J is then directed along the z-axis. There is another first integral corresponding to the conserved world-line energy. Denoting the magnitude of J per unit of mass by $\ell = J/m$, and the energy per unit of mass by ε, we have

$$\frac{d\phi}{ds} = \frac{\ell}{r^2} \quad \frac{dt}{ds} = \frac{\varepsilon}{1 - \frac{2MG}{r}}. \tag{3.3}$$

The equation for the radial coordinate r can be integrated owing to the conservation of the world-line Hamiltonian, i.e. the conservation of the absolute four-velocity:

$$\left(\frac{dr}{ds}\right)^2 = \varepsilon^2 - \left(1 - \frac{2MG}{r}\right)\left(1 + \frac{\ell^2}{r^2}\right). \tag{3.4}$$

From this a simplified expression for the radial acceleration is easily derived:

$$\frac{d^2 r}{ds^2} = -\frac{MG}{r^2} + \left(\frac{\ell^2}{r^3}\right)\left(1 - \frac{3MG}{r}\right). \tag{3.5}$$

The equation (3.4) can in principle be integrated; indeed, the orbital function $r(\phi)$ is given by an elliptic integral [19, 20]. However, to get explicitly an approximate parametric solution to the equations of motion one can also study *perturbations* of special simple orbits, namely, the circular ones.

Observe that for circular orbits $r = R =$ constant, the expressions for dr/ds, Eq. (3.4), and d^2r/ds^2, Eq. (3.5), must both vanish at all times. This produces two relations between the three dynamical quantities (R, ε, ℓ), showing that the circular orbits are characterized completely by specifying either the radial coordinate, or the energy, or the angular momentum of the planet.

In particular, the equation for null radial velocity gives

$$\varepsilon^2 = \left(1 - \frac{2MG}{R}\right)\left(1 + \frac{\ell^2}{R^2}\right). \tag{3.6}$$

Then the null radial acceleration condition (3.5) gives the well-known result

$$MGR^2 - \ell^2(R - 3MG) = 0 \quad \Rightarrow \quad R = \frac{\ell^2}{2MG}\left(1 + \sqrt{1 - \frac{12(MG)^2}{\ell^2}}\right), \tag{3.7}$$

leading to the requirement $R \geq 6MG$ for stable circular orbits to exist.

Let us write down the four differential equations that must be satisfied by the geodesic deviation 4-vector $n^\mu(s)$ close to a circular orbit. We recall that on the circular orbit of radius R (which is a geodesic in the background Schwarzschild metric) we have:

$$u^t = \frac{dt}{ds} = \frac{\varepsilon}{(1 - \frac{2MG}{R})}, \quad u^r = \frac{dr}{ds} = 0,$$
$$u^\phi = \frac{d\varphi}{ds} = \omega_0 = \frac{\ell}{R^2}, \quad u^\theta = \frac{d\theta}{ds} = 0, \tag{3.8}$$

because $r = R = \text{const.}$, $\theta = \pi/2 = \text{const.}$, so that $\sin\theta = 1$ and $\cos\theta = 0$.

The four equations are much easier to arrive at if we use the explicit form of the first-order deviation equation (2.7). We get without effort the first three equations, for the components n^θ, n^ϕ and n^t:

$$\frac{d^2 n^\theta}{ds^2} = -\,(u^\phi)^2\, n^\theta = -\frac{\ell^2}{R^4}\, n^\theta\,, \qquad \frac{d^2 n^\phi}{ds^2} = -\frac{2\ell}{R^3}\,\frac{dn^r}{ds}\,, \tag{3.9}$$

$$\frac{d^2 n^t}{ds^2} = -\frac{2MG\varepsilon}{R^2(1-\frac{2MG}{R})^2}\,\frac{dn^r}{ds}. \tag{3.10}$$

The deviation n^θ is independent of the remaining three variables n^t, n^r and n^φ. The harmonic oscillator equation (3.9) for n^θ displays the frequency which is equal to the frequency of the circular motion of the planet itself:

$$n^\theta\,(s) = n^\theta_0\,\cos(\omega_0\, s + \gamma) = n^\theta_0\,\cos\left(\frac{\ell}{R^2}s + \gamma\right). \tag{3.11}$$

This can be interpreted as the result of a change of the coordinate system, with a new z-axis slightly inclined with respect to the original one, so that the plane of the orbit does not coincide with the plane $z = 0$. In this case the deviation from the plane will be described by the above solution, i.e. a trigonometric function with the period equal to the period of the planetary motion. Being a pure coordinate effect, it allows us to eliminate the variable n^θ by choosing $n^\theta = 0$.

It takes a little more time to establish the equation for n^r, using Eq. (2.7):

$$\frac{d^2 n^r}{ds^2} + 2\Gamma^r_{\lambda\rho}\, u^\lambda\, \frac{dn^\rho}{ds} + \partial_\sigma \Gamma^r_{\lambda\rho}\, u^\lambda u^\rho n^\sigma = 0. \tag{3.12}$$

Taking into account that only the components u^t and u^ϕ of the four-velocity on the circular orbit are different from zero, and recalling that we have chosen to set $n^\theta = 0$, too, the only non-vanishing terms in the above equation are:

$$\frac{d^2 n^r}{ds^2} + 2\Gamma^r_{tt}\, u^t\, \frac{dn^t}{ds} + 2\Gamma^r_{\phi\phi}\, u^\phi\, \frac{dn^\phi}{ds} + \partial_r \Gamma^r_{tt}\, u^t u^t n^r + \partial_r \Gamma^r_{\phi\phi}\, u^\phi u^\phi n^r = 0. \tag{3.13}$$

Using the identities (3.7) and the definitions (3.8), we get

$$\frac{d^2 n^r}{ds^2} - \frac{3\ell^2}{R^4}\left(1 - \frac{2MG}{R}\right) n^r + \frac{2MG\varepsilon}{R^2}\,\frac{dn^t}{ds} - \frac{2\ell}{R}\left(1 - \frac{2MG}{R}\right)\frac{dn^\varphi}{ds} = 0. \tag{3.14}$$

The system of three remaining equations can be expressed in a matrix form:

$$\begin{pmatrix} \frac{d^2}{ds^2} & \frac{2MG\varepsilon}{R^2(1-\frac{2MG}{R})^2}\,\frac{d}{ds} & 0 \\ \frac{2M\varepsilon}{R^2}\,\frac{d}{ds} & \frac{d^2}{ds^2} - \frac{3\ell^2}{R^4}\,(1 - \frac{2MG}{R}) & -\frac{2\ell}{R}\,(1 - \frac{2MG}{R})\,\frac{d}{ds} \\ 0 & \frac{2\ell}{R^3}\,\frac{d}{ds} & \frac{d^2}{ds^2} \end{pmatrix} \begin{pmatrix} n^t \\ n^r \\ n^\varphi \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{3.15}$$

The characteristic equation of the above matrix is

$$\lambda^4 \left[\lambda^2 + \frac{\ell^2}{R^4}\left(1 - \frac{2MG}{R}\right) - \frac{4MG\varepsilon^2}{R^4(1-\frac{2MG}{R})^2}\right] = 0, \tag{3.16}$$

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which after using the identities (3.6) and (3.7) reduces to

$$\lambda^4 \left[\lambda^2 + \frac{\ell^2}{R^4} \left(1 - \frac{6MG}{R} \right) \right] = 0, \tag{3.17}$$

so that the characteristic circular frequency is

$$\omega = \frac{\ell}{R^2} \sqrt{1 - \frac{6MG}{R}} = \omega_0 \sqrt{1 - \frac{6MG}{R}}. \tag{3.18}$$

Obviously enough, the general solution contains oscillating terms $\cos(\omega s)$; however, before we analyse in detail this part of solution, let us consider the terms linear in the variable s or constants: as a matter of fact, because of the presence of first and second-order derivatives with respect to s in the matrix operator (3.15), the general solution may also contain the following vector:

$$\begin{pmatrix} (\Delta u^t)\, s + \Delta t \\ (\Delta u^r)\, s + \Delta r \\ (\Delta u^\varphi)\, s + \Delta \varphi . \end{pmatrix} \tag{3.19}$$

When inserted into the system (3.15), the solution is the following:

Δt and $\Delta \varphi$ are arbitrary;

$\Delta u^r = 0$, which means that the radial velocity remains null; and

$$\frac{3\ell^2}{R^4} \left(1 - \frac{2MG}{R} \right) \Delta r = \frac{2MG\varepsilon}{R^2} \Delta u^t - \frac{2\ell}{R} \left(1 - \frac{2MG}{R} \right) \Delta u^\varphi = 0. \tag{3.20}$$

This condition coincides with the transformation of the initial circular geodesic of radius R to a neighbor one, with radius $R + \Delta r$, with the subsequent variations Δu^t and Δu^φ added to the corresponding components of the 4-velocity in order to satisfy the condition $g_{\mu\nu} u^\mu u^\nu = 1$ in the linear approximation.

Let us choose the initial phase to have (with $n_0^r > 0$):

$$n^r(s) = -n_0^r \cos(\omega s), \tag{3.21}$$

which corresponds to the perihelion position. What remains to be done is to compare this frequency with the fundamental circular frequency $\omega_0 = \ell / R^2$ of the unperturbed circular orbital motion.

The discrepancy between the two circular frequencies ω and ω_0 is exactly what produces the perihelion advance, and its value coincides with the value obtained in the usual way (1.1) in the limit of quasi-circular orbits, i.e. when $e^2 \to 0$: we get both the correct value and the correct sign.

Let us display the complete solution for the first-order deviation vector $n^\mu(s)$ which takes into account only the non-trivial degrees of freedom:

$$n^\theta = 0,\ n^r(s) = -n_0^r \cos(\omega s),\ n^\varphi = n_0^\varphi \sin(\omega s),\ n^t = n_0^t \sin(\omega s). \tag{3.22}$$

The only independent amplitude is given by n_0^r, because we have

$$n_0^t = \frac{2MG\varepsilon}{R^2 (1 - \frac{2MG}{R})^2 \omega} n_0^r = \frac{2\sqrt{MG}}{\sqrt{R} \left(1 - \frac{2MG}{R}\right) \sqrt{1 - \frac{6M}{R}}} n_0^r, \tag{3.23}$$

$$n_0^\varphi = \frac{2\ell}{R^3 \omega} n_0^r = \frac{2\omega_0}{R\omega} n_0^r = \frac{2}{R\sqrt{1 - \frac{6MG}{R}}} n_0^r. \tag{3.24}$$

The trajectory and the law of motion are given by

$$r = R - n_0^r \cos(\omega s), \tag{3.25}$$

$$\varphi = \omega_0 s + n_0^\varphi \sin(\omega s) = \frac{\sqrt{MG}}{R^{3/2}\sqrt{1-\frac{3MG}{R}}} s + n_0^\varphi \sin(\omega s), \tag{3.26}$$

$$t = \frac{\varepsilon}{(1-\frac{2MG}{R})} s + n_0^t \sin(\omega s) = \frac{1}{\sqrt{1-\frac{3MG}{R}}} s + n_0^t \sin(\omega s), \tag{3.27}$$

where the phase in the argument of the *cosine* function was chosen so that $s = 0$ corresponds to the perihelion, and $s = \frac{\pi}{\omega}$ to the aphelion.

The coefficient n_0^r, which also fixes the values of the two remaining amplitudes, n_0^t and n_0^φ, defines the size of the actual deviation, so that the ratio $\frac{n_0^r}{R}$ becomes the dimensionless infinitesimal parameter controlling the approximation series with consecutive terms proportional to the consecutives powers of $\frac{n_0^r}{R}$.

What we see here is the approximation to an elliptic orbital movement as described by the presence of an *epicycle* (exactly like in the Ptolemean system, except for the fact that the Sun is placed in the center instead of the Earth, and that the epicycle happens to be an ellipse rather than a circle). As a matter of fact, the development into power series with respect to the eccentricity e considered as a small parameter, and truncating all the terms except the linear one, leads to the Kepler result [21],

$$r(t) = \frac{a(1-e^2)}{1+e\cos(\omega_0 t)} \simeq a\left[1 - e\cos(\omega_0 t)\right], \tag{3.28}$$

which looks almost as our formula (3.25) if we identify the eccentricity e with $\frac{n_0^r}{R}$ and the greater half-axis a with R; but there is also the additional difference, that the circular frequency of the epicycle is now slightly lower than the circular frequency of the unperturbed circular motion.

But if the circular frequency is lower, the period is slightly longer: in a linear approximation, we have

$$\omega = \sqrt{\frac{\ell^2}{R^4}\left(1 - \frac{6MG}{R}\right)}, \tag{3.29}$$

hence keeping the terms up to the third order in $\frac{MG}{R}$,

$$T \simeq T_0 \left(1 + \frac{3MG}{R} + \frac{27}{2}\frac{(MG)^2}{R^2} + \frac{135}{2}\frac{(MG)^3}{R^3} + \ldots\right). \tag{3.30}$$

Then obviously one must have $\frac{\Delta\varphi}{2\pi} = \frac{\Delta T}{T_0}$ from which we obtain the perihelion advance after one revolution

$$\Delta\varphi = \frac{6\pi MG}{R} + \frac{27\pi(MG)^2}{R^2} + \frac{135\pi(MG)^3}{R^3} + \ldots \tag{3.31}$$

Note that at this order of approximation we could not keep track of the factor $(1-e^2)^{-1}$, containing the eccentricity (here replaced by the ratio $\frac{n_0^r}{R}$) only through its square. In contrast, we obtain without effort the coefficients in front of quadratic or cubic terms in $\frac{MG}{R}$. This shows that our method can be of interest when one has to consider the low-eccentricity orbits in the vicinity of very massive and compact bodies, having a non-negligible ratio $\frac{MG}{R}$.

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In order to include this effect, at least in its approximate form as the factor $(1+e^2)$, we must go beyond the first-order deviation equations and investigate the solutions of the equations describing the quadratic effects.

After inserting the complete solution for the first-order deviation vector (3.22)–(3.24) into the system (2.11) and a tedious calculation, we find the following set of linear equations satisfied by the second-order deviation vector $b^\mu(s)$:

$$\begin{pmatrix} \frac{d^2}{ds^2} & \frac{2MG\varepsilon}{R^2(1-\frac{2MG}{R})^2}\frac{d}{ds} & 0 \\ \frac{2MG\varepsilon}{R^2}\frac{d}{ds} & \frac{d^2}{ds^2}-\frac{3\ell^2}{R^4}(1-\frac{2MG}{R}) & -\frac{2\ell}{R}(1-\frac{2MG}{R})\frac{d}{ds} \\ 0 & \frac{2\ell}{R^3}\frac{d}{ds} & \frac{d^2}{ds^2} \end{pmatrix} \begin{pmatrix} b^t \\ b^r \\ b^\varphi \end{pmatrix} = (n_0^r)^2 \begin{pmatrix} C^t \\ C^r \\ C^\varphi \end{pmatrix}, \tag{3.32}$$

The common factor $(n_0^r)^2$ shows the explicit quadratic dependence of the second-order deviation vector b^μ on the first-order deviation amplitude n_0^r. The expressions on the right-hand side C^t, C^r and C^φ are functions M, R, ω_0, ω, ε, $\sin(2\omega s)$ and $\cos(2\omega s)$:

$$C^t = -\frac{6(MG)^2(2-\frac{7MG}{R})\varepsilon\sin(2\omega s)}{(1-\frac{3MG}{R})(1-\frac{2MG}{R})^2R^6\omega}, \tag{3.33}$$

$$C^r = \frac{3MG\left[(2-\frac{5MG}{R}+\frac{18(MG)^2}{R^2})-(6-\frac{27MG}{R}+\frac{6MG^2}{R^2})\cos(2\omega s)\right]}{2(1-\frac{3MG}{R})(1-\frac{6MG}{R})R^4}, \tag{3.34}$$

$$C^\varphi = -\frac{6MG(1-\frac{MG}{R})\omega_0\sin(2\omega s)}{(1-\frac{3MG}{R})R^5\omega}. \tag{3.35}$$

