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Weyl-Guilfoyle fluids and quasiblack holes with pressure*

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A class of electrically charged systems with pressure in general relativity arises when the gravitational potential W^2 and the electric potential field ϕ obey a relation of the form $W^2 = a\left(-\varepsilon\sqrt{G}\,\phi+b\right)^2+c$, where a, b and c are arbitrary constants, $\varepsilon=\pm1$, and G is Newton's constant. This relation generalizes the usual Weyl relation (for which $a=1$), and we call it the Weyl-Guilfoyle relation. For both, Weyl and Weyl-Guilfoyle relations, the electrically charged fluid, if present, may have nonzero pressure. Fluids obeying the Weyl-Guilfoyle relation are called Weyl-Guilfoyle fluids which have very interesting features. In the present work we display some new properties of Weyl-Guilfoyle fluids. These fluids, under assumption of spherical symmetry, exhibit solutions which can be matched to the electrovacuum Reissner-Nordström spacetime to yield global asymptotically flat stars. We show that a particular spherically symmetric class of stars found by Guilfoyle has a well behaved limit which corresponds to a quasiblack hole (QBH) with pressure, i.e., in which the fluid inside has electric charge and pressure. The main physical properties of such charged stars and QBHs with pressure are analyzed.

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1. Introduction

It was Weyl while studying electric fields in vacuum Einstein-Maxwell theory who first perceived that it is interesting to consider a functional relation between the metric component $g_{tt} \equiv W^2(x^i)$ and the electric potential $\phi(x^i)$ (where x^i represent the spatial coordinates, $i = 1,2,3$) given by the ansatz $W = W(\phi)$. By assuming the system is vacuum and axisymmetric, Weyl found that such a relation must be of the form $W^2 = \left(-\varepsilon\sqrt{G}\phi + b\right)^2 + c$, where b and c are arbitrary constants, $\varepsilon = \pm 1$, G is Newton's constant, and we use units such that the speed of light equals unity. The corresponding spacetime metric may be written as

$$ds^2 = -W^2 dt^2 + h_{ij}\, dx^i dx^j\,, \qquad i,j = 1,2,3\,, \tag{1.1}$$

where h_{ij} is also a function of the spatial coordinates x^i. One can go beyond vacuum solutions, and consider fluids which obey the Weyl relation, obtaining their properties, see [1] for the original references, including the works of Majumdar and Papapetrou who studied a perfect square relationship between W and ϕ. An interesting development on Weyl's work was performed by Guilfoyle [2] who considered charged fluid distributions with the hypothesis that the functional relation between the gravitational and the electric potential, $W = W(\phi)$, is slightly more general than Weyl's. This Weyl-Guilfoyle relation has the form

$$W^2 = a\left(-\varepsilon\sqrt{G}\,\phi + b\right)^2 + c\,, \tag{1.2}$$

where a is another arbitrary constant. Guilfoyle [2] investigated several general properties of such matter systems, which we call Weyl-Guilfoyle fluids. In general, these fluids have electrical charge and pressure.

Drawing upon Guilfoyle's work we further construct a relationship between the various field and matter quantities [3], in much the same way as Gautreau and Hoffman [4] have done for fluids obeying a pure Weyl relation. Furthermore, in the very same work, Guilfoyle [2] found that Weyl-Guilfoyle fluids, under a static spherically symmetric assumption, exhibit interesting solutions which can be matched to the electrovacuum Reissner-Nordström spacetime, yielding global asymptotically flat solutions, i.e., charged stars with pressure. We explore one particular class of those spherically symmetric charged fluid stars and show that they display quasiblack hole (QBH) behavior, i.e., the matter boundary approaches its own horizon in a well behaved manner [5]. So, for the first time, a QBH with pressure is exhibited. QBHs purely supported by electrical charge are known, see [6] and references therein. The presence of pressure in QBH solutions is important, since it tends to stabilize the system.

