

Space-time Torsion and Neutrino Oscillations

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We verify the consistency of the prescription of alternative minimum coupling (connection) proposed by the Teleparallel Equivalent to General Relativity (TEGR) for the Dirac equation. With this aim, we study the problem of neutrino oscillations in the Schwarzschild metric in radial motion. The relationship between the neutrino spin and space-time torsion is clarified through the determination of the phase differences between the spin eigenstates of neutrinos. In particular, if both mass eigenstates of neutrinos have opposite spin directions there is a contribution to the oscillation of neutrinos that comes from torsion.

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1. Introduction

The description of the gravitational field in the Teleparallel Equivalent to General Relativity (TEGR) introduced by Maluf led to tensorial expressions for the energy, momentum and angular momentum of the gravitational field [1], [2]. This theory can be considered as a reformulation of Einstein's general relativity in terms of tetrad fields $e^a{}_\mu$, which is known as tetrad gravity.

We examine the consistency of the Dirac equation in the TEGR through a new prescription of minimal coupling with the Dirac spinor fields ψ . This alternative prescription involves the Levi-Civita connection ${}^0\omega_{\mu ab}$ rather than the spin connection $\omega_{\mu ab}$ (field variable independent of the tetrad field $e^a{}_\mu$). With respect to this connection, Maluf showed in 1994 [3] that it is possible to rule out $\omega_{\mu ab}$ both in Lagrangian and Hamiltonian formulation of the TEGR.

One motivation for this work, is the fact that, in the context of metric affine theories of gravitation (MAG), Obukhov and Pereira [4] found an inconsistency in the coupling of the Dirac spinor with the gravitational field when the spin connection $\omega_{\mu ab}$ was used in the covariant derivative. They found that the Dirac spinor fields couple to the gravitational field in a manner consistent only with a spinless matter or a matter with a conserved spin tensor. The TEGR was studied by Obukhov and Pereira and can be considered to be a special case of the general theory of MAG. In 2003, Maluf showed that it is possible to bypass this problem with the use of a new type of coupling to the Dirac spinor fields [5]. To check the coupling explicitly, we apply the Dirac equation of the TEGR with the connection ${}^0\omega_{\mu ab}$ (totally dependent on tetrad field) to the problem of oscillations of solar neutrinos or mixing in vacuum. In particular, we calculate the phase dynamics of neutrinos in space-time with torsion in radial motion. The contributions of torsion and their relationship with the directions of spin in mass eigenstates were determined. These effects are compared with the structure of Minkowski and Riemann-Cartan geometry. In the Riemann geometry, there is no coupling between the spin of the particle and the gravitational field.

The article is organized as follows: Section 2 presents a summary of the Hamiltonian formulation of the TEGR and the problem of using the spin connection $\omega_{\mu ab}$. In Section 3 we present the new minimum coupling to the Dirac equation in the TEGR and apply this equation to model neutrino oscillations in vacuum. The conclusions are presented in Section 4.

Notation: Space-time indices μ, ν, \dots and local Lorentz $SO(3, 1)$ indices a, b, \dots run from 0 to 3. Time and space indices are indicated according to $\mu = 0, i$; $a = (0), (i)$. The flat space-time is fixed by $\eta_{ab} = e_{a\mu} e_{b\nu} g^{\mu\nu} = (-+++)$. The tetrad field $e^a{}_\mu$ and the arbitrary spin affine connection $\omega_{\mu ab}$ yield the usual definitions of the torsion and curvature tensors:

$R_{ab\mu\nu} = \partial_\mu \omega_{\nu ab} - \partial_\nu \omega_{\mu ab} - \dots$, $T^a{}_{\mu\nu} = \partial_\mu e^a{}_\nu + \omega_\mu{}^a{}_b e^b{}_\nu - \dots$. The determinant of the tetrad field is represented by $e = \det(e^a{}_\mu)$. c is the speed of light, \hbar is Planck's constant and G is the Newtonian gravitational constant

2. The Hamiltonian formulation of theTEGR

In 1994, Maluf [3] implemented a Hamiltonian formulation of theTEGR with local symmetry by imposing Schwinger's *time gauge* [6] and found the following Hamiltonian density:

$$\begin{aligned} H(e_{(j)i}, \Pi^{(j)i}, \omega_{kab}, P^{kab}, \omega_{0ab}) = & -N^k C_k - NC + \omega_{0ab} J^{ab} + \\ & -Ne\lambda^{abik} R_{abik} - \Sigma_{(m)(n)} C^{(m)(n)} + \\ & + \partial_k (P^{kab} \omega_{0ab}) + \partial_i [N_k \Pi^{ki} + N(2eT^i)] - \\ & - \lambda^{ij} \Pi_{[ij]}, \end{aligned} \quad (2.1)$$

where: λ^{abik} , N^k , $C^{(m)(n)}$, J^{ab} , λ^{ij} and N are Lagrange multipliers; C_k , $\Sigma_{(m)(n)}$, C , $\Pi_{[ij]}$, J^{ab} , R_{abik} are constraints defined in [3]; P^{kab} and $\Pi^{(j)i}$ are components of the momentum canonically conjugate to ω_{kab} and $e_{(j)i}$, respectively.