The solution of the above matrix for $b^\mu(s)$ has the same characteristic equation of the matrix (3.15) for $n^\mu(s)$, and the general solution containing oscillating terms with angular frequency ω is of no interest because it is already accounted for by $n^\mu(s)$. But the particular solution includes the terms linear in the proper time s, constant ones, and the terms oscillating with angular frequency 2ω. Being interested in the quantities directly accessible to observation, we give here the eplicit expressions of $b^r(s)$ and $b^\varphi(s)$:

$$b^r = \frac{(n_0^r)^2}{2R(1-\frac{6MG}{R})}\left[\frac{3(2-\frac{5MG}{R}+\frac{18(MG)^2}{R^2})}{1-\frac{6MG}{R}}+\left(2+\frac{5MG}{R}\right)\cos(2\omega s)\right], \tag{3.36}$$

$$b^\varphi = \frac{(n_0^r)^2\,\omega_0}{R^2(1-\frac{6MG}{R})}\left[-\frac{3(2-\frac{5MG}{R}+\frac{18(MG)^2}{R^2})}{1-\frac{6MG}{R}}s+\frac{1-\frac{8MG}{R}}{2\omega}\sin(2\omega s)\right]. \tag{3.37}$$

Now we need to calculate $\frac{1}{2}\delta^2x^\mu$ in order to obtain the geodesic curve x^μ with second-order geodesic deviation; again, we show only the components $\delta^2 r$ and $\delta^2\varphi$:

$$\delta^2 r = \frac{(n_0^r)^2}{R(1-\frac{6M}{R})}\left[\frac{5-\frac{33M}{R}+\frac{90M^2}{R^2}-\frac{72M^3}{R^3}}{(1-\frac{2M}{R})(1-\frac{6M}{R})}-\left(1-\frac{7M}{R}\right)\cos(2\omega s)\right], \tag{3.38}$$

$$\delta^2\varphi = \frac{(n_0^r)^2\,\omega_0}{R^2(1-\frac{6M}{R})}\left[-\frac{3(2-\frac{5M}{R}+\frac{18M^2}{R^2})}{1-\frac{6M}{R}}s+\frac{5-\frac{32M}{R}}{2\omega}\sin(2\omega s)\right]. \tag{3.39}$$

The fact that the second-order deviation vector b^μ turns with angular frequency 2ω enables us to get a better approximation of the elliptic shape of the resulting orbit. The trajectory described by x^μ including second-order deviations is not an ellipse, but we can match the perihelion and aphelion distances to see that $R \neq a$ and $e \neq n_0^r/R$ when second-order deviation is used. The perihelion and aphelion distances of the Keplerian, i.e., elliptical orbit are $a(1-e)$ and $a(1+e)$. For x^μ, the perihelion is obtained when $\omega s = 2k\pi$ and the aphelion when $\omega s = (1+2k)\pi$, where $k \in \mathbb{Z}$. Matching the radius for perihelion and aphelion, we obtain the semimajor axis a and the eccentricity e of an ellipse that has the same perihelion and aphelion distances of the orbit described by x^μ:

$$a = R + 2\,\frac{(n_0^r)^2}{R} + \mathscr{O}\left(\frac{(n_0^r)^4}{R^3}\right)\, e = \frac{n_0^r}{R} + \mathscr{O}\left(\frac{(n_0^r)^3}{R^3}\right). \tag{3.40}$$

In the limit case of $\frac{MG}{R} \to 0$, there is no perihelion advance and $a = R\left[1+2(\frac{n_0^r}{R})^2\right]$ and $e = \frac{n_0^r}{R}$, so the second-order deviation increases the semimajor axis a of a matching ellipse compared to the first-order deviation, when $a = R$ and $e = \frac{n_0^r}{R}$.

4. Third-order terms and gravitational radiation

With the third-order approximation we are facing a new problem, arising from the presence of *resonance terms* on the right-hand side. It is easy to see that after reducing the expressions on the right-hand side, which contain the terms of the form

$$\cos^3\omega s, \quad \sin\omega s\cos^2\omega s$$

and the like, we shall get not only the terms containing

$$\sin 3\,\omega\, s\,, \quad \text{and} \quad \cos 3\,\omega\, s$$

which do not create any particular problem, but also the *resonance terms* containing the functions $\sin\omega s$ and $\cos\omega s$, whose circular frequency is the same as the eigenvalue of the matrix-operator acting on the left-hand side.

As a matter of fact, the equation for the covariant third-order deviation h^μ can be written in matrix form, with principal part linear in the third-order deviation h^μ, represented by exactly the same differential operator as in the lower-order deviation equations. The right-hand side is separated into two parts, one oscillating with frequency ω, and another with frequency 3ω:

$$\begin{pmatrix} \frac{d^2}{ds^2} & \frac{2M\varepsilon}{R^2(1-\frac{2M}{R})^2}\frac{d}{ds} & 0 \\ \frac{2M\varepsilon}{R^2}\frac{d}{ds} & \frac{d^2}{ds^2}-\frac{3\ell^2}{R^4}(1-\frac{2M}{R}) & -\frac{2\ell}{R}(1-\frac{2M}{R})\frac{d}{ds} \\ 0 & \frac{2\ell}{R^3}\frac{d}{ds} & \frac{d^2}{ds^2}\end{pmatrix}\begin{pmatrix} h^t \\ h^r \\ h^\varphi\end{pmatrix} =$$

$$= (n_0^r)^3\begin{pmatrix} B^t\sin(\omega s)+C^t\sin(3\omega s)+s\,D^t\cos(\omega s) \\ B^r\cos(\omega s)+C^r\cos(3\omega s)+s\,D^r\sin(\omega s) \\ B^\varphi\sin(\omega s)+C^\varphi\sin(3\omega s)+s\,D^\varphi\cos(\omega s)\end{pmatrix}, \tag{4.1}$$

where the coefficients B^k, C^k and D^k, $k = t, r, \varphi$ are complicated functions of $\frac{MG}{R}$.

The proper frequency of the matrix operator acting on the left-hand side is equal to ω; the terms containing the triple frequency $3\,\omega$ will give rise to the unique non-singular solution of the same frequency, but the resonance terms of the basic frequency on the right-hand side will give rise to secular terms, proportional to s, which is in contradiction with the bounded character of the deviation we have supposed from the beginning. The term proportional to s on the right-hand side is eliminated in the differential equation for h^r when $\frac{dh^\varphi}{ds}$ and $\frac{dh^t}{ds}$ are replaced by theirs values.

Poincaré [22] was first to understand that in order to solve this apparent contradiction, one has to take into account possible perturbation of the basic frequency itself, which amounts to the replacement of ω by an infinite series in powers of the infinitesimal parameter, which in our case is the eccentricity $e = \frac{n^r_0}{R}$:

$$\omega \to \omega + e\,\omega_1 + e^2\,\omega_2 + e^3\,\omega_3 + \ldots, \tag{4.2}$$

Then, developing both sides into a series of powers of the parameter e, we can not only recover the former differential equations for the vectors n^μ, b^μ, h^μ, but get also some algebraic relations defining the corrections ω_1, ω_2, ω_3, etc.

The equations resulting from the requirement that all resonant terms on the right-hand side be canceled by similar terms on the left-hand side are rather complicated. We do not attempt to solve them here. However, one easily observes that the absence of resonant terms in the second-order deviation equations forces ω_1 to vanish, while the next term ω_2 is different from 0.

Similarly, as there are no resonant terms in the equations determining the fourth-order deviation, because all four-power combinations of sine and cosine functions will produce terms oscillating with frequencies 2ω and 4ω; as a result, the correction ω_3 will be also equal to 0. Next secular terms appear at the fifth-order approximation, as products of the type $cos^5\omega s$, $sin^3\omega s\, cos^2\omega s$, etc, produce resonant terms again, which will enable us to find the correction ω_4, and so on, so that the resulting series representing the frequency ω contains only even powers of the small parameter $\frac{n^r_0}{R}$.

The decomposition of the elliptic trajectory turning slowly around its focal point into a series of epicycles around a circular orbit can also serve for obtaining an approximate spectral decomposition of gravitational waves emitted by a celestial body moving around a very massive attracting center.

It is well known that gravitational waves are emitted when the quadrupole moment of a mass distribution is different from zero, and the amplitude of the wave is proportional to the third derivative of the quadrupole moment with respect to time (in the reference system in which the center of mass coincides with the origin of the Cartesian basis in three dimensions, see Ref. [**?**]).

Of course, it is only a linear approximation, but it takes the main features of the gravitational radiation emitted by the system well into account, provided the velocities and the gravitational fields are not relativistic and the wavelength of gravitational radiation is large compared to the dimensions of the source (quadrupole approximation).

More precisely, let us denote the tensor Q_{ij} of a given mass distribution $\mu\,(x_i)$, where $i, j, = 1, 2, 3$, see Ref. [38]:

$$Q_{ij} = \int \mu\; x_i x_j dV = \sum_\alpha m_\alpha x_{\alpha i} x_{\alpha j}, \tag{4.3}$$

where m_α are point masses.

Let $\overrightarrow{OP}$ be the vector pointing at the observer (placed at the point P), from the origin of the coordinate system coinciding with the center of mass of the two orbiting bodies whose motion is approximately described by our solution in a Fourier series form. It is also supposed that the length of this vector is much greater than the characteristic dimensions of the radiating system, i.e. $|\overrightarrow{OP}| \gg R$.

Then the total power of gravitational radiation P emitted by the system over all directions is given by the following expression (see Ref. [38]):

$$P = \frac{G}{5c^5}\left(\frac{d^3Q_{ij}}{dt^3}\frac{d^3Q_{ij}}{dt^3} - \frac{1}{3}\frac{d^3Q_{ii}}{dt^3}\frac{d^3Q_{jj}}{dt^3}\right). \tag{4.4}$$

When applied to Keplerian motion of two masses m_1 and m_2, with orbit equation and angular velocity given by

$$r = \frac{a(1-e^2)}{1+e\cos\varphi}, \quad \frac{d\varphi}{dt} = \frac{\sqrt{G(m_1+m_2)a(1-e^2)}}{r^2}, \tag{4.5}$$

the total power P now reads

$$P = \frac{8}{15}\frac{G^4}{c^5}\frac{m_1^2 m_2^2(m_1+m_2)}{a^5(1-e^2)^5}(1+e\cos\varphi)^4\left[12(1+e\cos\varphi)^2 + e^2\sin^2\varphi\right]. \tag{4.6}$$

We shall calculate the P in Eq. (4.4) with our solution x^μ using second-order geodesic deviation, to inspect the non-negligible effects of the ratio $\frac{M}{R}$. We have the explicit solutions $r(s)$, $\varphi(s)$ and $t(s)$, so to calculate $\frac{dQ_{ij}}{dt}$ we need only the derivatives with respect to s, i.e., $\frac{df}{dt} = \frac{df}{ds}/\frac{dt}{ds}$ can be applied successively to obtain $\frac{d^3Q_{ij}}{dt^3}$. So we finally get P as function of s, which is not shown here because it is a very large expression that nevertheless can be easily obtained using a symbolic calculus computer program.

As we want to compare the two total powers P during one orbital period (between perihelions), P in the Kepler case is obtained from the numerical solution for $\varphi(t)$ calculated from Eq. (4.5), and P of the geodesic deviation case has to use $s(t)$ obtained from $t(s)$ by means of successive approximations, starting with $s = \frac{t}{\varepsilon}\sqrt{1-\frac{2M}{R}}$.

There are many possible ways to compare a Keplerian orbit with a relativistic one. Here we assume $m_1 \gg m_2$ and fix the values of a, e, m_1; the values of R and n_0^r are calculated to obtain an exact ellipse (up to the second order in e) in the limit $\frac{M}{R} \to 0$, so $R = \frac{(1-e^2)}{(1+e^2)}a$ and $n_0^r = R\,e$. Up to first order in e, we have $R = a$. The choice of $M = m_1$ allows the two total powers P to be equal when $e = 0$ and $\frac{M}{R} \to 0$. Figure 1 shows this comparison for a small eccentricity and a non-negligible $\frac{MG}{R}$ ratio.

Because the emitted total powers P calculated with geodesic deviations depend on the $\frac{M}{R}$ ratio, we see that the period is not $T = \frac{2\pi a^{3/2}}{\sqrt{Gm_1}}$ (third Kepler's law), but an increased one,

$$T = \frac{2\pi R^{3/2}}{\sqrt{GM}\sqrt{1-\frac{6GM}{R}}} + \mathscr{O}\left(\frac{(n_0^r)^2}{R^2}\right). \tag{4.7}$$

This effect is the direct consequence of the form of angular frequency ω that appears in the first and higher-order geodesic deviations.

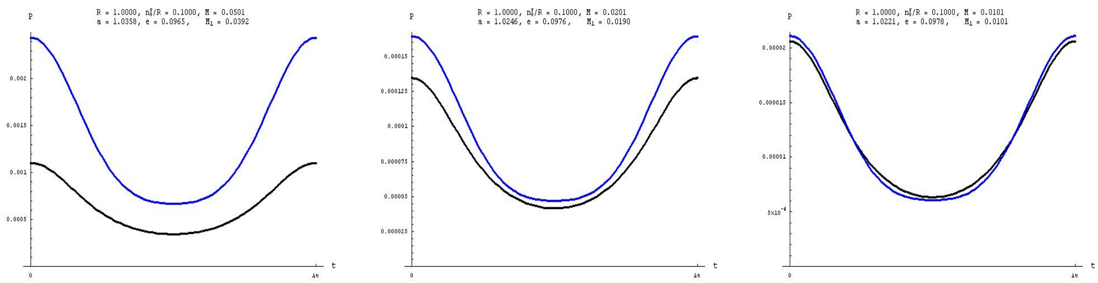

Figure 1: The total power P in three cases as function of t during one orbital period T, with $M = m_1$. The upper (blue) curve represent the result obtained by post-post-Newtonian approximations, serving as a reference. The lower (black) curves correspond to the approximations obtained via the epicycle method, with first order only (left), with the second-order approximation (middle) and the third-order approximation (right). The excentricity has a common value $e = 0.1$. The curves are taken from the reference [23]

Another expected feature of Figure 1: as e (i.e., $\frac{n_0^r}{R}$) is kept small, the P using geodesic deviations converge very fast in respect of the orders of geodesic deviation.

Caution is required as the use of quadrupole approximation is not allowed for high values of $\frac{M}{R}$, so the exact amplitude and shape of P using geodesic deviations can only be calculated if additional $\frac{M}{R}$ contributions to the gravitational radiation formula are included. This approach, but using the post-Newtonian expansion scheme, is well developed in Refs. [39, 40, 41].

The two approaches are complementary in the following sense: the post-Newtonian scheme gives better results for small values of $\frac{MG}{R}$ and arbitrary eccentricity, whereas our scheme is best adapted for small eccentricities, but arbitrary values of $\frac{MG}{R} < \frac{1}{6}$. In both approaches the emission of gravitational radiation is estimated using the quadrupole formula, based on a flat-space approximation.

The next challenge is to include finite-size and radiation back-reaction effects. In the post-Newtonian scheme some progress in this direction has already been made. In this aspect our result may be regarded as the first term in an expansion in $\frac{m}{M}$. Other applications can be found in problems of gravitational lensing and perturbations by gravitational waves.

The shortcoming of this method, which was entirely focused on the trajectories, was the total lack of any variation of the gravitational field, i.e. the Schwarzschild metric which was maintained invariable for all orders of geodesic deviation. This fact reduced the validity of the method only to the case of test particles with mass m negligible when compared with the mass M of the central body appearing in the Schwarzschild background metric. More precisely, the successive approximations of planet's trajectory remain valid as long as the dimensionless parameter (m/M) can be considered as negligibly small, i.e. $(m/M) << 1$. When the mass m is not a negligible quantity, its presence must inevitably alter the geometry of the initial Schwarzschild metric, and its influence can be therefore represented by a power series in the small parameter (m/M).

Now we shall describe the departure from the initial Minkowskian or Schwarzschild metrics in terms of the embedding functions. Embeddings of the exterior Schwarzschild geometry in pseudo-Euclidean flat spaces are known since a long time ([28], [29], [27]), and once such an embedding is given, all intrinsic geometric quantities of the embedded manifold can be expressed in terms of derivatives of the embedding functions which depend on four "internal" parameters which are the space-time coordinates. Embedding techniques were also used in certain models of primordial cosmology, like the change of signature, as in ([24]).

A general analysis of deformations of the embedded Einstein spaces was given in [25]; nevertheless, only the theoretical setup was considered, without any concrete solution describing Ricci-flat deformations of known exact Einstein spaces, and in the first place, Minkowskian or Schwarzschild.