2. Weyl-Guilfoyle fluids and their properties

To study some properties of Weyl-Guilfoyle fluids we assume that the spacetime is static and that the metric can be written in the form of equation (1.1), with (1.2) holding. The gauge field A_μ and the four-velocity of the fluid U_μ are then given by $A_\mu = -\phi\,\delta^0_\mu$ and $U_\mu = -W\,\delta^0_\mu$. Then, from the Einstein-Maxwell equations one finds that the metric potential W and the electric potential ϕ

satisfy the following equations,

$$\nabla^2 W = 4\pi G W \left(\rho_{\rm m} + \frac{1}{3}\,p + \rho_{\rm em}\right), \qquad \nabla_i \left(\frac{1}{W}\,\nabla^i \phi\right) = -4\pi\,\rho_{\rm e}\,, \tag{2.1}$$

where we have defined the electromagnetic energy density $\rho_{\rm em}$ by $\rho_{\rm em} = \dfrac{1}{4\pi}\dfrac{(\nabla_i \phi)^2}{W^2}$, with ∇_i denoting the covariant derivative with respect to the coordinate x^i, whose connection coefficients are given in terms of the metric h_{ij}. The two equations in (2.1) may be combined to produce the following relation

$$\nabla_i \left(\tfrac{1}{W}\,\nabla^i \left[W^2 - a\Phi^2 + c\right]\right) = 8\pi G \left[\left(\rho_{\rm m} + \tfrac{1}{3}\,p + (1-a)\,\rho_{\rm em}\right) W - \varepsilon\, a\, \rho_{\rm e}\Phi\right], \tag{2.2}$$

where Φ is given by $\Phi = -\varepsilon\sqrt{G}\,\phi + b$, and a, b and c are arbitray constants. On basis of this equation the following theorem can be stated.

Theorem 1. Lemos and Zanchin (2009)
(i) In any Einstein-Maxwell charged pressure fluid, if the metric and electric potentials are such that $W^2 - a\left(-\varepsilon\sqrt{G}\,\phi + b\right)^2 - c$ vanishes everywhere, then the fluid quantities satisfy the constraint

$$a\, b\, \rho_{\rm e} = \varepsilon\sqrt{G}\left[\left(\rho_{\rm m} + \frac{1}{3}\,p\right) W + \phi\rho_{\rm e} + (a-1)\left(\phi\rho_{\rm e} - W\rho_{\rm em}\right)\right]. \tag{2.3}$$

(ii) Conversely, in any Einstein-Maxwell charged pressure fluid, if the fluid quantities are such that equation (2.3) holds and there is a closed surface, with no singularities, holes, or alien matter inside it, where $W^2 - a\left(-\varepsilon\sqrt{G}\,\phi + b\right)^2 - c$ vanishes, then it vanishes everywhere inside the surface.

Proof. The proof of this theorem is given in [3] for d-dimensional spacetimes, and will not be reproduced here. Of course it also holds for $\mathrm{d} = 4$.

As we can see, Theorem 1 gives an explicit relation among the fluid quantities which satisfy the Weyl-Guilfoyle relation (1.2), and so it can be considered as an equation of state satisfied by the charged fluid with pressure. Earlier, Gautreau and Hoffman [4] had investigated the structure of the sources that produce Weyl type fields which satisfy the Weyl quadratic relation (i.e., equation (1.2) wit $a = 1$), in the case the matter has stresses, i.e., the pressures, do not vanish. They found that the fluid obeys the condition $b\,\rho_{\rm e} = \varepsilon\sqrt{G}\left[\left(\rho_{\rm m} + \frac{1}{3}\,p\right) W + \phi\rho_{\rm e}\right]$. Comparing our theorem to the theorem by Gautreau and Hoffman [4] we see that in their analysis only the binding energy was taken into account, and hence the constant a was forced to be equal to unity. Our theorem generalizes thus the Gautreau-Hoffman theorem.

3. Spherical solutions and quasiblack holes with pressure

3.1 The solution

Given an ansatz of the type (1.2) one is tempted to find exact solutions with matter. This is what Guilfoyle [2] has done by assuming spherical symmetry among other things. Guilfoyle has found several classes of solutions of charged matter with pressure. Furthermore, all the solutions found by Guilfoyle in [2] obey the constraint provided by equation (2.3), see [3]. When matched to the Reissner-Nordström exterior solution, the exact solutions found in [2] yield charged stars with pressure.