Maluf showed that the constraints in Eq. (2.1) are first class by fixing $\omega_{0ab} = 0$. Then we can eliminate the momentum P^{kab} from the Hamiltonian density. The evolution equation for ω_{kab} leads to:

$$\dot{\omega}_{kab} = \{\mathbf{H}, \omega_{kab}\} = 0, \quad (2.2)$$

and, for the equation to be consistent, it is necessary that $\omega_{kab} = 0$ and then $\omega_{\mu ab}$ is ruled out of the theory. Then, the Lagrangian density and the field equations are invariant under global Lorentz transformations.

3. Neutrino oscillations in theTEGR

The tetrad field associated with the Schwarzschild metric was determined using Schwinger's *time gauge* [6] $e^{(0)}{}_{i-} = 0$, $e_{(k)}{}^0 = 0$ and the condition for spatial symmetry [2]:

$$e_{(i)j} = e_{(j)i}$$

$$e^a{}_\mu = \begin{pmatrix} f & 0 & 0 & 0 \\ 0 & f^{-1} & 0 & 0 \\ 0 & 0 & r & 0 \\ 0 & 0 & 0 & r \sin \theta \end{pmatrix}, \quad (3.1)$$

where $f = (1 - \frac{2GM}{c^2 r})^{\frac{1}{2}}$ and M is the mass of the sun. The determinant of the tetrad field is given by $e = r^2 \sin \theta$.

The equation proposed by Maluf [5] for minimum coupling is described by a covariant derivative:

$$D_\mu \psi = \partial_\mu \psi - \frac{i}{4} \omega_{\mu ab} \Sigma^{ab} \psi. \quad (3.2)$$

By substituting (3.2) in the Dirac equation $i\hbar \gamma^\mu D_\mu \psi - m_0 c \psi = 0$ and using $\omega_{\mu ab} = -K_{\mu ab}$ where $K_{\mu ab}$ is the contortion tensor $K_{\mu ab} = \frac{1}{2} e_a^\lambda e_b^\nu (T_{\lambda\mu\nu} + T_{\nu\lambda\mu} - T_{\mu\nu\lambda})$, we have:

$$i\hbar \gamma^\mu (\partial_\mu \psi + \frac{i}{4} K_{\mu ab} \Sigma^{ab} \psi) - m_0 c \psi = 0. \quad (3.3)$$

By using the identity $\varepsilon_{abcf} \varepsilon^{abcd} = -3! \delta_f^d$, and by considering only the axial part of the contortion tensor represented by the vector $A^a = (A^{(0)}, \vec{A})$, we obtain:

$$K_{abc} \varepsilon^{abcd} = \varepsilon_{abcf} A^f \varepsilon^{abcd} = \varepsilon_{abcf} \varepsilon^{abcd} A^f = -3! \delta_f^d A^f. \quad (3.4)$$

By using identities involving the Dirac matrices $\gamma^a = e^a_\mu \gamma^\mu = (\gamma^{(0)}, \gamma^{(i)})$:

$$\gamma^a [\gamma^b, \gamma^c] = -2\eta^{ab} \gamma^c + 2\eta^{ac} \gamma^b + 2i \varepsilon^{abcd} \gamma_{(5)} \gamma^{(0)} \gamma^{(1)} \gamma^{(2)} \gamma^{(3)} \gamma_d, \dots \quad (3.5)$$

$$\gamma_{(5)} = -i \gamma^{(0)} \gamma^{(1)} \gamma^{(2)} \gamma^{(3)}, \quad \Sigma^{ab} = \frac{i}{2} [\gamma^a, \gamma^b] \quad (3.6)$$

$$\gamma^{(0)} \gamma_{(5)} \gamma_a A^a = \gamma_{(5)} A^{(0)} + \vec{\Sigma} \cdot \vec{A}, \quad (3.7)$$

$$\vec{\Sigma} = \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}, \quad \gamma^a \gamma^b + \gamma^b \gamma^a = -2\eta^{ab}, \quad (3.8)$$

and by defining the components of the momentum as:

$$p_r = -i\hbar \left(\frac{\partial}{\partial r} - \frac{1}{r} - \frac{1}{2f} \frac{\partial}{\partial r} f \right), \quad p_\theta = -\frac{i\hbar}{r} \left(\frac{\partial}{\partial \theta} - \frac{\cot \theta}{2} \right) \text{ and } p_\varphi = -\frac{i\hbar}{r \sin \theta} \frac{\partial}{\partial \varphi}, \quad (3.9)$$

we can write the Dirac Hamiltonian matrix with the help of Eq. (3.4):