Here we shall explore not only the first linear approximation, but also the effects of second and third order in the expansion of deformations in powers of small parameter ε, including the corrections describing gravitational waves. We start with the investigation of wave-like deformations of flat Minkowskian space-time embedded in five or, if necessary, more dimensional pseudo-Euclidean space. It will appear that the wave-like behavior of the Riemann tensor can be obtained only in third order terms proportonal to ε^3. In the Minkowskian case such a wave can appear even if the embedding is in five dimensions; but in the case of spherical gravitational wave, more extra dimensions are necessary in order to accomodate it.

In the case of spherical gravitational waves the exact solution cannot be found at once; instead, we shall find successive approximations considering first the dominant terms at infinity, behaving as r^{-1}; then we consider the effects of short-range behavior of the type r^{-2}, r^{-3}, etc., which can give more information about the source.

5. Isometric embeddings and their properties

Consider the embedding of a four-dimensional Riemannian space parametrized by local coordinates (denoted by x^μ, $\mu, \nu = 0, 1, 2, 3$ as usual) in a pseudo-Euclidean space E^N of dimension N. The dimension N, yet unspecified, depends on the topology of the Riemannian space under consideration, and may be quite high, as acknowledged in [27]. Locally, any n-dimensional Riemannian manifold can be embedded in a (pseudo)-Euclidean space of dimension $N = n(n+1)/2$. Here we are interested in *global* embeddings, which may require a relatively low dimension of the "host" space if the Riemannian space to be embedded possesses some particular symmetry. For example, the de Sitter space can be embedded globally in a five-dimensional pseudo-Euclidean space with signature $(+----)$, and exterior or interior Schwarzschild solutions can be embedded in a six-dimensional E^N with signatures $(++----)$ or $(+-----)$.

Consider a global embedding of a Riemannian space V_4 given by the set of *embedding functions* z^A:

$$z^A = z^A\,(x^\mu), \text{ with } \; A, B, ... = 1, 2, ...N, \quad \mu, \nu = 0, 1, 2, 3. \tag{5.1}$$

The metric tensor of V_4 is the *induced metric* defined as

$$\overset{\circ}{g}_{\mu\nu} = \eta_{AB}\,\partial_\mu z^A\,\partial_\nu z^B \tag{5.2}$$

The inverse metric tensor $\overset{\circ}{g}{}^{\mu\nu}$ cannot be obtained directly from the embedding functions, but should be computed from the covariant components as their inverse matrix. From now on we use the superscript notation in order to make difference between the "basic" induced metric $\overset{\circ}{g}_{\mu\nu}$ which will be considered as a background, and its infinitesimal deformations expanded in terms of an infinitesimal parameter ε as follows:

$$g_{\mu\nu} = \overset{\mathrm{o}}{g}_{\mu\nu} + \varepsilon \overset{1}{g}_{\mu\nu} + \varepsilon^2 \overset{2}{g}_{\mu\nu} + \ldots \tag{5.3}$$

induced by the following deformation of the initial embedding functions:

$$z^A\,(x^\mu) \to z^A\,(x^\mu) + \varepsilon\, v^A\,(x^\mu) + \varepsilon^2\, w^A\,(x^\mu) + \ldots \tag{5.4}$$

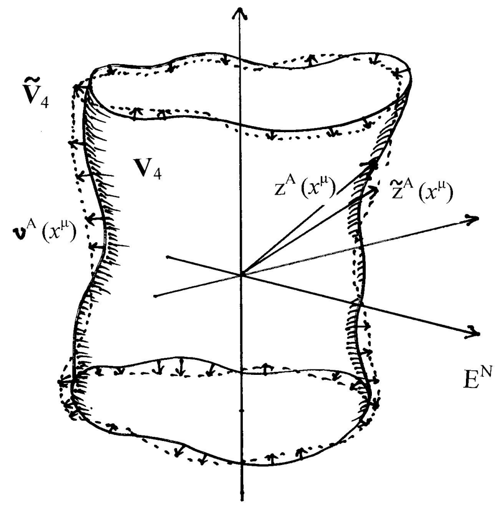

Figure 2: The initial embedding of a Riemannian manifold V_4 and an infinitesimal deformation producing the new embedding $\tilde{V}_4$

When seen from the ambient pseudo-Euclidean space, the new embedded manifold $\tilde{V}_4$ is the result of an infinitesimal deformation of the initial manifold V_4 induced by a vector field in $E^N_{(p,q)}$. It is obvious that such a field, which is defined on the embedded submanifold, can be decomposed into its normal part (in the sense of the pseudo-Euclidean metric) and a part tangent to V_4. The last part induces an internal diffeomorphism of V_4 and can be implemented as a local coordinate transformation. Such deformations do not have any physical meaning, but it is not always necessary to consider exclusively the deformations orthogonal to the embedded V_4; sometimes a deformation having non-vanishing parallel and orthogonal parts can have less non-zero components in the ambient space $E^N_{(p,q)}$ than its part orthogonal to the embedded V_4 manifold.

Our first aim is to express all important geometrical quantities e.g. the connection coefficients and the curvature tensor, in terms of embedding functions z^A and their partial derivatives. Let us start with Christoffel connection

$$\overset{\mathrm{o}}{\Gamma}{}^{\lambda}_{\mu\nu} = \frac{1}{2}\, \overset{\mathrm{o}}{g}{}^{\lambda\rho} \left(\partial_\mu \overset{\mathrm{o}}{g}_{\nu\rho} + \partial_\nu \overset{\mathrm{o}}{g}_{\mu\rho} - \partial_\rho \overset{\mathrm{o}}{g}_{\mu\nu} \right). \tag{5.5}$$

From the definition of $\overset{\mathrm{o}}{g}_{\mu\nu}$ (5.2) we have the expression for its partial derivatives:

$$\partial_\lambda\, \overset{\mathrm{o}}{g}_{\mu\nu} = \eta_{AB} \left(\partial^2_{\lambda\mu}\, z^A\, \partial_\nu\, z^B + \partial_\mu\, z^A\, \partial^2_{\lambda\nu}\, z^B \right). \tag{5.6}$$

When substituted into the definition (5.5) it gives

$$\overset{\circ}{\Gamma}{}^{\lambda}_{\mu\nu} = \eta_{AB}\, \overset{\circ}{g}{}^{\lambda\rho}\, \partial_\rho z^A\, \partial^2_{\mu\nu} z^B. \tag{5.7}$$

Now, an alternative (although implicit) definition of Christoffel symbols is contained in the equation that states the vanishing of covariant derivatives of the metric:

$$\nabla_\lambda\, g_{\mu\nu} = 0, \tag{5.8}$$

which after substitution $g^{\circ}{}_{\mu\nu} = \eta_{AB}\, \partial_\mu z^A\, \partial_\nu z^B = \eta_{AB}\, \nabla_\mu z^A\, \nabla_\nu z^B$ leads to the identity

$$\eta_{AB} \left[\nabla_\lambda \nabla_\mu z^A\, \nabla_\nu z^B + \nabla_\mu z^A\, \nabla_\lambda \nabla_\nu z^B \right] = 0. \tag{5.9}$$

Taking the combination $\nabla_\lambda\, g_{\mu\nu} + \nabla_\mu\, g_{\lambda\nu} - \nabla_\nu\, g_{\mu\lambda} = 0$, we get the identity for arbitrary indices μ, ν, λ:

$$\eta_{AB} \left[\nabla_\lambda \nabla_\mu z^A\, \nabla_\nu z^B \right] = 0. \tag{5.10}$$

Using this result, let us form the following combination of covariant derivatives which vanishes identically:

$$\eta_{AB} \left[\nabla_\mu (\nabla_\rho\, z^A\, \nabla_\nu \nabla_\sigma\, z^B) - \nabla_\nu (\nabla_\rho\, z^A\, \nabla_\mu \nabla_\sigma\, z^B) \right] = 0. \tag{5.11}$$

Applying the derivation and using the Leibniz rule we get:

$$\eta_{AB} \left[\nabla_\mu \nabla_\rho\, z^A\, \nabla_\nu \nabla_\sigma\, z^B - \nabla_\nu \nabla_\rho\, z^A\, \nabla_\mu \nabla_\sigma\, z^B) + (\nabla_\rho\, z^A) \left[\nabla_\mu \nabla_\nu \nabla_\sigma\, z^B - \nabla_\nu \nabla_\mu \nabla_\sigma\, z^B \right] \right] = 0.$$

Recalling that

$$\left[\nabla_\mu\, \nabla_\nu - \nabla_\nu\, \nabla_\mu \right] \nabla_\rho\, z^B = \overset{\circ}{R}{}_{\mu\nu}{}^{\lambda}{}_{\rho}\, \nabla_\lambda\, z^B, \tag{5.12}$$

so that we can write

$$\overset{\circ}{R}{}_{\mu\nu\lambda\rho} = -\eta_{AB} \left[\nabla_\mu \nabla_\lambda\, z^A \nabla_\nu \nabla_\rho\, z^B - \nabla_\nu \nabla_\lambda\, z^A \nabla_\mu \nabla_\rho\, z^B \right]. \tag{5.13}$$

which is the well known Gauss-Codazzi equation.

The definition of the Riemann tensor by means of derivatives of the embedding functions given by formula (5.13) looks compact, but is in fact highly non-linear and complicated. This is so because it contains many Christoffel symbols involved in the second covariant derivatives, which contain the contravariant metric tensor $\overset{\circ}{g}{}^{\mu\nu}$. The components of the contravariant metric tensor are obtained as rational expressions in third and fourth powers of $\nabla_\mu\, z^A$. Nevertheless, the most important point here is that the Riemann tensor depends only on first and second derivatives of embedding functions, so that the Einstein equations expressed in terms of the embedding functions will lead to second-order partial differential equations.

The expressions derived in this section will be very useful in the development of a power series expansion of infinitesimally deformed embedding.

6. Infinitesimal deformations of embeddings

Let us consider an isometric embedding of an Einsteinian manifold $\overset{\circ}{V}_4$ in a pseudo-Euclidean space $E^N_{p,q}$ with signature $(p+,q-)$, with $p+q=N$:

$$\overset{\circ}{V}_4 \quad \rightarrow \quad E^N_{p,q}, \quad z^A = z^A\,(x^\mu)$$
$$\text{with} \quad A,B,... = 1,2,...N, \qquad \mu,\nu = 0,1,2,3. \tag{6.1}$$

Consider now an infinitesimal deformation of the embedding defined by a converging series of terms proportional to the consecutive powers of a small parameter ε. The deformed embedding defines an Einsteinian space $\tilde{V}_4$;

$$z^A\,(x^\mu) \rightarrow \tilde{z}^A\,(x^\mu) = z^A\,(x^\mu) + \varepsilon\, v^A\,(x^\mu) + \varepsilon^2\, w^A\,(x^\mu) + \ldots \tag{6.2}$$

The induced metric on $\tilde{V}_4$ can also be developed in a series of powers of ε:

$$\begin{aligned}
\tilde{g}_{\mu\nu} &= \overset{\circ}{g}_{\mu\nu} + \varepsilon \overset{1}{g}_{\mu\nu} + \varepsilon^2 \overset{2}{g}_{\mu\nu} + ... \\
&= \eta_{AB}\left[\partial_\mu z^A \partial_\nu z^B + \varepsilon\left(\partial_\mu z^A \partial_\nu v^B + \partial_\mu v^A \partial_\nu z^B\right) + \right. \\
&+ \left. \varepsilon^2\left(\partial_\mu v^A \partial_\nu v^B + \partial_\mu z^A \partial_\nu w^B + \partial_\mu w^A \partial_\nu z^B\right)\right].
\end{aligned} \tag{6.3}$$

Among possible infinitesimal deformations of the embedding functions $z^A\,(x^\mu) + \varepsilon\, v^A\,(x^\mu)$ there is a large class of functions $v^A\,(x^\mu)$ which will not alter the intrinsic geometry of the embedded manifold. The translations v^A =Const. obviously do not change the internal metric $\overset{\circ}{g}_{\mu\nu} = \eta_{AB}\,\partial_\mu z^A \partial_\nu z^B$. Also the generalized Lorentz transformations of the pseudo-Euclidean space $E^N_{(p,q)}$ keep the internal metric unchanged. Indeed, if we set

$$z^A \rightarrow \tilde{z}^A = z^A + \varepsilon\,\Lambda^A_B\, z^B, \tag{6.4}$$

with Λ^A_B constant matrix,

then the first-order correction vanishes if the matrices Λ^A_B satisfy the identity

$$\eta_{AB}\,\Lambda^B_C + \eta_{CB}\Lambda^C_A = 0,$$

defining infinitesimal rigid rotations (Lorentz transformations) of the pseudo-Euclidean space $E^N_{(p,q)}$.

The geometric character of our approach enables us to eliminate unphysical degrees of freedom using simple geometrical arguments. Remember that in the traditional approach leading to linearized equations for gravitational fields the starting point is the following development of the metric tensor:

$$g_{\mu\nu} = \overset{\circ}{g}_{\mu\nu} + \varepsilon\, h_{\mu\nu}, \tag{6.5}$$

thus introducing *ten* components of $h_{\mu\nu}$ as dynamical fields. We know however that most of them do not represent real dynamical degrees of freedom due to the gauge invariance. The metric tensor itself does not correspond to any directly measurable quantity. In fact, its components may be changed by a gauge transformation without changing the components of the Riemann tensor which is the source of measurable gravitational effects. In particular, the gauge transformation

$$g_{\mu\nu} \to \tilde{g}_{\mu\nu} = g_{\mu\nu} + \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu \tag{6.6}$$

does not alter the Riemann tensor so that both $\tilde{g}_{\mu\nu}$ and $g_{\mu\nu}$ describe the same gravitational field. The arbitrary vector field ξ^μ generating gauge transformation (6.6) represents four degrees of freedom which are redundant in $g_{\mu\nu}$; this is why in the linearized Einstein equations one may impose four gauge conditions e.g.

$$\nabla_\mu h^{\mu\nu} = 0. \tag{6.7}$$

The unphysical degrees of freedom can be easily eliminated from the embedding deformation functions $v^A(x^\mu)$ if we note that any vector field in the embedding space E^N that is *tangent* to the embedded Riemannian space V_4 describes nothing else but a diffeomorphism of V_4, in other words a coordinate change, which has no influence on any physical or intrinsic geometrical quantities.

Vector fields tangent to the four-dimensional embedded manifold V_4 can be decomposed along four arbitrarily chosen independent smooth vector fields in E^N tangent to V_4. On the other hand, vector fields *transversal* to the embedded hypersurface V_4 must satisfy the following obvious *orthogonality conditions*:

$$\eta_{AB}\, \partial_\mu z^A\, v^B = \eta_{AB}\, v^A\, \nabla_\mu z^B = 0. \tag{6.8}$$

For any value of A the four partial derivatives (let us remind that $\nabla_\mu z^A = \partial_\mu z^A$) span a basis of four vector fields in E^N tangent to the submanifold V_4; therefore any vector v^B satisfying the orthogonality condition (6.8) is transversal to V_4 (as seen in E^N).

The orthogonality condition (6.8) imposes four independent equations, which reduce the number of independent deformation functions v^A to $N-4$. This means that general non-redundant deformations can be decomposed along $N-4$ independent fields $X^A_{(k)}$, $\quad k = 1,2,...,N-4$:

$$v^A(x^\mu) = \sum_{k=1}^{N-4} v^k(x^\nu))\, X^A_{(k)}\,(x^\lambda). \tag{6.9}$$

The basic fields $X^A_{(k)}\,(x^\lambda)$ can be chosen at will provided they induce a non-singular global vector field on V_4, while the relevant degrees of freedom are contained in $N-4$ functions $v^k(x^\lambda)$. To take an example, the de Sitter space can be globally embedded in a five-dimensional pseudo-Euclidean space $E^5_{1,4}$ with signature $(+----)$; therefore its global deformations can be described by a single function $v(x^\lambda)$ (see [25]).

One may ask the following question: what if the deformation destroys the initial symmetry of the embedded manifold so that the deformed manifold cannot be embedded in the initially sufficient N-dimensional pseudo-Euclidean space but needs a flat embedding space of higher dimension ? It is known that global embeddings of the Kerr metric need more than six flat dimensions sufficient for the embedding of the exterior (or interior) Schwarzschild solution (see [34],[36], [37]), although Schwarzschild's metric can be obtained from Kerr's metric as a limit when the Kerr parameter a (the angular momentum) tends to zero.