In order to display a particular class of Guilfoyle's solutions one writes the metric in the spherically symmetric form

$$ds^2 = -B(r)dt^2 + A(r)dr^2 + r^2 d\Omega\,, \tag{3.1}$$

where r is the radial coordinate, A and B are function of r only, and $d\Omega$ is the metric of the unit sphere S^2. This form of the metric is more useful to explore the physical properties of the system than the form given in equation (1.1). The charged pressure fluid is bounded by a spherical surface of radius $r = r_0$, and for $r > r_0$ the metric and the electric potential are given by Reissner-Nordström solution $ds^2 = -\left(1 - \frac{2m}{r} + \frac{q^2}{r^2}\right)dt^2 + \left(1 - \frac{2m}{r} + \frac{q^2}{r^2}\right)^{-1} dr^2 + r^2 d\Omega$, and $\phi = \frac{q}{r} + \phi_0$, with ϕ_0 being an arbitrary constant which defines the zero of the electric potential, and that, in asymptotically Reissner-Nordström spacetimes as we consider here, can be set to zero. Here m and q are the mass and charge at infinity, respectively. When $q = m$ the solution is said extremal. The Guilfoyle's solutions are found under the assumptions that the effective energy density $\rho_{\rm eff}(r) = \rho_{\rm m}(r) + Q^2(r)/(8\pi r^4)$ is a constant, and that the metric coefficient $A(r)$ is given by $A(r)^{-1} = 1 - r^2/R^2$, R^2 being a constant. The solutions we are interested here is the class Ia solutions, in which the constant $c = 0$. These solutions are given by [2]

$$B(r) = \left[\frac{2-a}{a^{1+1/a}}\left(k_0 R^2\sqrt{1 - r^2/R^2} - k_1\right)\right]^{2a/(a-2)}, \qquad A(r)^{-1} = 1 - \frac{r^2}{R^2}, \tag{3.2}$$

$$8\pi\,\rho_{\rm m}(r) = \frac{3}{R^2} - \frac{Q^2(r)}{r^4}, \qquad Q(r) = \frac{\varepsilon\sqrt{a}}{2-a}\,\frac{k_0\, r^3}{k_0\, R^2\sqrt{1 - r^2/R^2} - k_1}, \tag{3.3}$$

$$8\pi p(r) = -\frac{1}{R^2} + \frac{a}{(2-a)^2}\,\frac{k_0^2\, r^2}{\left(k_0\, R^2\sqrt{1 - r^2/R^2} - k_1\right)^2} + \frac{2k_0\, a}{2-a}\,\frac{\sqrt{1 - r^2/R^2}}{k_0\, R^2\sqrt{1 - r^2/R^2} - k_1}\,, \tag{3.4}$$

where k_0 and k_1 are integration constants. These constants are determined by using the continuity of the metric potentials $A(r)$ and $B(r)$ and the first derivative of $B(r)$ with respect to r at the boundary $r = r_0$, and so k_0 and k_1 are given in terms of m, q and r_0.

3.2 Quasiblack holes with pressure

Given the star solutions provided by Guilfoyle, equations (3.2)-(3.4), we now show some properties of their metric, electric and matter fields, and that under sufficient compactification the charged stars turn into quasiblack holes (QBHs) with pressure, in which the fields and matter are regular everywhere. The presence of pressure in QBH solutions is new.

In principle, the solution (3.2)-(3.4) is valid for all $a > 0$, but there are some regions in the domain of the parameter a in which they are unphysical. An important quantity is the speed of sound $c_{\rm s}$, because when compared to the other fluid quantities, it imposes the strongest bounds on the range of values of a. We take the usual definition for the speed of sound, $c_{\rm s}^2 = \delta p/\delta\rho_{\rm m}$, and consider variations of the pressure p and of the energy density $\rho_{\rm m}$ in terms of the radial coordinate r, i.e., $\delta p = p'(r)\delta r$ and $\delta\rho_{\rm m} = \rho_{\rm m}'(r)\delta r$. The final result yields the speed of sound as a function of the radial coordinate. We find that it is well behaved and smaller than the speed of light for the configurations for which the parameter a is in the interval $1 \leq a \leq 4/3$. Thus we study only star and QBH solutions in the parameter region given by $1 \leq a \leq 4/3$.

To show analytically that QBH solutions with pressure really exist we follow the definition of QBHs given in [5]. A QBH is neither a usual regular spacetime, as for instance a star, nor a black hole. Roughly speaking a QBH is an object on the verge of becoming an extremal black hole but actually is distinct from it in many ways, for instance, it has matter inside its own quasihorizon and is regular everywhere. More precisely, according to the definition in [5], for zero surface stresses (a condition verified in the solutions above since $p = 0$ at the surface), the QBH should be extremal, so that $m \to q\,(1+\delta)$, with very small non-negative δ. Moreover, there must be a quasihorizon r^*, and then the radius of the star r_0 must coincide with r^*. As a consequence, we must have $m \to q \to r_0\,(1-\delta)$. In the present case, once we have $q \simeq r_0\,(1-\delta)$, the boundary conditions at $r = r_0$ imply in $m/q \sim 1 + (a-1)\delta^2/2a$, and then $m/r_0 \sim 1-\delta$. Taking these considerations into account we find that $B(r)$ is of the order of δ^2, for all r in the interval $0 \leq r \leq r_0$. Moreover, we find $1/A(r_0) \sim \delta^2$. These conditions satisfy the properties of a QBH as defined in [5].

