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Symmetry approach and the generalized Korteweg-de Vries equation with variable coefficients

Wescley L. Souza* **and Érica M. Silva†**
Instituto de Física - UFMT, Brazil
E-mail: wescley@fisica.ufmt.br, erica@fisica.ufmt.br

Using Lie symmetry arguments, we consider a class of nonlinear KdV-type equations which are usually denoted by $K(m,n)$. We start with the 4-dimensional Lie algebra of the ordinary KdV equation to derive and classify $K(m,n)$ equations with space- and time-dependent coefficients.

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*Poster Section
†Corresponding author

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1. Introduction

More than one decade ago, the celebrated Korteweg-de Vries (KdV) equation,

$$u_t + \left[u^2 + u_{xx}\right]_x = 0, \tag{1.1}$$

was generalized to a class of nonlinear equations, named as $K(m,n)$ equations [1], given by

$$u_t + [u^m + (u^n)_{xx}]_x = 0\ ,\ \ m > 0\ ,\ 1 < n \leq 3\ . \tag{1.2}$$

For some values of m and n, solutions of $K(m,n)$ equations have compact support and independent wave amplitude width [1]. This kind of solution is called *compacton* and, in nature, is different from a KdV soliton, that narrows as the amplitude increases.

In the classical soliton theory, integrability and elastic collisions are closely connected but, in the realm of the $K(m,n)$ equations, albeit some conservation laws have been derived, it is not known whether these equations are integrable [2]. A lot of effort has been carried out in order to understand the nonlinear mechanism that underlies processes described by $K(m,n)$ equations [3, 4, 5], including an analogous generalization of the Sine-Gordon equation [6]. Lie symmetry methods have also been used for this purpose, and a partial symmetry classification of $K(m,n)$ equations has been achieved [7, 8, 9].

In order to derive generalizations of a partial differential equation, a standard procedure is to choose a starting symmetry, that usually is considered as the symmetry of a more restrictive set of equations. This approach has been adopted to obtain, for instance, generalized Fokker-Planck equations which admits the Lie symmetry of a specific diffusion equation [10]. Our goal here is to follow along this line and study and classify some $K(m,n)$ equations. Considering the Lie symmetry algebra of the classical KdV equation, say ℓ_{KdV}, then we proceed to find all equations in a given class of $K(m,n)$ equations that are invariant under ℓ_{KdV}. The class we have studied is the nonlinear $K(m,n)$ equations with space- and time-dependent coefficients.

2. Determining equations

Let us start by noting that, associated with Eq. (1.1), there is a set of generators of Lie symmetries given by

$$\begin{aligned} X_1 &= \partial_x\ , \\ X_2 &= \partial_t\ , \\ X_3 &= 2t\partial_x + \partial_u\ , \\ X_4 &= x\partial_x + 3t\partial_t - 2u\partial_u\ . \end{aligned}$$

These generators fulfill the following commutation relations

$$\begin{aligned} [X_1, X_4] &= X_1\ , \\ [X_2, X_3] &= 2X_1\ , \\ [X_2, X_4] &= 3X_2\ , \\ [X_3, X_4] &= -2X_3\ , \\ [X_1, X_2] &= [X_1, X_3] = 0\ . \end{aligned} \tag{2.1}$$

We use this KdV equation Lie algebra, denoted here by ℓ_{KdV}, to derive $K(m,n)$ equations. Let us consider a nonlinear generalization of Eq. (1.2) with space- and time-dependent coefficients, that is,

$$u_t + [fu^m + g(u^n)_{xx}]_x = 0 , \tag{2.2}$$

where $f = f(x,t)$ and $g = g(x,t)$. This equation is written as

$$u_t + a_0 u^m + a_1 u^{m-1} u_x + a_2 u^{n-2} u_x^2 + a_3 u^{n-1} u_{xx} + a_4 u^{n-3} u_x^3 + a_5 u^{n-2} u_x u_{xx} + a_6 u^{n-1} u_{xxx} = 0 , \tag{2.3}$$

where

$$\begin{aligned} a_0 &= f_x , \\ a_1 &= mf , \\ a_2 &= n(n-1)g_x , \\ a_3 &= ng_x , \\ a_4 &= n(n-1)(n-2)g , \\ a_5 &= 3n(n-1)g , \\ a_6 &= ng . \end{aligned} \tag{2.4}$$