The answer is that as long as we investigate only the first-order corrections to geometry, we should not worry about this issue for the following two reasons: first, when a global embedding is given, its infinitesimal deformations cannot lead to a global modification of the embedding; second, if the bigger embedding space was introduced, say E^{N+m}, it would contain the initial embedding space E^N as its linear subspace, so that

$$E^{N+m} = E^N \oplus E^m,$$

and its pseudo-Euclidean metric could be represented as a blockwise reducible matrix

$$\eta_{\alpha\beta} = \begin{pmatrix} \eta_{AB} & 0 \\ 0 & \eta_{ij} \end{pmatrix}, \tag{6.10}$$

with $A, B, ... = 1, 2, ...N,\ i, j, ... = 1, 2, ..m,\ \alpha, \beta, ... = 1, 2, ..., N+m$. Accordingly, any deformation of the initial embedding can be decomposed in two parts, one contained in the initial embedding space E^N and another one in the complementary subspace E^m:

$$v^\alpha = [v^A,\, v^m]. \tag{6.11}$$

But the initial embedding functions had their components entirely in the first subspace E^N, $z^\alpha = [z^A,\, 0]$, therefore the deformed embedding functions can be written as

$$\tilde{z}^\alpha = [z^A + \varepsilon\, v^A,\, \varepsilon\, v^k], \tag{6.12}$$

so that the induced metric of the deformed embedding will be

$$\begin{aligned} g_{\mu\nu} &= \overset{\circ}{g}_{\mu\nu} + \varepsilon \overset{1}{g}_{\mu\nu} + \varepsilon^2 \overset{2}{g}_{\mu\nu} + ... \\ &= \eta_{AB}\left(\partial_\mu z^A \partial_\nu z^B\right) + \varepsilon\, \eta_{AB}\left(\partial_\mu z^A \partial_\nu v^B + \partial_\mu v^A \partial_\nu z^B\right) \\ &+ \varepsilon^2 \left[\eta_{AB}\left(\partial_\mu v^A \partial_\nu v^B + \partial_\mu z^A \partial_\nu w^B + \partial_\mu w^A \partial_\nu z^B\right) + \eta_{ij}\left(\partial_\mu\, v^i \partial_\nu\, v^j\right)\right] ... \end{aligned} \tag{6.13}$$

From this one can see that the deformations towards extra dimensions do not contribute to the first-order corrections of any geometrical quantities obtained from the deformed embedding functions. This is why we shall not consider such deformations while investigating at first only the terms linear in the infinitesimal parameter ε.

Our principal aim now is to establish the explicit form of connection and curvature components induced on the infinitesimally deformed embedding $\tilde{V}_4$. To this end we must calculate the approximate expression of the contravariant metric tensor $g^{\mu\nu}$. If the covariant metric is decomposed as in (5.3),

$$g_{\mu\nu} = \overset{\circ}{g}_{\mu\nu} + \varepsilon \overset{1}{g}_{\mu\nu} + \varepsilon^2 \overset{2}{g}_{\mu\nu} + ...$$

then we have the following formulae defining the corresponding decomposition of $g^{\mu\nu}$:

$$\begin{aligned} g^{\mu\nu} &= \overset{\circ}{g}{}^{\mu\nu} + \varepsilon \overset{1}{g}{}^{\mu\nu} + \varepsilon^2 \overset{2}{g}{}^{\mu\nu} + \dots \\ &= g^{\mu\nu} - \varepsilon \overset{\circ}{g}{}^{\mu\rho} \overset{\circ}{g}{}^{\nu\sigma} \overset{1}{g}_{\rho\sigma} - \varepsilon^2 \left[\overset{\circ}{g}{}^{\mu\rho} \overset{\circ}{g}{}^{\nu\sigma} \overset{2}{g}_{\rho\sigma}\right] + \varepsilon^2 \left[\overset{\circ}{g}{}^{\mu\rho} \overset{\circ}{g}{}^{\nu\sigma} \overset{\circ}{g}{}^{\lambda\kappa} \overset{1}{g}_{\rho\lambda} \overset{1}{g}_{\sigma\kappa}\right], \end{aligned} \tag{6.14}$$

which shows that also the contravariant metric tensor depends exclusively on the first derivatives of the embedding functions, although in a quite complicated manner.

In what follows we shall keep only the first order terms linear in ε.

Let us start by computing the first (linear) correction to the components of the Christoffel connection, which develops in Taylor series as

$$\Gamma^{\lambda}_{\mu\nu} = \overset{o}{\Gamma}{}^{\lambda}_{\mu\nu} + \varepsilon \overset{1}{\Gamma}{}^{\lambda}_{\mu\nu} + \varepsilon^2 \overset{2}{\Gamma}{}^{\lambda}_{\mu\nu} + ..., \tag{6.15}$$

then by definition we have:

$$\begin{aligned}\overset{1}{\Gamma}{}^{\lambda}_{\mu\nu} &= \frac{1}{2}\overset{o}{g}{}^{\lambda\rho}\left(\partial_\mu \overset{1}{g}_{\nu\rho} + \partial_\nu \overset{1}{g}_{\mu\rho} - \partial_\rho \overset{1}{g}_{\mu\nu}\right) + \frac{1}{2}\overset{1}{g}{}^{\lambda\rho}\left(\partial_\mu \overset{o}{g}_{\nu\rho} + \partial_\nu \overset{o}{g}_{\mu\rho} - \partial_\rho \overset{o}{g}_{\mu\nu}\right) \qquad (6.16)\\ &= \frac{1}{2}\overset{o}{g}{}^{\lambda\rho}\left(\partial_\mu \overset{1}{g}_{\nu\rho} + \partial_\nu \overset{1}{g}_{\mu\rho} - \partial_\rho \overset{1}{g}_{\mu\nu}\right) - \frac{1}{2}\overset{o}{g}{}^{\lambda\sigma}\overset{o}{g}{}^{\rho\kappa}\overset{1}{g}_{\sigma\kappa}\left(\partial_\mu \overset{o}{g}_{\nu\rho} + \partial_\nu \overset{o}{g}_{\mu\rho} - \partial_\rho \overset{o}{g}_{\mu\nu}\right).\end{aligned}$$

One easily checks that

$$\frac{1}{2}\overset{o}{g}{}^{\lambda\rho}\left(\partial_\mu \overset{1}{g}_{\nu\rho} + \partial_\nu \overset{1}{g}_{\mu\rho} - \partial_\rho \overset{1}{g}_{\mu\nu}\right) = \eta_{AB}\,\overset{o}{g}{}^{\lambda\rho}\left[\partial_\rho z^A\,\partial^2_{\mu\nu} v^B + \partial_\rho v^A\,\partial^2_{\mu\nu} z^B\right], \tag{6.17}$$

while the second term after some algebra gives

$$\frac{1}{2}\overset{1}{g}{}^{\lambda\rho}\left(\partial_\mu \overset{o}{g}_{\nu\rho} + \partial_\nu \overset{o}{g}_{\mu\rho} - \partial_\rho \overset{o}{g}_{\mu\nu}\right) = -\eta_{AB}\,\overset{o}{g}{}^{\lambda\rho}\left[\partial_\rho z^A\,\overset{o}{\Gamma}{}^{\sigma}_{\mu\nu}\,\partial_\sigma v^B + \partial_\rho v^A\,\overset{o}{\Gamma}{}^{\sigma}_{\mu\nu}\,\partial_\sigma z^B\right]. \tag{6.18}$$

Combining together (6.17), (6.17) and (6.18) we find the final expression

$$\overset{1}{\Gamma}{}^{\lambda}_{\mu\nu} = \eta_{AB}\,\overset{o}{g}{}^{\lambda\rho}\left[\nabla_\rho z^A\,\nabla_\mu\nabla_\nu v^B + \nabla_\rho v^A\,\nabla_\mu\nabla_\nu z^B\right]. \tag{6.19}$$

This expression has a tensorial character as it should be, because by definition both quantities $\Gamma^{\lambda}_{\mu\nu}$ and $\tilde{\Gamma}^{\lambda}_{\mu\nu}$, transform as connection coefficients, therefore their difference must transform as a tensor, and this is true for any term of the development into series of powers of ε.

The coefficients $\overset{1}{\Gamma}{}^{\lambda}_{\mu\nu}$ will be useful for the derivation of geodesic equations in the deformed space-time, but they are not necessary for the computation of the first-order deformation of the Riemann tensor, which can be determined as follows. Let us develop second covariant derivatives of the deformed embedding functions $\tilde{z}^A$ yields:

$$\begin{aligned}\tilde{\nabla}_\mu\tilde{\nabla}_\nu\tilde{z}^A &= \tilde{\nabla}_\mu\tilde{\nabla}_\nu z^A + \varepsilon\,\tilde{\nabla}_\mu\tilde{\nabla}_\nu v^A + O(\varepsilon^2)\\ &= \nabla_\mu\nabla_\nu z^A + \varepsilon\left[\nabla_\mu\nabla_\nu v^A - \overset{1}{\Gamma}{}^{\lambda}_{\mu\nu}\,\nabla_\lambda z^A\right] + O(\varepsilon^2). \qquad (6.20)\end{aligned}$$

The Riemann tensor induced on the deformed embedding is defined by the same formula as in the previous section (5.13):

$$\tilde{R}_{\nu\mu\lambda\rho} = \eta_{AB}\left[\tilde{\nabla}_\mu\tilde{\nabla}_\lambda\tilde{z}^A\,\tilde{\nabla}_\nu\tilde{\nabla}_\rho\tilde{z}^B - \tilde{\nabla}_\nu\tilde{\nabla}_\lambda\tilde{z}^A\,\tilde{\nabla}_\mu\tilde{\nabla}_\rho\tilde{z}^B\right] \tag{6.21}$$

Note that in order to calculate the components of the Riemann tensor induced on the deformed manifold $\tilde{V}_4$ we use not only the deformed embedding functions $\tilde{z}^A$, but also the "deformed" covariant derivations $\tilde{\nabla}_\mu$. Now, when we insert the expressions like (6.20) into the definition of Riemann tensor components (6.21), we shall encounter, besides the zeroth-order initial Riemann tensor $\overset{o}{R}_{\mu\nu\lambda\rho}$ and the second-order corrections proportional to ε^2, just two types of terms linear in ε:

$$\varepsilon\,\eta_{AB}\,\nabla_\mu\nabla_\lambda\,z^A\,\nabla_\nu\nabla_\rho\,v^B \quad \text{and} \quad \varepsilon\,\eta_{AB}\,\nabla_\mu\nabla_\lambda\,z^A\overset{1}{\Gamma}{}^{\lambda}_{\nu\rho}\,\nabla_\lambda\,z^B. \tag{6.22}$$

The terms of the second type vanish by virtue of the identity (5.10); therefore the first-order correction to the components of Riemann tensor can be written as follows:

$$\begin{aligned}\overset{1}{R}{}_{\nu\mu\lambda\rho} = \eta_{AB}\Big[&\nabla_\mu\nabla_\lambda\,z^A\,\nabla_\nu\nabla_\rho\,v^B + \nabla_\mu\nabla_\lambda\,v^A\,\nabla_\nu\nabla_\rho\,z^B \\ &-\nabla_\nu\nabla_\lambda\,z^A\,\nabla_\mu\nabla_\rho\,v^B - \nabla_\nu\nabla_\lambda\,v^A\,\nabla_\mu\nabla_\rho\,z^B\Big]\end{aligned} \tag{6.23}$$

To establish the form of linear correction to Einstein's equations we need to know the components of the first-order correction to the Ricci tensor and the Riemann scalar. These quantities are readily computed as follows:

$$\overset{1}{R}{}_{\mu\rho} = \overset{o}{g}{}^{\nu\lambda}\,\overset{1}{R}{}_{\mu\nu\lambda\rho} + \overset{1}{g}{}^{\nu\lambda}\,\overset{o}{R}{}_{\mu\nu\lambda\rho}, \tag{6.24}$$

Consequently, the first-order correction to the Riemann scalar is:

$$\overset{1}{R} = \overset{o}{g}{}^{\mu\nu}\,\overset{1}{R}{}_{\mu\nu} + \overset{1}{g}{}^{\mu\nu}\,\overset{o}{R}{}_{\mu\nu}. \tag{6.25}$$

Finally, The first-order correction to the Einstein tensor, i.e. the left-hand side of Einstein's equations is:

$$\begin{aligned}\overset{1}{G}{}_{\mu\nu} &= \overset{1}{R}{}_{\mu\nu} - \frac{1}{2}\overset{1}{g}{}_{\mu\nu}\overset{o}{R} - \frac{1}{2}\overset{o}{g}{}_{\mu\nu}\overset{1}{R} = \\ &= \overset{1}{R}{}_{\mu\nu} - \frac{1}{2}\overset{o}{g}{}_{\mu\nu}\overset{o}{g}{}^{\lambda\rho}\overset{1}{R}{}_{\lambda\rho} - \frac{1}{2}\overset{o}{g}{}_{\mu\nu}\overset{1}{g}{}^{\lambda\rho}\overset{o}{R}{}_{\lambda\rho} - \frac{1}{2}\overset{1}{g}{}_{\mu\nu}\overset{o}{R}.\end{aligned} \tag{6.26}$$

In what follows, we shall always suppose that the initial Riemannian manifold is a solution of Einstein's equations, i.e. an Einstein space which is Ricci-flat and consequently has zero scalar curvature, too. Therefore the linear correction (of the first order in small parameter ε) to the Einstein tensor will reduce to:

$$\overset{1}{R}{}_{\mu\nu} - \frac{1}{2}\overset{o}{g}{}_{\mu\nu}\overset{o}{g}{}^{\lambda\rho}\overset{1}{R}{}_{\lambda\rho} \tag{6.27}$$

In the absence of any extra gravitating matter (besides the matter generating the basic solution, e.g. the central spherical body for Schwarzschild's solution) the equations to solve can be written in form of a matrix acting on the first-order correction to the Ricci tensor:

$$\left(\delta^\lambda_\mu\,\delta^\rho_\nu - \frac{1}{2}\overset{o}{g}{}_{\mu\nu}\overset{o}{g}{}^{\lambda\rho}\right)\overset{1}{R}{}_{\lambda\rho} = 0 \tag{6.28}$$

But this amounts to the Ricci flatness up to the first order, because the operator acting on the right on the Ricci tensor in (6.28) is non-singular; in fact, it is its own inverse:

$$\left(\delta^{\lambda}_{\kappa}\,\delta^{\rho}_{\sigma} - \frac{1}{2}\overset{\mathrm{o}}{g}_{\kappa\sigma}\overset{\mathrm{o}}{g}{}^{\lambda\rho}\right)\left(\delta^{\kappa}_{\mu}\,\delta^{\sigma}_{\nu} - \frac{1}{2}\overset{\mathrm{o}}{g}_{\mu\nu}\overset{\mathrm{o}}{g}{}^{\kappa\sigma}\right) = \delta^{\lambda}_{\mu}\,\delta^{\rho}_{\nu} \tag{6.29}$$

From this we infer that in an Einsteinian background the first-order correction *in vacuo* should satisfy the equation

$$\overset{1}{R}_{\lambda\rho} = 0. \tag{6.30}$$

In the case when the energy-momentum tensor is present (supposing however that it describes the influence of matter weak enough in order to keep the basic solution unchanged), one must use the full Einstein's tensor on the right-hand side. The first correction, linear in ε, reduces then to only two terms due to the fact that the initial solution is an Einstein space in vacuo so that $\overset{o}{R}_{\lambda\rho} = 0$ and $\overset{o}{R} = 0$:

$$\overset{1}{R}_{\mu\nu} - \frac{1}{2}\overset{\mathrm{o}}{g}_{\mu\nu}\overset{\mathrm{o}}{g}{}^{\lambda\rho}\overset{1}{R}_{\lambda\rho} = \left[\delta^{\lambda}_{\mu}\,\delta^{\rho}_{\nu} - \frac{1}{2}\overset{\mathrm{o}}{g}_{\mu\nu}\overset{\mathrm{o}}{g}{}^{\lambda\rho}\right]\overset{1}{R}_{\mu\nu} = -\frac{8\pi G}{c^4}T_{\mu\nu}, \tag{6.31}$$

and this in turn, due to the idempotent property (6.29), can be written equivalently as

$$\overset{1}{R}_{\mu\nu} = -\frac{8\pi G}{c^4}\left[\delta^{\lambda}_{\mu}\,\delta^{\rho}_{\nu} - \frac{1}{2}\overset{\mathrm{o}}{g}_{\mu\nu}\overset{\mathrm{o}}{g}{}^{\lambda\rho}\right]T_{\lambda\rho} \tag{6.32}$$

which may prove to be more practical for further calculations especially when the energy-momentum tensor has a particularly simple form.