Now in possession of these results we can study numerically an interesting typical case. We choose here $a = 4/3$ because, besides being typical, it is the case in which the speed of sound approaches the speed of light close to the surface of the star, independently of how compressed the star is. All other cases $1 \leq a \leq 4/3$ are also typical and have the speed of sound less than 1 at the surface, the speed of sound being zero for $a = 1$. We study star solutions and the corresponding QBH limit. First we analyze the metric potentials, then the matter fields.

(i) The metric potentials $B(r)$ and $A(r)$ for the typical interesting case $a = 4/3$ are shown in figure 1. The exterior metric is Reissner-Nordström metric, and then the curves tend to unity for large values of the normalized radial coordinate r/q. It is seen that for the quasi-extremal case where $r_0/q = 1.00001$, the QBH features show up. Namely, $B(r) \to 0$ for the whole interior region, $0 \leq r < r_0$ and $1/A(r) \to 0$ at $r = r_0$.

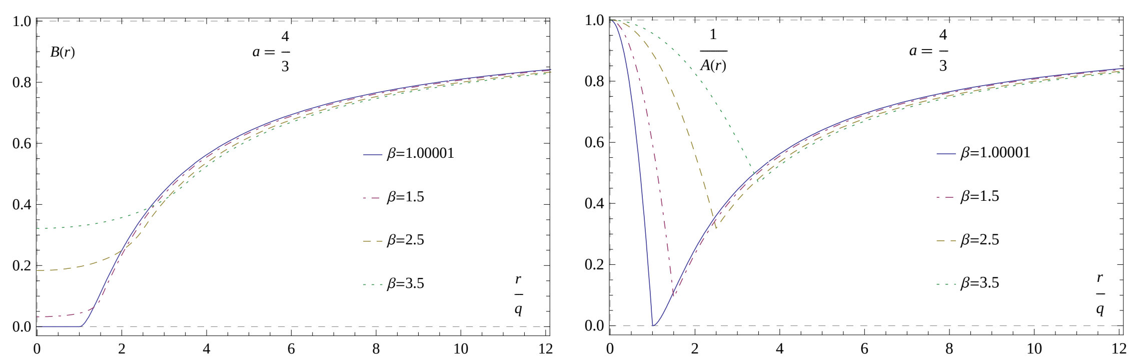

Figure 1: The metric potentials $B(r)$ and $1/A(r)$ as a function of the normalized radial coordinate r/q, for four values of $\beta = r_0/q$ in each graph. The case $\beta = 1$, which gives $q = m = r_0$, is a QBH.

(ii) The matter fields $\rho_{\rm m}$ and p for the case $a = 4/3$ are shown in figure 2. The study of the physical properties satisfied by these matter fields was in part done in ref. [2] (see figure 2). We see that the pressure is always smaller than the energy density for all $r_0/q \geq 1$. The ratio m/q runs from unity, at the QBH limit, to $\sqrt{a}$ for a extremely sparse star, with $r_0 \longrightarrow \infty$. We also find that, for the class of solutions we are discussing, i.e., those for which $c = 0$, the boundary conditions

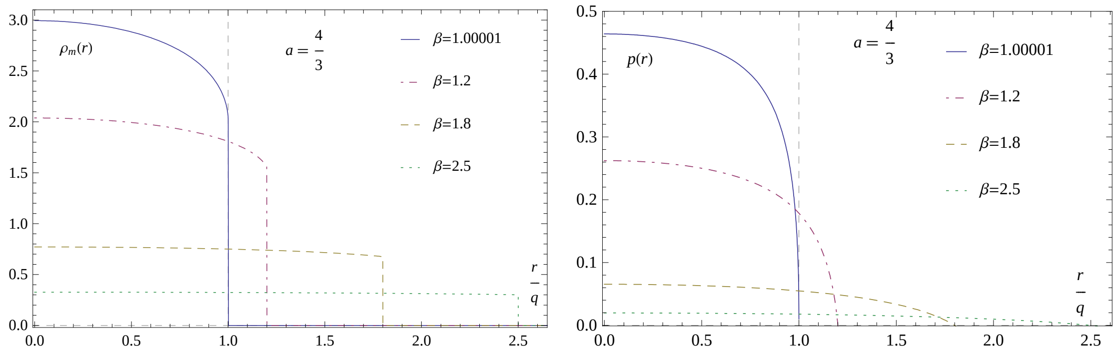

Figure 2: The relativistic mass density $\rho_{\rm m}(r)$, and the pressure $p(r)$ as a function of the normalized r/q for four different values of $\beta = r_0/q$.