$$H = \begin{pmatrix} f l m_0 c^2 + \frac{3}{2} \hbar c f (\sigma^{(1)} A^{(1)} + \sigma^{(2)} A^{(2)} + \sigma^{(3)} A^{(3)}) & f^2 I \sigma^{(1)} p_r c + f l \sigma^{(2)} p_\theta c + f l \sigma^{(3)} p_\varphi c - \frac{3}{2} \hbar c f A^{(0)} I \\ f^2 I \sigma^{(1)} p_r c + f l \sigma^{(2)} p_\theta c + f l \sigma^{(3)} p_\varphi c - \frac{3}{2} \hbar c f A^{(0)} I & -f l m_0 c^2 + \frac{3}{2} \hbar c f (\sigma^{(1)} A^{(1)} + \sigma^{(2)} A^{(2)} + \sigma^{(3)} A^{(3)}) \end{pmatrix}, \quad (3.10)$$

where $\sigma^{(i)}$ are the Pauli matrices, and I is the identity matrix.

We analyze here the radial motion which was considered in Ref. [7] and [8] in a different formalism.

3.1 Radial motion

Zhang and Pereira in Ref. [9] was one of the first to calculate the mass neutrino oscillation induced by torsion in the context of the weak gravitational expansion method. For the calculation of the phase difference in radial motion in the presence of the torsion of space-time, we proceed in the same way as in Ref. [10]. Substituting $\vec{p} = (p_r, p_\theta, p_\varphi) = (p, 0, 0)$ in the

expression (3.10) and using the ultra-relativistic approximation $pc \cong E$, we obtain the following phase differences for the neutrino mass eigenstates:

1. For eigenstates with the same spin:

$$\Delta\Phi = \Phi_2^\uparrow - \Phi_1^\uparrow = \frac{\Delta m^2 c^3}{2E\hbar} \Delta r, \quad (3.1.1)$$

where $\Delta m^2 = m_2^2 - m_1^2$ and $\Delta r = r_B - r_A$. Here r_A is the radius of the sun where the neutrino was produced and r_B is the distance from the center of the sun to the earth's surface where the neutrino was measured.

2. For the first eigenstate with spin down and the second eigenstate with spin up:

$$\Delta\Phi = \Phi_2^\downarrow - \Phi_1^\uparrow = \frac{\Delta m^2 c^3}{2E\hbar} \Delta r - 3A \left(\Delta r - \frac{MG}{c^2} \ln \frac{r_B}{r_A} \right), \quad (3.1.2)$$

3. For the first eigenstate with spin up and the second eigenstate with spin down:

$$\Delta\Phi = \Phi_2^\uparrow - \Phi_1^\downarrow = \frac{\Delta m^2 c^3}{2E\hbar} \Delta r + 3A \left(\Delta r - \frac{MG}{c^2} \ln \frac{r_B}{r_A} \right), \quad (3.1.3)$$

where $A = \sqrt{(A^{(0)} - A^{(1)})^2 + (A^{(2)})^2 + (A^{(3)})^2}$. Here, we used $cdt \cong dr$ and $f = \left(1 - \frac{2MG}{rc^2}\right)^{\frac{1}{2}} \approx 1 - \frac{MG}{rc^2}$. Note that the magnitude of the contribution from torsion is very small compared with the mass differences between neutrinos [10].

If electron neutrinos are produced at time $t = 0$, the time evolution of the flavor eigenstate ν_e is given by:

$$\nu_e(t) = \cos \Theta e^{-i\Phi_1(t)} \nu_1 + \sin \Theta e^{-i\Phi_2(t)} \nu_2, \quad (3.1.4)$$

where ν_1 and ν_2 are called mass eigenstates at $t = 0$ and Θ is called the mixing angle. The probability of measuring a muon or tau neutrino is given by:

$$P(\nu_e \rightarrow \nu_{\mu,\tau}) = \sin^2 2\Theta \sin^2 \frac{\Delta\Phi}{2}. \quad (3.1.5)$$

4. Conclusions

In radial motion, the torsion makes a contribution to the phase dynamics of neutrinos that depends on the directions of spin in the mass eigenstates. If both mass eigenstates have the same direction of spin, there is no contribution to oscillation from torsion, but if the mass eigenstates have opposite spin directions there is a contribution to the oscillation of neutrinos that comes from torsion.

Without torsion or curvature (with $f = 1$), our results coincide with results for neutrino oscillations in vacuum in flat space-time [7].

Our result formally coincides with a result obtained by Adak et al. [10] from the Dirac equation in a Riemann-Cartan geometry with the connection of spin $\omega_{\mu ab}$.

The prescription of minimum coupling adopted by Maluf does not lead to inconsistencies of the type that Obukhov and Pereira found and are qualitatively consistent with the literature, at least for the problem of neutrino oscillation. The formal inconsistency in the Dirac equation is removed if we adopt the Levi-Civita connection of the theory as

${}^0\omega_{\mu ab} = -K_{\mu ab}$ leaving the total Lagrangian density in the TEGR in the presence of Dirac spinor fields invariant under global Lorentz transformations.

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