Let us expand the deformed embedding functions into the power series of small parameter ε as before, this time keeping the terms up to the third order:

$$\tilde{z}^A\,(x^\mu) = z^A\,(x^\mu) + \varepsilon\, v^A\,(x^\mu) + \varepsilon^2\, w^A\,(x^\mu) + \varepsilon^3\, h^A\,(x^\mu)\ldots \tag{6.33}$$

The metric tensor $\tilde{g}_{\mu\nu}$ of the deformed embedding is given by the formula (6.3), and the first order correction to the Christoffel connection coefficients was already given in eq. (6.19).

The second order correction to the Christoffel connection can be found from the expansion in powers of ε of the formula (5.10) which is valid also on the deformed manifold:

$$\eta_{AB}\left[\tilde{\nabla}_\lambda\tilde{\nabla}_\mu\tilde{z}^A\,\tilde{\nabla}_\nu\tilde{z}^B\right] = 0. \tag{6.34}$$

From the expansion, identifying the terms proportional to ε^2, one readily gets:

$$\overset{2}{\Gamma}{}^{\lambda}_{\mu\nu} = \eta_{AB}\,\overset{\mathrm{o}}{g}{}^{\lambda\rho}\Big[\nabla_\rho v^A\,\nabla_\mu\nabla_\nu\, v^B + \nabla_\rho w^A\,\nabla_\mu\nabla_\nu\, z^B + \nabla_\rho z^A\,\nabla_\mu\nabla_\nu\, w^B -$$

$$-\overset{1}{\Gamma}{}^{\sigma}_{\mu\nu}\,(\nabla_\rho\, v^A\nabla_\sigma\, z^B + \nabla_\rho\, z^A\nabla_\sigma\, v^B)\Big]. \tag{6.35}$$

As we shall see, the third order corrections to the Christoffel symbols do not appear in the third order expansion term of the Riemann tensor, therefore we do not write them down explicitly here. The second order terms in the expansion of the Riemann tensor are given by the following formula:

$$\begin{aligned}\overset{2}{R}_{\mu\nu\sigma\rho} \; = \; & \eta_{AB}\Big[\nabla_\nu\nabla_\sigma z^A\left(\nabla_\mu\nabla_\rho w^B - \overset{1}{\Gamma}{}^\kappa_{\mu\rho}\nabla_\kappa v^B - \overset{2}{\Gamma}{}^\kappa_{\mu\rho}\nabla_\kappa z^B\right) + \\ & + \left(\nabla_\nu\nabla_\sigma v^A - \overset{1}{\Gamma}{}^\kappa_{\nu\sigma}\nabla_\kappa z^A\right)\left(\nabla_\mu\nabla_\rho v^B - \overset{1}{\Gamma}{}^\kappa_{\mu\rho}\nabla_\kappa z^B\right) + \\ & + \left(\nabla_\nu\nabla_\sigma w^A - \overset{1}{\Gamma}{}^\kappa_{\nu\sigma}\nabla_\kappa v^A - \overset{2}{\Gamma}{}^\kappa_{\nu\sigma}\nabla_\kappa z^A\right)\nabla_\mu\nabla_\rho z^B - (\mu\leftrightarrow\nu)\Big]\end{aligned} \tag{6.36}$$

The terms proportional to $\overset{2}{\Gamma}$ are zero according to (5.10), because they contain the products of first and second covariant derivatives of the original embedding functions contracted with the Euclidean metric, $\eta^{AB}\nabla_\mu\nabla_\lambda z^A\nabla_\nu z^B = 0$. With a little algebra we find:

$$\begin{aligned}\overset{2}{R}_{\mu\nu\sigma\rho} \; = \; & \eta_{AB}\left[\nabla_\nu\nabla_\sigma z^A\nabla_\mu\nabla_\rho w^B + \nabla_\nu\nabla_\sigma w^A\nabla_\mu\nabla_\rho z^B - \right. \\ & - \left.\nabla_\mu\nabla_\sigma z^A\nabla_\nu\nabla_\rho w^B - \nabla_\mu\nabla_\sigma w^A\nabla_\nu\nabla_\rho z^B\right] + \\ & + \eta_{AB}\left[\nabla_\nu\nabla_\sigma v^A\nabla_\mu\nabla_\rho v^B - \nabla_\mu\nabla_\sigma v^A\nabla_\nu\nabla_\rho v^B\right] + \overset{o}{g}_{\kappa\lambda}\left[\overset{1}{\Gamma}{}^\kappa_{\mu\sigma}\overset{1}{\Gamma}{}^\lambda_{\nu\rho} - \overset{1}{\Gamma}{}^\kappa_{\nu\sigma}\overset{1}{\Gamma}{}^\lambda_{\mu\rho}\right],\end{aligned}$$

where the $\overset{1}{\Gamma}\overset{1}{\Gamma}$ terms contain only z and v functions. To write down the second-order correction to Einstein equations we need also the explicit form of the correction to the Ricci tensor:

$$\overset{2}{R}_{\nu\rho} = \overset{o}{g}{}^{\mu\sigma}\overset{2}{R}_{\mu\nu\sigma\rho} + \overset{1}{g}{}^{\mu\sigma}\overset{1}{R}_{\mu\nu\sigma\rho} + \overset{2}{g}{}^{\mu\sigma}\overset{o}{R}_{\mu\nu\sigma\rho} \tag{6.37}$$

and the scalar curvature

$$\overset{2}{R} = \overset{o}{g}{}^{\mu\sigma}\overset{2}{R}_{\mu\sigma} + \overset{1}{g}{}^{\mu\sigma}\overset{1}{R}_{\mu\sigma} + \overset{2}{g}{}^{\mu\sigma}\overset{o}{R}_{\mu\sigma} \tag{6.38}$$

with the w-functions contained only in $\overset{2}{R}_{\mu\nu\sigma\rho}$ and in $\overset{2}{g}{}^{\mu\sigma}$.

Obviously, if the first-order deviation was Einsteinian, i.e. not only the initial space-time was Ricci-flat, but also $R^1{}_{\nu\rho} = 0$, the second-order correction to the scalar curvature contains only the first contribution, $g^{o\,\mu\sigma}R^2{}_{\mu\sigma}$, and the correction to Einstein's equations reduces again to the vanishing second-order Ricci tensor, $R^2{}_{\mu\sigma} = 0$.

Finally, the third order correction to the Riemann tensor can be obtained by expanding the Gauss-Codazzi formula applied to the deformed manifold:

$$\tilde{R}_{\mu\nu\lambda\rho} = -\eta_{AB}\left[\tilde{\nabla}_\mu\tilde{\nabla}_\lambda\,\tilde{z}^A\tilde{\nabla}_\nu\nabla_\rho\,\tilde{z}^B - \tilde{\nabla}_\nu\tilde{\nabla}_\lambda\,\tilde{z}^A\,\tilde{\nabla}_\mu\tilde{\nabla}_\rho\,\tilde{z}^B\right]. \tag{6.39}$$

and expanding all the quantites in powers of ε. This gives the following result:

$$\begin{aligned}\overset{3}{R}_{\mu\nu\sigma\rho} \; = \; & \eta_{AB}\left[\nabla_\nu\nabla_\sigma z^A\nabla_\mu\nabla_\rho h^B + \nabla_\nu\nabla_\sigma h^A\nabla_\mu\nabla_\rho z^B + \right. \\ & + \left.\nabla_\mu\nabla_\sigma v^A\nabla_\nu\nabla_\rho w^B + \nabla_\mu\nabla_\sigma w^A\nabla_\nu\nabla_\rho v^B - (\mu\leftrightarrow\nu)\right] + \\ & + \left[\overset{o}{g}_{\kappa\lambda}\left(\overset{2}{\Gamma}{}^\kappa_{\mu\sigma}\overset{1}{\Gamma}{}^\lambda_{\nu\rho} + \overset{1}{\Gamma}{}^\kappa_{\mu\sigma}\overset{2}{\Gamma}{}^\lambda_{\nu\rho}\right) + \overset{1}{g}_{\kappa\lambda}\overset{1}{\Gamma}{}^\kappa_{\mu\sigma}\overset{1}{\Gamma}{}^\lambda_{\nu\rho} - (\mu\leftrightarrow\nu)\right],\end{aligned} \tag{6.40}$$

Again, in order to write down the third correction to Einstein's equations, we should get the explicit expression for the third order correction to the Ricci tensor.

$$\overset{3}{R}_{\nu\rho} = \overset{o}{g}{}^{\mu\sigma}\overset{3}{R}_{\mu\nu\sigma\rho} + \overset{1}{g}{}^{\mu\sigma}\overset{2}{R}_{\mu\nu\sigma\rho} + \overset{2}{g}{}^{\mu\sigma}\overset{1}{R}_{\mu\nu\sigma\rho} + \overset{3}{g}{}^{\mu\sigma}\overset{o}{R}_{\mu\nu\sigma\rho}, \tag{6.41}$$

Again, because the third-order correction to the scalar curvature is given by the formula

$$\overset{3}{R} = \overset{o}{g}{}^{\mu\sigma}\overset{3}{R}_{\mu\sigma} + \overset{1}{g}{}^{\mu\sigma}\overset{2}{R}_{\mu\sigma} + \overset{2}{g}{}^{\mu\sigma}\overset{1}{R}_{\mu\sigma} + \overset{3}{g}{}^{\mu\sigma}\overset{o}{R}_{\mu\sigma}, \tag{6.42}$$

and if the previous two orders were Ricci-flat, the third order correction to Einstein's equations will reduce to the Ricc-flatness of the third order, $\overset{3}{R}_{\mu\nu} = 0$.

7. Plane waves in a flat background space-time

In a Minkowskian space-time M_4 parameterized by cartesian coordinates $x^\mu = [ct, x, y, z]$ all connection coefficients identically vanish, as well as the components of the Riemann and Ricci tensors. The flat Minkowskian space can be embedded as a hyperplane in any pseudo-Euclidean space with more than four dimensions and signature $(1+, (N-1)-)$. Let us choose the simplest case of embedding in five dimensions:

$$M_4 \to E^5_{1,4}$$

with the first four components denoting a Minkowskian space-time vector in cartesian coordinates:

$$z^1 = ct,\, z^2 = x,\, z^3 = y,\, z^4 = z, \quad z^5 = 0, \tag{7.1}$$

the last cartesian coordinate considered as an extra dimension of $E^5_{1,4}$ orthogonal to the M_4 hyperplane. All covariant derivatives in (6.21) can be replaced by partial derivatives, and all second derivatives of linear embedding functions are identically zero. Therefore in order to investigate non trivial deformations of the Minkowskian space embedded as a hyperplane we must go the second order in ε. This leads to the following equation resulting from the requirement of vanishing of the Ricci tensor:

$$\overset{2}{R}_{\mu\rho} \; = 0 \Longrightarrow \overset{o}{g}{}^{\lambda\nu}\eta_{AB}\Big[\nabla_\mu\nabla_\lambda\, v^A\,\nabla_\nu\nabla_\rho\, v^B - \nabla_\nu\nabla_\lambda\, v^A\,\nabla_\mu\nabla_\rho\, v^B\Big] = 0$$

We shall not consider infinitesimal deformations of the first four coordinates because they coincide with coordinate transformations in V_4; therefore the only non vanishing component of v^A is the remaining fifth coordinate deformation, expanded in a series of powers of ε:

$$z^5 = \varepsilon\, v(x^\mu) + \varepsilon^2\, w(x^\mu) + \varepsilon^3\, h(x^\mu) + \ldots$$

In order to keep the Einstein equations satisfied after deformation up to the second order terms, we must have

$$\overset{2}{R}_{\mu\rho} = \overset{o}{g}{}^{\lambda\nu}\Big[\nabla_\mu\nabla_\lambda\, v\,\nabla_\nu\nabla_\rho\, v - \nabla_\nu\nabla_\lambda\, v\,\nabla_\mu\nabla_\rho\, v\Big] = 0 \tag{7.2}$$

Any function of linear combination of cartesian coordinates is an obvious solution of Eq. (7.2). Indeed, if we set:

$$v(x^{\mu}) = f\,(k_{\mu}\,x^{\mu}) \tag{7.3}$$

inserting the derivatives of $v(k_{\mu}\,x^{\mu})$ into (7.2) results in the following simple equation :

$$\overset{o}{g}{}^{\lambda\nu}\left[k_{\mu}k_{\lambda}\,k_{\nu}k_{\rho}\,v'^{2} - k_{\nu}k_{\lambda}\,k_{\mu}k_{\rho}\,v'^{2}\right] = k_{\nu}k^{\nu}\,v'^{2}\left[k_{\mu}k_{\rho} - k_{\rho}k_{\mu}\right] = 0, \tag{7.4}$$

But in fact, this deformation does not have any physical meaning, because the Riemann tensor, which is the only observable quantity, identically vanishes:

$$\overset{2}{R}_{\mu\nu\lambda\rho} = \left[k_{\mu}\,k_{\lambda}\,k_{\nu}\,k_{\rho} - k_{\nu}\,k_{\lambda}\,k_{\mu}\,k_{\rho}\right] = 0 \tag{7.5}$$

The vanishing of the Riemann tensor is not surprising, because the deformation considered looks like a deformation of a plane into a cylinder, which does not alter its intrinsic flat geometry.

The fact that there are no wave-like solutions at the first order of deformation of Minkowskian spacetime suggests that the same situation will prevail when we shall investigate other Einsteinian manifolds embedded in a pseudo-Euclidan flat space, e.g. the Schwarzschild solution. If the contrary was true, one could keep the wave-like propagating deformations also in the flat limit, which would contradict the absence of such solutions among the first-order deformations of the Minkowskian space-time.

This means that the only hope to produce contributions to the Riemann tensor behaving like a propagating gravitational field, i.e. the gravitational waves, is to consider the third (and higher) order deformations of embedded Einsteinian manifolds. The third order variation for the Riemann tensor in the case of deformations of all orders orthogonal to the embedded manifold reduces to the formula (6.40) given in the previous section.

The linear contribution coming from the expressions containing third-order deviation linearly does vanish because the derivatives of the corresponding z^5 coordinate are identically zero.

A wave-like behavior of the Riemann tensor can be produced if we assume that w depends on variables orthogonal to the worldlines parallel to the vector k. For the sake of simplicity, let us start with the first order deformation in the direction of fifth coordinate, i.e. orthogonal to the embedded Minkowskian hyperplane M_4 as a plane wave propagating along the z-axis:

$$v(x^{\mu}) = A\,\cos\left(\omega t - kz\right).$$

According to our general analysis, by virtue of (7.5), this deformation does not contribute to the Riemann tensor, which remains zero even at the second order. Now let us add up the second order deformation depending on the variables x and y only:

$$z^5 = \varepsilon A\,\cos\left(\omega t - kz\right) + \varepsilon^2\,w(x,y) \tag{7.6}$$

The only contribution to the third order correction to the Riemann tensor has the form given by the formula (6.40) in which the covariant derivatives can be replaced by partial derivatives given that all Christoffel symbols vanish in cartesian coordinates. The function $w(x,y)$ must have some non vanishing second order derivatives; let us make the simplest choice and set $w(x,y) = Bxy$, with B = Const having the dimension cm^{-1}.

Then the only non vanishing second derivative is $\partial^2_{xy} w = B$. Taking into account the form of (6.40), the only non vanishing components are:

$$\overset{3}{R}_{\mu x y \rho} = \left(\partial^2_{\mu y}\, v\, \partial^2_{x\rho}\, w + \partial^2_{\mu y}\, w\, \partial^2_{x\rho}\, v - \partial^2_{xy}\, v\, \partial^2_{\mu\rho}\, w - \partial^2_{xy}\, w\, \partial^2_{\mu\rho}\, v \right) \tag{7.7}$$

and all other components obtained from this one by permutations of indexes allowed by the well known symmetries of Riemann's tensor, like e.g. $\overset{3}{R}_{x\mu\rho y}$, etc.