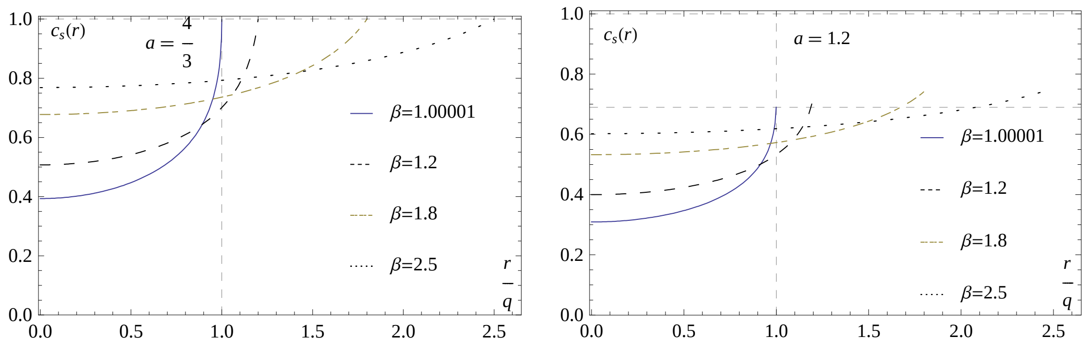

Figure 3: Two typical cases for the radial speed of sound as a function of the normalized r/q, for four values of $\beta = r_0/q$.

at $r = r_0$ give $\frac{m}{q} = (1-a)\,\frac{q}{r_0} + \sqrt{a\left[1+(a-1)\,\frac{q^2}{r_0^2}\right]}$. The maximum value of m/q is in the limit $r_0 \to \infty$ and is given by $\left.\frac{m}{q}\right|_{\rm max} = \sqrt{a}$. Since we restrict the range of variation of the parameter a within the interval $(1, 4/3)$ the maximum possible value for the ratio m/q is $2\sqrt{3}/3$. We can also obtain analytic expressions for the fluid quantities like mass and charge densities and pressure in the QBH limit, but we do not write such expressions here. An analysis on the speed of sound $c_{\rm s}$ of these solutions is new and done now. The condition that $c_{\rm s}$ is well defined and smaller than the speed of light guarantees, of course, further interesting physical properties for the solutions. We restrict here our analysis for the QBH limit of these charged stars. In this limit the speed of sound $c_{\rm s}$ is given by $c_{\rm s}^2 = \frac{a+(2-a)\sqrt{a}\sqrt{1-r^2/r_0^2}}{2-a+\sqrt{a}\sqrt{1-r^2/r_0^2}} - 1$. The case $a = 1$ (and $c = 0$) corresponds to the Majumdar-Papapetrou solutions, which have zero pressure. Thus, as expected the speed of sound of the QBH is zero when $a = 1$. At the surface, $c_{\rm s}^2$ tends to $-1 + a/(2-a)$ which reaches unity for $a = 4/3$. The speed of sound at the center ($r = 0$) of the QBH is such that $c_{\rm s}^2(0)$ is bounded to -1 from below as a tends to zero. In fact we see that $c_{\rm s}^2(0)$ is zero for $a = 1$ and tends monotonically to -1 as a goes to zero. Hence, $c_{\rm s}$ is undefined for all $a < 1$. The behavior of the speed of sound inside the stars with $a = 4/3$ and $a = 1.2$ can be seen in figure 3. As seen in the figure, for $a = 4/3$

the speed of sound approaches the speed of light close the surface of the star, independently of how compressed the star is.

4. Conclusions

Charged matter systems with pressure present novel and interesting features. A novel feature is the beautiful simple relation (2.3) displayed by the fields and matter quantities. An interesting feature is that when the matter is matched to a proper vacuum, the corresponding star solution, under appropriate compactification, can turn into a QBH with pressure. The presence of pressure in QBH solutions is new, and its importance comes from the fact that it tends to stabilize the system, erasing thus the criticism against the possible existence of QBHs based on the previous solutions without pressure (e.g., those solutions found in [6]).

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