Now, given that v does not depend on x and on y, the only non vanishing term in (7.7) is the one containing $\partial_x \partial_y w = B$; so that we have

$$\overset{3}{R}_{\mu x y \rho} = -\partial^2_{xy}\, w\, \partial^2_{\mu\rho}\, v = -B\, \partial^2_{\mu\rho}\, v \tag{7.8}$$

There is no contribution to the Ricci tensor coming from $\overset{o}{g}{}^{xy}\, \overset{3}{R}_{\mu x y \rho}$ because the Minkowskian metric tensor is diagonal and $\overset{o}{g}{}^{xy} = 0$; therefore, to make the Ricci tensor vanish up to the third order means that the following equation must be satisfied:

$$\overset{o}{g}{}^{\mu\rho}\, \overset{3}{R}_{\mu x y \rho} = -B\, \overset{o}{g}{}^{\mu\rho}\, \partial^2_{\mu\rho}\, v = 0 \tag{7.9}$$

This is the wave equation for v, imposing the dispersion relation $\omega^2 = c^2 k^2$.

This particular form of the "modulating" function $w(x,y)$ can be easily generalized. As a first step, let us consider an arbitrary quadratic form in variables x and y: let us put

$$w = Ax^2 + Bxy + Cy^2$$

Besides the non vanishing component $\overset{3}{R}_{\mu x y \rho}$, two other components of Riemann tensor will appear now:

$$\overset{3}{R}_{\mu x x \rho} = 2A, \text{ and } \overset{3}{R}_{\mu y y \rho} = 2C$$

which have the same structure as the (x,y) component (7.8):

$$\begin{aligned} \overset{3}{R}_{\mu x x \rho} &= -\partial^2_{xx}\, w\, \partial^2_{\mu\rho}\, v = -2A\, \partial^2_{\mu\rho}\, v, \\ \overset{3}{R}_{\mu y y \rho} &= -\partial^2_{yy}\, w\, \partial^2_{\mu\rho}\, v = -2C\, \partial^2_{\mu\rho}\, v, \end{aligned} \tag{7.10}$$

The components (xx), (xy) and (yy) of the Ricci tensor vanish if the same condition (7.9) is satisfied; but now we shall also make sure that all other components of the Ricci tensor vanish, too, which will be true if the following trace is zero:

$$\begin{aligned} \overset{o}{g}{}^{xx} \overset{3}{R}_{\mu x x \rho} + \overset{o}{g}{}^{yy} \overset{3}{R}_{\mu y y \rho} &= -\partial^2_{xx}\, w\, \partial^2_{\mu\rho}\, v - \partial^2_{xx}\, w\, \partial_\mu \partial_\rho v \\ &= -(2A + 2C)\, \partial^2_{\mu\rho}\, v \end{aligned} \tag{7.11}$$

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leading to the extra condition on the coefficients A and C, namely, $A = -C$, thus leaving only two degrees of freedom for the function w. This suggests the *quadrupolar* character of the gravitational wave, which deforms the space simultaneously in two directions perpendicular to the direction of propagation; notice that if w depended only on one transversal variable, say x, the vanishing of the Ricci tensor would impose $w = 0$ (or a constant, which would not have any physical meaning at all).

The same is true for any homogeneous polynomial of two variables x and y, provided it satisfies the two-dimensional Laplace equation $\partial^2_{xx} w + \partial^2_{yy} w = 0$. Finally, we can generalize our result by stating that the deformation of Minkowskian space-time embedded as a hyperplane in an five-dimensional Euclidean ambient space leads to the vanishing of the Ricci tensor up to the third order in small parameter ε if it has the form

$$z^5 = \varepsilon \cos(\omega t - kz) + \varepsilon^2 w(x,y) + O(\varepsilon^3) \tag{7.12}$$

provided $\omega^2 = c^2 k^2$ and w satisfies the two-dimensional Laplace equation $\nabla^2 w = 0$. The only non-vanishing components of the Riemann tensor are then

$$\overset{3}{R}_{txyt},\ \overset{3}{R}_{txxz},$$

Taking into account that the corresponding Riemann tensor $\varepsilon^3 \overset{3}{R}_{\mu\nu\lambda\rho}$ is linear both in v and w, we can compose by superposition a transversally polarized plane wave of arbitrary shape and spectrum, propagating with the phase velocity equal to the speed of light.

However, although from a purely mathematical point of view an $w(x,y)$ satisfying the two-dimensional Laplace equation becomes a solution to the third-order correction to Einstein's equations, we claim that only the quadratic functions of x and y represent a physically acceptable case. This is because constant and linear functions w make vanish not only the Ricci tensor, but also Riemann's tensor as well (due to the second derivatives of the zero-order *linear* embedding functions z^A).

The quadratic functions w lead to a constant (on the x,y-plane) Riemann tensor, multiplied by the oscillating factor $\cos(\omega t - kz)$, which is acceptable for an infinite plane wave.

But if we take a cubic or higher degree polynomial in x,y as the modulating function w, the Riemann tensor will become linear (or higher degree) in x,y, becoming infinite at spatial infinity, which is physically unacceptable. Therefore only the second degree polynomial fully characterizes the plane gravitational wave, conveying it only two degrees of freedom. The two independent polynomials being $x^2 - y^2$ and xy, one can form their complex combination $x^2 - y^2 + 2ixy$, which under the rigid rotation in the $x - y$ plane, $x + iy \rightarrow e^{i\phi}\,(x + iy$ will take on the factor $e^{2i\phi}$, typical for the spin 2fields.

8. Spherical waves in a flat background

The particular form of plane wave solution suggests also the form of a spherical wave. The first-order deformation v^A far from the source should contain a factor propagating in radial direction, while the second-order deformation w^A should depend on the angular variables. We should not expect total vanishing of the second-order correction to the Riemann tensor like it happened in the case of plane waves. It is important that there will be no propagating terms at that order of approximation; static terms vanishing at spatial infinity like r^{-2} or r^{-3} can be neglected and in fact describe the approximation to the static part of the space-time deformation inevitably produced by the source of spherical gravitational waves.

Let us start with the first-order deformation of Minkowskian space-time embedded as a hyperplane in some pseudo-Euclidean space; it has one component along one extra dimension perpendicular to the Minkowskian hyperplane M_4. We suppose that is depends on the variables r and t only:

$$v^5 = v^5\,(t,\,r) \tag{8.1}$$

Being perpendicular to the embedded manifold M_4 as seen from the host space, this deformation does not contribute to the first-order correction to the Riemann tensor. In order to evaluate the second-order correction to the Riemann tensor, $\overset{2}{R}_{\mu\nu\lambda\rho}$, we need to insert the expressions for second covariant derivatives of v. In a flat space parameterized by spherical coordinates the non-vanishing Christoffel symbols are:

$$\begin{array}{llll} \Gamma^r_{\theta\theta} & = -r & \Gamma^r_{\varphi\varphi} & = -r\sin^2\theta \\ \Gamma^\theta_{r\theta} & = r^{-1} & \Gamma^\varphi_{r\phi} & = r^{-1} \\ \Gamma^\theta_{\varphi\varphi} & = -\sin\theta\cos\theta & \Gamma^\phi_{\theta\varphi} & = \frac{\cos\theta}{\sin\theta}. \end{array}$$

and in the case when v is a function only of t and r the non-vanishing combinations are:

$$\nabla_t\nabla_r\, v = \partial^2_{tr}\, v, \quad \nabla_t\nabla_t\, v = \partial^2_{tt}\, v$$

$$\nabla_r\nabla_r\, v = \partial^2_{rr}\, v, \quad \nabla_\theta\nabla_\theta\, v = r\,\partial_r v, \;\; \nabla_\varphi\nabla_\varphi\, v = r\,\sin^2\theta\,\partial_r v \tag{8.2}$$

We consider the same deformation of the fifth coordinate as before:

$$z^5 = \varepsilon\, v^5(x^\mu) + \varepsilon^2\, w^5(x^\mu) + \varepsilon^3\, h^5(x^\mu) + \ldots$$

The first order deformation of the Ricci tensor is still zero, because the deformation is performed orthogonally to the embedded manifold. In the case of the Minkowskian background even the first-order correction to the Riemann tensor is automatically zero. This is easy to see in cartesian coordinates, when all second derivatives of the *linear* embedding functions do vanish identically. This will remain true in any curvilinear coordinate system, too.

In order to make vanish also the the second order correction to the Einstein equations *in vacuo* we have to satisfy the eq. (7.2). Because we are looking for radiative solutions, we shall neglect all terms which decay at spatial infinity more rapidly than $1/r$, keeping only the radiative part. Let us set

$$v^5(x^\mu) = \frac{A\,\cos\,(\omega t - kr)}{r}. \tag{8.3}$$

With the choice for $v^5(t,r)$ given above (8.3), it is easy to see that $\overset{1}{R}_{\mu\nu\lambda\rho} = 0$ and the only components of $\overset{2}{R}_{\mu\,\nu\,\lambda\,\rho}$ that we need to calculate are:

$$\overset{2}{R}_{t\,r\,r\,t} \quad \overset{2}{R}_{t\,\theta\,\theta\,t} \quad \overset{2}{R}_{t\,\theta\,\theta\,r} \quad \overset{2}{R}_{r\,\theta\,\theta\,r} \quad \overset{2}{R}_{\theta\,\phi\,\phi\,\theta},$$

The explicit calculus yields the following result (in what follows, in order to make the formulae shorter, we have put $\Omega = \omega t - kr$):

$$\overset{2}{R}_{t\,r\,r\,t} = -\frac{A^2\omega^2}{c^2 r^4} - \frac{A^2\omega^2\cos^2\Omega}{c^2 r^4}$$

$$\overset{2}{R}_{t\,\theta\,\theta\,t} = -\frac{A^2\omega^2 k}{c^2 r}\sin\Omega\cos\Omega + \frac{A^2\omega^2\cos^2\Omega}{c^2 r^2}$$

$$\overset{2}{R}_{t\,\theta\,\theta\,r} = \frac{A^2\omega k^2}{cr}\sin\Omega cos\Omega + \frac{A^2\omega k}{cr^2}\left(\sin^2\Omega - \cos^2\Omega\right) - \frac{A^2\omega}{cr^3}\sin\Omega\cos\Omega$$

$$\overset{2}{R}_{r\,\theta\,\theta\,r} = -\frac{A^2k^3}{r}\sin\Omega\cos\Omega + \frac{A^2k^2}{r^2}\cos^2\Omega - \frac{2A^2k^2}{r^2}\sin^2\Omega +$$

$$+\frac{4A^2k}{r^3}\sin\Omega\cos\Omega - \frac{2A^2}{r^4}\cos^2\Omega$$

$$\overset{2}{R}_{\theta\,\varphi\,\varphi\,\theta} = \sin^2\theta\left(A^2k^2\cos^2\Omega + \frac{2Ak}{r}\sin\Omega\cos\Omega + \frac{A^2}{r^2}\sin^2\Omega\right)$$

Among the second-order corrections to the Ricci tensor there are only two non-vanishing components of radiative character, i.e. behaving at infinity like $1/r$:

$$\overset{2}{R}_{\varphi\varphi} = \sin^2\theta\,\overset{2}{R}_{\theta\theta} \sim \left(k^2c^2 - \omega^2\right)\,\sin^2\theta\,\frac{\sin(\Omega)\,\cos(\Omega)}{r} \tag{8.4}$$

and they do vanish provided that $\omega^2 = k^2c^2$. Nevertheless, although the presence of terms behaving at infinity like r^{-1} could be interpreted as a gravitational wave, it should not be there at spatial infinity, i.e. in the plane wave limit, where the wave-like behavior could be observed only in the corrections of the third order; what is worse, we get everywhere the quadratic expressions like $\sin\Omega$ and $\cos\Omega$, which will give rise to terms like $\sin 2\Omega$ and $\cos 2\Omega$. whose frequency is the double of the basic frequency Ω, and there is no reason to observe such frequencies in a gravitational wave whose source was supposed to oscillate with the frequency Ω. The same is also true for the components of the second-order correction to the metric tensor, which is also quadratic in the first derivatives of v.

The very form of these undesirable terms suggests how they can be suppressed. It is quite clear that in order to cancel double frequencies (quadratic terms) we should add a deformation toward an additional (here sixth) dimension, of the form:

$$v^6(t,r) = A\,\frac{\sin(\omega t - kr)}{r} \equiv A\,\frac{\sin(\Omega)}{r}$$

(recall that $v^5(t,r) = A\,r^{-1}\,\cos\Omega$). The contributions to the second-order corrections to Riemann's tensor $\overset{2}{R}_{\mu\,\nu\,\lambda\,\rho}$ are the same as the ones coming from $v^5(t,r)$, with the substitution

$$\cos\Omega \to \sin\Omega, \quad \sin\Omega \to -\cos\Omega$$

Now all the cross terms $\sin\Omega\cos\Omega$ cancel each other, and for each term containing $\sin^2\Omega$ there will be a corresponding term with $\cos^2\Omega$ instead, so that their sum will produce a constant.

With two simultaneous deformations in two extra dimensions, opposite in phase, $v^5(t,r) = Ar^{-1}\cos\Omega$ and $v^6(t,r) = Ar^{-1}\sin\Omega$, there are no more radiative terms in the second-order Riemann tensor, whose only non-vanishing components are the following:

$$\overset{2}{R}_{trrt} = -\frac{3A^2\omega^2}{c^2r^4} \quad \overset{2}{R}_{t\theta\theta t} = \frac{A^2\omega^2}{c^2r^2}$$

$$\overset{2}{R}_{r\theta\theta r} = -\frac{2A^2k^2}{r^2} - \frac{2A^2}{r^4} \quad \overset{2}{R}_{\theta\varphi\varphi\theta} = \sin^2\theta\,\left(A^2k^2 + \frac{A^2}{r^2}\right)$$

$$\overset{2}{R}_{t\theta\theta r} = 0,\ \overset{2}{R}_{t\varphi\varphi r} = 0. \tag{8.5}$$

and of course, $\overset{2}{R}_{t\varphi\varphi t} = \sin^2\theta\,\overset{2}{R}_{t\theta\theta t},\quad \overset{2}{R}_{r\varphi\varphi r} = \sin^2\theta\,\overset{2}{R}_{r\theta\theta r}$.

The terms behaving like r^{-2} or r^{-4} can be dealt with and eventually cancelled via introducing extra dimensions. By the same token, the correction to the metric tensor does not contain any time-dependent functions, being now reduced to the following form:

$$\overset{2}{g}_{tt} = \frac{\omega^2A^2}{c^2r^2},\quad \overset{2}{g}_{tr} = \overset{2}{g}_{rt} = \frac{\omega kA^2}{cr^2},\quad \overset{2}{g}_{rr} = \frac{k^2A^2}{r^2}. \tag{8.6}$$

A short calculus gives the non vanishing components of the second-order Ricci tensor:

$$\overset{2}{R}_{tt} = \frac{A^2\omega^2}{c^2r^4},\qquad \overset{2}{R}_{rr} = -\frac{3A^2\omega^2}{c^2r^4} + \frac{2A^2k^2}{r^4} + \frac{4A^2}{r^6}$$
$$\overset{2}{R}_{\theta\theta} = \frac{A^2\omega^2}{c^2r^2} + \frac{A^2}{r^4},\qquad \overset{2}{R}_{\varphi\varphi} = \sin^2\theta\,\left(\frac{A^2\omega^2}{c^2r^2} + \frac{A^2}{r^4}\right). \tag{8.7}$$

As in the plane wave case, we expect the radiative behavior to appear in the third order of approximation. But now Einstein's equations in vacuo can be expanded not only in a series of powers of small deformation parameter ε, but each term of this expansion can be represented as a series of negative powers of r. Therefore, our strategy is based on the analysis of the radiative terms first (behaving at spatial infinity as r^{-1}), considering all more rapidly decaying terms as negligible at this stage.

The third order correction to the Riemann tensor is given by the equation (6.40), from which we can easily calculate the third-order correction to the Ricci tensor. Note that in the third order, the Ricci tensor contains only the contraction of the unperturbed metric tensor $\overset{o}{g}{}^{\mu\nu}$ with the third-order Riemann tensor, because in this particular case $\overset{1}{g}{}^{\mu\nu} = 0$. We shall suppose that the second-order deformations represent as before a monochromatic spherical wave, whereas the third-order deformation functions w^A do not depend on time t, $w^A = w^A(r,\theta,\varphi)$, $A = 5$ or 6.. Under these assumptions the ten independent components of the third-order correction to the Ricci tensor are as follows

$$\overset{3}{R}_{tt} = \nabla_t\nabla_t v\left[-\nabla_r\nabla_r w + \overset{o}{g}{}^{\theta\theta}\nabla_\theta\nabla_\theta w + \overset{o}{g}{}^{\varphi\varphi}\nabla_\varphi\nabla_\varphi w\right],$$

$$\overset{3}{R}_{tr} = \nabla_t\nabla_r v\left[\overset{o}{g}{}^{\theta\theta}\nabla_\theta\nabla_\theta w + \overset{o}{g}{}^{\varphi\varphi}\nabla_\varphi\nabla_\varphi w\right],$$

$$\overset{3}{R}_{rr} = \nabla_r\nabla_r v \left[\overset{\circ}{g}{}^{\theta\theta}\nabla_\theta\nabla_\theta w + \overset{\circ}{g}{}^{\varphi\varphi}\nabla_\varphi\nabla_\varphi w\right] + \nabla_r\nabla_r w \left[\nabla_t\nabla_t v + \overset{\circ}{g}{}^{\theta\theta}\nabla_\theta\nabla_\theta v + \overset{\circ}{g}{}^{\varphi\varphi}\nabla_\varphi\nabla_\varphi v\right],$$

$$\overset{3}{R}_{t\theta} = \nabla_t\nabla_r v\nabla_r\nabla_\theta\, w \qquad \overset{3}{R}_{t\varphi} = \nabla_t\nabla_r v\nabla_r\nabla_\varphi\, w,$$

$$\overset{3}{R}_{r\theta} = \left[\nabla_t\nabla_t\, v + \overset{\circ}{g}{}^{\varphi\varphi}\nabla_\varphi\nabla_\varphi\, v\right]\,\nabla_r\nabla_\theta\, w, \quad \overset{3}{R}_{r\varphi} = \left[\nabla_t\nabla_t\, v + \overset{\circ}{g}{}^{\theta\theta}\nabla_\theta\nabla_\theta\, v\right]\,\nabla_r\nabla_\varphi\, w,$$

$$\overset{3}{R}_{\theta\theta} = [\nabla_r\nabla_r\, v - \nabla_t\nabla_t\, v]\;\nabla_\theta\nabla_\theta\, w + \frac{\partial_r v}{r}\left[\nabla_\theta\nabla_\theta\, w + \frac{1}{\sin^2\theta}\,\nabla_\varphi\nabla_\varphi\, w\right] - \nabla_r\nabla_r w\nabla_\theta\nabla_\theta v$$

$$\overset{3}{R}_{\theta\varphi} = [\nabla_r\nabla_r\, v - \nabla_t\nabla_t\, v]\;\nabla_\theta\nabla_\varphi\, w$$

$$\overset{3}{R}_{\varphi\varphi} = [\nabla_r\nabla_r\, v - \nabla_t\nabla_t\, v]\;\nabla_\varphi\nabla_\varphi\, w + \sin^2\theta\,\frac{\partial_r\, v}{r}\left[\nabla_\theta\nabla_\theta\, w + \frac{1}{\sin^2\theta}\,\nabla_\varphi\nabla_\varphi\, w\right]$$

In these formulae we displayed only the generic form, common for both indices $A = 5$ or 6, and replaced $\overset{\circ}{g}{}^{tt}$ by 1 and $\overset{\circ}{g}{}^{rr}$ by -1. It is quite easy to determine the behavior of each of these components at spatial infinity, $r \to \infty$. Let us do it systematically, by groups of terms displaying similar behavior:

- The last three components, $\overset{3}{R}_{\theta\theta}$, $\overset{3}{R}_{\theta\varphi}$ and $\overset{3}{R}_{\varphi\varphi}$, contain the common factor

$$[\nabla_r\nabla_r\, v - \nabla_t\nabla_t\, v] = \left(\frac{\omega^2}{c^2} - k^2\right)\frac{A\cos\Omega}{r} - \frac{2kA\,\sin\Omega}{r^2} - \frac{2A\cos\Omega}{r^3}. \tag{8.8}$$

The r^{-1}-like term is cancelled if the dispersion relation $k^2c^2 = \omega^2$ is satisfied, which we shall supposed true from now on. Then only the terms behaving like r^{-2} are left.

Other two terms present in $\overset{3}{R}_{\theta\theta}$ and $\overset{3}{R}_{\varphi\varphi}$ contain the factor $r^{-1}\,\partial_r\, v$, which also behaves like r^{-2} at least. The last remaining expression that may cause trouble is $-\nabla_r\nabla_r w\nabla_\theta\nabla_\theta v$; by an appropriate choice of the function $w(r,\theta,\varphi)$we shall ensure that it also behaves like r^{-3} at least.

Therefore, there are no radiative terms at infinity in the last three components of the Ricci tensor

- The first three components, $\overset{3}{R}_{tt}$, $\overset{3}{R}_{tr}$ and $\overset{3}{R}_{rr}$, contain the common factor corresponding to the angular part of the three-dimensional Laplace operator,

$$\left[\overset{\circ}{g}{}^{\theta\theta}\nabla_\theta\nabla_\theta w + \overset{\circ}{g}{}^{\varphi\varphi}\nabla_\varphi\nabla_\varphi w\right] = -\frac{1}{r^2}\left[\partial^2_{\theta\theta}\, w + \frac{\cos\theta}{\sin\theta}\,\partial_\theta\, w + \frac{1}{\sin^2\theta}\,\partial^2_{\varphi\varphi}\, w\right], \tag{8.9}$$

which behaves like r^{-2}; besides, the factors containing the derivatives of the first-order correction v all start with the term proportional to r^{-1}, which makes the components $\overset{3}{R}_{tt}$, $\overset{3}{R}_{tr}$ and $\overset{3}{R}_{rr}$ behave at least like r^{-3}.

- Finally, the remaining four components, $\overset{3}{R}_{t\theta}$, $\overset{3}{R}_{t\varphi}$, $\overset{3}{R}_{r\theta}$ and $\overset{3}{R}_{r\varphi}$, contain, besides the factors with second derivatives of v, the factor $\nabla_r\nabla_\theta\, w$ or $\nabla_r\nabla_\varphi\, w$. Explicitly, these expressions are given by the formulae

$$\nabla_r\nabla_\theta\, w = \partial^2_{r\theta}\, w - \frac{1}{r}\,\partial_\theta\, w, \;\; \nabla_r\nabla_\varphi\, w = \partial^2_{r\varphi}\, w - \frac{1}{r}\,\partial_\varphi\, w. \tag{8.10}$$

Now, there is an obvious solution to the above equations which consists in choosing the linear dependence on r,

$$w(x^\mu) = r\,Q\,(\theta,\phi) \tag{8.11}$$

This ansatz cancels identically the four components of the Ricci tensor, $\overset{3}{R}_{t\theta}$, $\overset{3}{R}_{t\varphi}$, $\overset{3}{R}_{r\theta}$ and $\overset{3}{R}_{r\varphi}$, but the function $Q(\theta,\varphi)$ inserted in the last three components will produce the leading factor behaving like r^{-1}, which we wanted to avoid.

However, there is no reason to require from these four components of the third-order correction to the Ricci tensor to vanish identically, while keeping the terms behaving like r^{-2} in other components. In fact, there is an alternative choice, admitting that the functions w^A can be developed into series of negative powers of r, starting with the zero-power term:

$$w^A\,(r,\theta,\varphi) = \sum_{p=0}^{\infty} r^{-p}\, w^A_p(\theta,\varphi), \tag{8.12}$$

This form of the second-order deformation makes the terms (8.10) decay at spatial infinity like r^{-1} or faster, and taking into account that the terms with the second derivatives of v's also start with the dependence r^{-1} , we are sure that all the components of the third-order correction to the Ricci tensor have their leading term at infinity behaving as r^{-2}; still, certain components present in the Riemann tensor will behave like r^{-1}, which can be interpreted as a spherical gravitational wave.

What we should prove now is that although we have the approximate Ricci-flatness satisfied asymptotically (up to the terms behaving like r^{-1} at spatial infinity), the Riemann tensor has at least some components behaving like r^{-1}, so that there is a spherical gravitational wave at spatial infinity. Inserting the second-order deformations $(A,B=5,6)$ given by (8.12) along with the spherical wave found in the second-order approximation into the formula for the third-order correction to the Riemann tensor (6.40), and keeping only the leading terms in negative powers of r, we observe (see the full list of terms in the Appendix 1) that only the first two terms in the expansion of the second-order ($\sim \varepsilon^2$) deformations w^A, namely

$$w^A_0(\theta,\varphi) + \frac{w^A_1(\theta,\varphi)}{r},$$

can produce, when combined with the first-order deformation $v^B \sim \frac{A\cos(\omega t - kr)}{r}$, the terms behaving at infinity like r^{-1}. But there is one component,

$$\overset{3}{R}_{\theta\varphi\theta\varphi} = \nabla_\theta \nabla_\theta\, v \nabla_\varphi \nabla_\varphi\, w + \nabla_\varphi \nabla_\varphi\, v \nabla_\theta \nabla_\theta\, w$$

whose leading term is independent of r even at spatial infinity:

$$k\sin(\Omega)\left[\partial^2_{\varphi\varphi} w_0(\theta,\varphi) - \sin\theta\cos\theta\,\partial_\theta\, w_0(\theta,\varphi) + \sin^2\theta\,\partial^2_{\theta\theta} w_0(\theta,\varphi)\right]$$

The obvious solution is $w_0 = Constant$, but this corresponds to a pure gauge and does not change the induced metric. Other solutions should contain the dependence on the azimuthal angle φ respecting periodicity, i.e. they must be proportional to $e^{im\varphi}$, which will result in the following condition

$$\left[-m^2\, w_0(\theta,\varphi) - \sin\theta\cos\theta\,\partial_\theta w_0(\theta,\varphi) + \sin^2\theta\,\partial^2_{\theta\theta} w_0(\theta,\varphi)\right] = 0, \tag{8.13}$$

that will ensure the vanishing of the radiative part of the component $\overset{3}{R}_{\theta\varphi\theta\varphi}$ of the Riemann tensor. This is in agreement with our result for the plane gravitational wave, for which the component

$\overset{3}{R}_{xyxy}$ was zero when the direction of propagation was along the z-axis; in the case of a spherical wave the propagation is along the radial direction, so that the "purely transversal" component of the Riemann tensor becomes precisely $R_{\theta\varphi\theta\varphi}$.

Now, if we set $m = 0$, the obvious solution is $Q = K\cos\theta$; if $m \neq 0$, the solution is the combination of two functions,

$$C_1 \left[\frac{\cos\theta - 1}{\cos\theta + 1}\right]^{\frac{\sqrt{m^2+1}}{2}} + C_2 \left[\frac{\cos\theta - 1}{\cos\theta + 1}\right]^{-\frac{\sqrt{m^2+1}}{2}}$$

The functions $w^A_0(\theta,\varphi)$ (as well as the functions and $w^A_1(\theta,\varphi)$ and the subsequent terms of the development (8.12) can be decomposed into the sum of the Legendre polynomials as follows:

$$w^A(\theta,\varphi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} C^A_{lm} P^m_l(\cos\theta)\, e^{im\phi} \tag{8.14}$$

They appear in the expressions for the third-order correction to the Ricci tensor via the combination $\left[\overset{\circ}{g}{}^{\theta\theta}\nabla_\theta\nabla_\theta w + \overset{\circ}{g}{}^{\varphi\varphi}\nabla_\varphi\nabla_\varphi w\right]$, which is the angular part of the Laplace operator in spherical coordinates. Therefore, as each of the associated Legendre polynomials is a solution of the full Laplace equation with the corresponding radial part included, which gives the factor $l(l+1)$, we shall get

$$\left[\overset{\circ}{g}{}^{\theta\theta}\nabla_\theta\nabla_\theta + \overset{\circ}{g}{}^{\varphi\varphi}\nabla_\varphi\nabla_\varphi\right] P^m_l(\cos\theta)\, e^{im\phi} = \frac{1}{r^2}\left[\frac{m^2}{\sin^2\theta} - l(l+1)\right] P^m_l(\cos\theta)\, e^{im\phi},$$

so that these terms become the sum of linear combinations of the second-order deformations w^A. These in turn can be superposed with the first-order deformations v^A in form of portent waves of different frequencies in order to produce the wave packets with an arbitrary frequency spectrum.

The knowledge of the asymptotic angular behavior should enable us to reconstruct the angular distribution of matter in the source of the gravitational wave.

Let us summarize the result of our approximation at this stage in the following table:

	ε^0	ε^1	ε^2	ε^3
$\frac{1}{r}$	0	0	0	$\overset{3}{R}{}^{\rho}_{\ \mu\nu\lambda} \neq 0,\ \ \overset{3}{R}_{\mu\nu} = 0.$
$\frac{1}{r^2}$	0	0	$\overset{2}{R}{}^{\rho}_{\ \mu\nu\lambda} \neq 0,\ \ \overset{2}{R}_{\mu\nu} \neq 0$	$\overset{3}{R}{}^{\rho}_{\ \mu\nu\lambda} \neq 0,\ \ \overset{3}{R}_{\mu\nu} \neq 0.$
$\frac{1}{r^3}$	0	0	0	$\overset{3}{R}{}^{\rho}_{\ \mu\nu\lambda} \neq 0,\ \ \overset{3}{R}_{\mu\nu} \neq 0.$
$\frac{1}{r^4}$	0	0	$\overset{2}{R}{}^{\rho}_{\ \mu\nu\lambda} \neq 0,\ \ \overset{2}{R}_{\mu\nu} \neq 0$	$\overset{3}{R}{}^{\rho}_{\ \mu\nu\lambda} \neq 0,\ \ \overset{3}{R}_{\mu\nu} \neq 0.$
$\frac{1}{r^5}$	0	0	0	$\overset{3}{R}{}^{\rho}_{\ \mu\nu\lambda} \neq 0,\ \ \overset{3}{R}_{\mu\nu} \neq 0.$
$\frac{1}{r^6}$	0	0	$\overset{2}{R}{}^{\rho}_{\ \mu\nu\lambda} \neq 0,\ \ \overset{2}{R}_{\mu\nu} \neq 0.$	$\overset{3}{R}{}^{\rho}_{\ \mu\nu\lambda} \neq 0,\ \ \overset{3}{R}_{\mu\nu} \neq 0.$

Table I: The non-vanishing components of Riemann and Ricci tensors for the approximate spherical wave solution in Minkowskian background

The table above shows that in the asymptotic radiative region the dominant terms satisfy Einstein's equations, i.e. up to the terms behaving at infinity like r^{-1} the space-time containing the spherical wave contribution is Ricci-flat. The remaining terms, be it the Riemann or the Ricci tensors, are similar to the Coulomb-like terms in the electromagnetic radiation case in the zone near the source, where they dominate. However, although the contributions to the Riemann tensor may (and should be) not equal to zero, The Ricci tensor should vanish at all orders, or at least up to the maximal number of orders for our approximation to be considered as valid. The non-vanishing components of the Ricci tensor, starting from the terms decaying like r^{-2} at infinity and in the first three lowest powers of the small parameter ε are displayed in the following Table II:

	$\frac{1}{r}$	$\frac{1}{r^2}$	$\frac{1}{r^3}$	$\frac{1}{r^4}$	$\frac{1}{r^5}$	$\frac{1}{r^6}$
$\overset{2}{R}_{tt}$	0	0	0	$\frac{A^2\omega^2}{c^2 r^4}$	0	0
$\overset{2}{R}_{rr}$	0	0	0	$-\frac{3A^2\omega^2-2A^2k^2c^2}{c^2r^4}$	0	$\frac{4A^2}{r^6}$
$\overset{2}{R}_{\theta\theta}$	0	$\frac{A^2\omega^2}{c^2r^2}$	0	$\frac{A^2}{r^4}$	0	0

Table II: The non-vanishing components of second-order Ricci tensor
for the approximate spherical wave solution in Minkowskian background ($R_{\varphi\varphi} = \sin^2\theta\, R_{\theta\theta}$)

Our strategy, in what follows, will consist in creating other deformations of the embedding functions, perhaps in extra dimensions, in order to cancel the non-zero components of the Ricci tensor displayed in the Table II. To make the Ricci tensor vanish at all orders of r^{-1} and in all powers of ε is beyond our possibilities at the moment; still, we proceeding step by step, we shall cancel at least the dominant terms, of which the most important is $\overset{2}{R}_{\theta\theta}$ proportional to r^{-2}.

Before going into details, let us draw attention to the physical nature of the problem. The spherical gravitational wave was produced by the deformations of the flat Minkowskian background, and is valid only asymptotically, at spatial infinity. But the presence of spherical radiation coming from the center of coordinates cannot be justified if there is no matter near the center. There must be some time-dependent mass distribution, because we can be sure that a static mass will not radiate. Whatever the distribution, if it is bounded in space, from a very great distance it will be seen as an almost point-like mass varying with time, $M(t)$.

Now, contrary to what may happen in electrodynamics, where the radiation may come from a charge distribution with total average zero, the function $M(t)$ can never become negative or even hit the zero value. This means that there is some minimal value below which $m(t)$ can never descend, so that we can write $M(t) = M + m(t)$, with $m(t)$ usually much smaller than M. This leads us to the conclusion that the natural background for a spherical gravitational wave is the Schwarzschild metric, with time-dependent perturbations which are the real source of radiation.

This is why we should combine the embedding of our approximate spherical wave with a Schwarzschild background. The embedding of the combined solution will certainly need more than six flat dimensions; let us recall the properties of embeddings of the exterior Schwarzschild solution.

9. Spherical wave and the Schwarzschild background

The yet unspecified small parameter ε can be now identified with the measure of the ratio $\mid max(m(t)) \mid /M$. The rapidity of the time variation will also play an important role, as can be inferred from the presence of the factor ω^2 in the non-vanishing components of the Riemann and Ricci tensors generated by the wave.

Isometric embeddings of Einstein spaces in pseudo-Euclidean flat spaces of various dimensions and signatures can be found in J. Rosen's paper in [27]. An embedding of the exterior Schwarzschild solution which is of particular interest to us , cited in Rosen's paper, has been found by Kasner [28], who also proved that the embedding of Schwarzschild's solution in a five-dimensional pseudo-Euclidean space is impossible. Kasner's embedding uses a pseudo-Euclidean space E^6 with signature $(++----)$, using trigonometric functions of time in order to parametrize the two timelike dimensions, and admits closed time-like curves. In 1959 C. Fronsdal [29] proposed a similar embedding into pseudo-Euclidean space with the signature $(1+,5-)$, using hyperbolic functions instead the trigonometric ones. Fronsdal's embedding is defined as follows:

$$z^1 = MG\left(1-\frac{2MG}{r}\right)^{\frac{1}{2}} \sinh\left(\frac{ct}{MG}\right), \quad z^2 = MG\left(1-\frac{2MG}{r}\right)^{\frac{1}{2}} \cosh\left(\frac{ct}{MG}\right),$$

$$z^3 = \int \left[\frac{1-\left(\frac{MG}{r}\right)^4}{1-\frac{2MG}{r}} - 1\right]^{\frac{1}{2}} \mathrm{d}r,$$

$$z^4 = r\sin\theta\,\cos\phi, \quad z^5 = r\sin\theta\,\sin\phi, \quad z^6 = r\cos\theta$$

Here M is the mass of the central gravitating body and G denotes Newton's gravitational constant. (Note the dimensional factor MG in front of the definitions of z^1 and z^2 in order to give these coordinates the dimension of length).

The embedded four-dimensional manifold V_4 is parameterized by the coordinates x^μ, with $\mu = 0,1,2,3$ so that $x^0 = ct,\ x^1 = r,\ x^2 = \theta, \quad x^3 = \phi$.

Let us define the flat metric by $\eta_{AB} = diag\,(+-----)$, with $A,B,... = 1,2,...6$.

Then it is easy to check that the induced metric on the embedded manifold has indeed the usual Schwarzschild form:

$$ds^2 = \eta_{AB}\,\partial_\mu z^A\,\partial_\nu z^B\,dx^\mu dx^\nu = g_{\mu\nu}\,(x^\lambda)\,dx^\mu dx^\nu = \tag{9.1}$$

$$= \left(1-\frac{2MG}{r}\right)c^2dt^2 - \left(1-\frac{2MG}{r}\right)^{-1} dr^2 - r^2(d\theta^2 - \sin^2\theta d\phi^2)$$

In what follows, we use Fronsdal's embedding of the exterior Schwarzschild space-time into a pseudo-Euclidean flat space with one time-like and five space-like dimensions.

It is worthwhile to look at the non-vanishing components of the Riemann tensor in the Schwarzschild space-time. They are displayed in the following Table III:

	r	1	$\frac{1}{r}$	$\frac{1}{r^2}$	$\frac{1}{r^3}$	$\frac{1}{r^4}$
$\overset{o}{R}_{trtr}$	0	0	0	0	$\frac{2MG}{r^3}$	0
$\overset{o}{R}_{t\theta t\theta}$	0	0	$-\frac{MG}{r}$	$\frac{2M^2G^2}{r^2}$	0	0
$\overset{o}{R}_{r\theta r\theta}$	0	0	$\frac{MG}{r}$	$\frac{2M^2G^2}{r^2}$	$\frac{4M^3G^3}{r^3}$	$\frac{8M^4G^4}{r^4}$
$\overset{o}{R}_{\theta\phi\theta\phi}$	$-2MGr\sin^2\theta$	0	0	0	0	0

Table III: The Riemann tensor for the Schwarzschild background

The Ricci tensor is null, of course. Now, a small deformation of the Schwarzschild embedding may be produced that alters the components of the Riemann tensor keeping the space-time Ricci flat. It consists in a small variation of the total mass, $M \to M + \delta M$. The resulting embedding induces the Schwarzschild metric corresponding to the total mass $M + \delta M$. The resulting first-order correction to the Riemann tensor (linear in δM) are displayed in the following Table IV:

	r	1	$\frac{1}{r}$	$\frac{1}{r^2}$	$\frac{1}{r^3}$	$\frac{1}{r^4}$
$\overset{1}{R}_{trtr}$	0	0	0	0	$\frac{2\delta MG}{r^3}$	0
$\overset{1}{R}_{t\theta t\theta}$	0	0	$-\frac{\delta MG}{r}$	$\frac{4MG\delta MG}{r^2}$	0	0
$\overset{1}{R}_{r\theta r\theta}$	0	0	$\frac{\delta MG}{r}$	$\frac{4MG\delta MG}{r^2}$	$\frac{12M^2G^2\delta MG}{r^3}$	$\frac{32M^3G^3\delta MG}{r^4}$
$\overset{1}{R}_{\theta\phi\theta\phi}$	$-2\delta MGr\sin^2\theta$	0	0	0	0	0

Table IV: First order Riemann tensor due to a small mass deformation around the Schwarzschild background

One can note that although the non-vanishing components of the $R_{\mu\nu\lambda\rho}$ correspond to the same set of indices as the contributions of a spherical wave in Minkowskian background, the repartition among the negative powers of r is quite different, as shown in the Table V below:

	r	1	$\frac{1}{r}$	$\frac{1}{r^2}$	$\frac{1}{r^3}$	$\frac{1}{r^4}$
$\overset{2}{R}_{trtr}$	0	0	0	0	0	$\frac{3A^2\omega^2}{c^2r^4}$
$\overset{2}{R}_{t\theta t\theta}$	0	0	0	$-\frac{A^2\omega^2}{c^2r^2}$	0	0
$\overset{2}{R}_{r\theta r\theta}$	0	0	0	$\frac{2A^2k^2}{r^2}$	0	$\frac{2A^2}{r^4}$
$\overset{2}{R}_{\theta\phi\theta\phi}$	0	$-A^2k^2\sin^2\theta$	0	$-\frac{A^2\sin^2\theta}{r^2}$	0	0

Table V: Second order Riemann tensor for a spherical wave in Minkowskian background

We saw already in the previous section that six pseudo-Euclidean dimensions were needed to embed the spherical gravitational wave; the exterior Schwarzschild requires also six dimensions for global embedding. Most probably, it is impossible to accomodate the two embeddings in the same six-dimensional flat space without their metrics being modified even at the lowest orders. This is why we shall look for the common embedding of spherical gravitational waves and the Schwarzschild background in an eight-dimensional pseudo-Euclidean space with one time-like and seven space-like directions.

We shall choose the first six dimensions (with the $(+-----)$ signature) for the embedding of the Schwarzschild space-time defined as in (9.1), and the remaining two space-like dimensions, z^7 and z^8, will be supposed to be flat and the corresponding deformations of first and second order in ε will represent the two degrees of freedom of the spherical wave, producing the contributions to Riemann and Ricci tensors displayed in the Tables II and IV.

Our aim now is to improve the approximation via cancelling more terms in the Ricci tensor. The most important terms after the radiative ones are the terms behaving like r^{-2} at infinity, found in the second-order Ricci tensor (proportional to ε^2). Supposing that in the radiative zone $r \to \infty$ Schwarzschild metric is asymptotically Minkowskian, we neglect the terms coming from the correction MG/r, but keep the embedding using Fronsdal's hyperbolic functions. Producing extra deformations in the first two pseudo-Euclidean dimensions carrying the signature $(+-)$ is the only way to produce negative contribution to the Ricci tensor able to cancel the positive-definite terms displayed in the Table IV.

Let us produce a first-order deformation of the first three embedding functions, deforming the embedding of Minkowskian space-time embedded in six pseudo-Euclidean dimensions. Here λ is yet unspecified constant with dimension of length:

$$v^1 = \lambda \sinh \frac{ct}{\lambda} f(r), \qquad v^2 = \lambda \cosh \frac{ct}{\lambda} f(r), \qquad v^3 = v^3(r) \tag{9.2}$$

with all other components v^A equal to zero.

Expanding the function f in a series of negative powers of r as follows:

$$f(r) = \frac{B}{r} + \frac{C}{r^2} + \frac{D}{r^3} + \frac{E}{r^4} + \dots$$

we obtain the following corrections to the Ricci tensor in four dimensions:

	$\frac{1}{r}$	$\frac{1}{r^2}$	$\frac{1}{r^3}$	$\frac{1}{r^4}$	$\frac{1}{r^5}$	$\frac{1}{r^6}$	other
$\overset{2}{R}_{tt}$	0	0	0	0	$-\frac{2BC}{r^5}$	$-\frac{6BD+2C^2}{r^6}$	0
$\overset{2}{R}_{rr}$	0	0	0	$-\frac{2B^2}{r^4}$	$-\frac{8BC}{r^5}$	$-\frac{14BD+6C^2-4\lambda^2B^2}{r^6}$	$-\frac{2v_3'v_3''}{r}$
$\overset{2}{R}_{\theta\theta}$	0	$\frac{B^2}{r^2}$	$\frac{3BC}{r^3}$	$\frac{B^2\lambda^2+4BD+2C^2}{r^4}$	0	0	$-rv_3'v_3'' - (v_3')^2$

Table VI: The contributions to the second-order Ricci tensor, coming from a v^A deformation

As one can see, all the contributions to the second-order correction to the Ricci tensor behaving at infinity as r^{-2} are positive-definite, and cannot cancel similar terms generated by the wave-like deformation. Moreover, they produce a first-order contribution to the Ricci tensor, behaving like r^{-1} at infinity, which was avoided up to now. This is why we shall set the function f equal to zero.

But there is no reason to neglect the second-order deformation of the first two embedding functions. Let us choose the second-order correction of the same form as the first-order one:

$$w^1 = \lambda \sinh \frac{ct}{\lambda}\, g(r), \qquad w^2 = \lambda \cosh \frac{ct}{\lambda}\, g(r), \qquad w^3 = w^3(r) \tag{9.3}$$

and the other components zero. The function g is also a series of negative powers of r:

$$g(r) = \frac{P}{r^2} + \frac{Q}{r^3} + \frac{R}{r^4} + \frac{S}{r^5} + \frac{T}{r^6} + \dots$$

Inserting this deformation alone into the formulae for the second-order correction to the Ricci tensor we obtain the following result, displayed in the Table VII below:

	$\frac{1}{r}$	$\frac{1}{r^2}$	$\frac{1}{r^3}$	$\frac{1}{r^4}$	$\frac{1}{r^5}$	$\frac{1}{r^6}$
$\overset{2}{R}_{tt}$	0	0	0	$-\frac{2P}{r^4}$	$-\frac{6Q}{r^5}$	$-\frac{12R}{r^6}$
$\overset{2}{R}_{rr}$	0	0	0	$\frac{6P}{r^4}$	$\frac{12Q}{r^5}$	$\frac{20R}{r^6}$
$\overset{2}{R}_{\theta\theta}$	0	$-\frac{2P}{r^2}$	$-\frac{3Q}{r^3}$	$-\frac{4R}{r^4}$	$-\frac{5S}{r^5}$	$-\frac{6T}{r^6}$

Table VII: The contributions at the second-order Ricci tensor, coming from a w^A deformation

Combining the results in the Tables II, VI and VII, we see that all the components of the second-order correction to the Ricci tensor can be cancelled provided the constants P and Q satisfy the following two relations:

$$A^2 k^2 - 2P = 0, \qquad Q = 0, \tag{9.4}$$

which cancels the components of $\overset{2}{R}_{\mu\lambda}$ behaving at infinity like r^{-2} and r^{-3}.

At this stage of approximation we get $P = \frac{1}{2} A^2 k^2$; This leads to the conclusion that the leading term in the second-order correction to the embedding functions of Schwarzschild (or Minkowskian) background is proportional to the square of the wave number, k^2.

The combined embedding of the Schwarzschild metric in six pseudo-Euclidean dimensions and the wave-like deformations of two extra dimensions produce a four dimensional manifold embedded in a pseudo-Euclidean space of eight dimensions, with the signature $(1+, 7-)$. This manifold represents an approximate radiative solution of Einstein's equations similar, up to the terms behaving like r^{-4} at spatial infinity, to the solutions found by Boardman and Bergmann ([31]) or by Robinson and Trautman ([32]).

10. Discussion and conclusions

In this article we have set forth a new formalism that makes it easier to consider small perturbations of a given Einsteinian background without unphysical degrees of freedom that mar traditional computations based on the deformations of the metric. Here the embedding provides us with clear geometric criterion selecting physical degrees of freedom and eliminating the unphysical ones.

We have succeeded the construction of wave solutions in flat space, or in an asymptotically flat Schwarzschild manifold at spatial infinity. These solutions can display any imposed form due to the possibility of linear superposition of Legendre polynomials. Once a radiative solution is chosen, we can extrapolate it towards the smaller values of r where the non-radiative terms prevail. These can be seen as the corrections to Schwarzschild metric close to the central body that are responsible for the emission of gravitational waves detected at spatial infinity.

The treatment of this problem is the subject of the work in progress.

Now that the Einstein equations are satisfied up to the second order (ε^2) and up to the terms decaying asymptotically as r^{-4} or faster, we can try to go farther and cancel the third-order (proportional to ε^3) terms of the Ricci tensor in the vicinity of the central mass, i.e. in the non-radiative region. Note that up to the second order, there was no information about any characteristic of the wave, which is a third order effect. Let us consider the combination of the Schwarzschild background with the outcoming gravitational wave: the corresponding Riemann tensors can be superimposed without provoking any modification in the lowest orders of the expansion, as it follows from the Tables IV and VI.

In order to produce a contribution to the Ricci tensor that would cancel the third-order terms generated as the side-effects of the wave-like deformation, we may follow the example of the Reissner-Nordstroem metric, which is not Ricci-flat because of the presence in the Einstein equations of the energy-momentum tensor of the electromagnetic field generated by the central electric charge Q. The exact solution is then given by the following modification of Schwarzschild's metric:

$$ds^2 = \left(1 - \frac{2MG}{r} + \frac{Q^2}{r^2}\right) c^2 dt^2 - \left(1 - \frac{2MG}{r} + \frac{Q^2}{r^2}\right)^{-1} dr^2 - r^2 (d\theta^2 - \sin^2\theta d\phi^2) \tag{10.1}$$

This metric can be induced by an embedding identical with Fronsdal's one, with new coefficients of the first two embedding functions. It is not Ricci-flat, because its Ricci tensor is proportional to the energy-mementum tensor of the electromagneetci field of a sperically symmetric distribution of electric charge density, Q bring its total charge. The Maxwell tensor of such a distribution behaves at infinity as r^{-2}, and the energy-momentum tensor is proportional to r^{-4}.

In our case, the yet non-vanishing components of the Ricci tensor are exactly of this form.

Because we are looking for the third-order effect, we shall modify Schwarzschild's embedding functions by replacing the term Q^2/r^2 by a term proportional to ε^3, of the form $\frac{W(t,\theta,\phi)}{r^p}$.

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