A vector field of the form

$$X = \eta(u,x,t)\partial_u + \theta_1(u,x,t)\partial_x + \theta_2(u,x,t)\partial_t \tag{2.5}$$

is a symmetry generator of Eq. (1.1) if Eq. (2.3) is form invariant under the infinitesimal transformation $x' = x + \varepsilon\theta_1$, $t' = t + \varepsilon\theta_2$, and $u' = u + \varepsilon\eta$. This leads to the following set of determining equations:

$$\begin{aligned}
eq_1 &: \left(a_6 u^{n-1}\partial_{uuu} + a_5 u^{n-2}\partial_{uu} + 2a_4 u^{n-3}\partial_u\right)\theta_2(u,x,t) = 0 , \\
eq_2 &: \left(6a_6 u^{n-1}\partial_{ux} + 2a_5 u^{n-2}\partial_x + 2a_3 u^{n-1}\partial_u\right)\theta_2(u,x,t) = 0 , \\
eq_3 &: \left(a_6 u^{n-1}\partial_{uuu} + a_5 u^{n-2}\partial_{uu} - a_4 u^{n-3}\partial_u\right)\theta_1(u,x,t) = 0 , \\
eq_4 &: \left(3a_6 u^{n-1}\partial_u\right)\theta_1(u,x,t) = 0 , \\
eq_5 &: \left(3a_6 u^{n-1}\partial_{uu} + a_5 u^{n-2}\partial_u\right)\theta_2(u,x,t) = 0 , \\
eq_6 &: \left(3a_6 u^{n-1}\partial_{xx} + 2a_3 u^{n-1}\partial_x\right)\theta_2(u,x,t) = 0 , \\
eq_7 &: \left(3a_6 u^{n-1}\partial_{ux} + a_5 u^{n-2}\partial_x\right)\theta_2(u,x,t) = 0 , \\
eq_8 &: \left(3a_6 u^{n-1}\partial_{uu} + 2a_5 u^{n-2}\partial_u\right)\theta_2(u,x,t) = 0 , \\
eq_9 &: \left(3a_6 u^{n-1}\partial_u\right)\theta_2(u,x,t) = 0 , \\
eq_{10} &: \left(6a_6 u^{n-1}\partial_{uu}\right)\theta_1(u,x,t) = 0 , \\
eq_{11} &: 3a_6 u^{n-1}\partial_x\theta_2(u,x,t) = 0 , \\
eq_{12} &: (3\partial_u)\theta_1(u,x,t) - \left(3a_6 u^{n-1}\partial_{uxx} + a_5 u^{n-2}\partial_{xx} + 2a_3 u^{n-1}\partial_{ux} + 2a_2 u^{n-2}\partial_x\right)\theta_2(u,x,t) = 0 , \\
eq_{13} &: \left(3a_6 u^{n-1}\partial_{uux} + 2a_5 u^{n-2}\partial_{ux} + 3a_4 u^{n-3}\partial_x + a_3 u^{n-1}\partial_{uu} + a_2 u^{n-2}\partial_u\right)\theta_2(u,x,t) = 0 ,
\end{aligned}$$

$$
\begin{aligned}
eq_{14} \ : \ & \left(nu^{-1} - u\right) \eta(u,x,t) + \left(\frac{(\partial_x a_6)}{a_6} - 3\partial_x\right) \theta_1(u,x,t) \\
& + \left(a_6 u^{n-1}\partial_{xxx} + \frac{(\partial_t a_6)}{a_6} + a_3 u^{n-1}\partial_{xx} + a_1 u^{m-1}\partial_x - \partial_u + \partial_t\right) \theta_2(u,x,t) = 0 \ , \\
eq_{15} \ : \ & \left(3a_6 u^{n-1}\partial_{uxx} + a_5 u^{n-2}\partial_{xx} + 2a_3 u^{n-1}\partial_{ux} + a_1 u^{m-2} m\right) \eta(u,x,t) \\
& - \left(a_6 u^{n-1}\partial_{xxx} + a_3 u^{n-1}\partial_{xx} + a_1 \frac{(\partial_x a_6)}{a_6} u^{m-1} - 4a_0 u^m \partial_u + \partial_t\right) \theta_1(u,x,t) \\
& - \left(a_1 \left(\frac{(\partial_t a_6)}{a_6} - \frac{(\partial_t a_1)}{a_1}\right) u^{m-1}\right) \theta_2(u,x,t) = 0 \ , \\
eq_{16} \ : \ & \left(3a_6 u^{n-1}\partial_{uux} + 2a_5 u^{n-2}\partial_{ux} + 3a_4 u^{n-3}\partial_x - a_3 u^{n-1}\partial_{uu} + a_2\left(u^{n-2}\partial_u - u^{n-3}\right)\right) \eta(u,x,t) \\
& - \left(3a_6 u^{n-1}\partial_{uxx} + a_5 u^{n-2}\partial_{xx} + 2a_3 u^{n-1}\partial_{ux} + a_2 u^{n-2}\left(\frac{(\partial_x a_6)}{a_6} - \partial_x\right)\right) \theta_1(u,x,t) \\
& - \left((\partial_x a_2) u^{n-2} - 3a_1 u^{m-1}\partial_u\right) \theta_1(u,x,t) - \left(a_2\left(\frac{(\partial_t a_6)}{a_6} - \frac{(\partial_t a_2)}{a_2}\right) u^{n-2}\right) \theta_2(u,x,t) = 0 \ , \\
eq_{17} \ : \ & \left(a_6 u^{n-1}\partial_{uuu} + a_5 u^{n-2}\partial_{uu} + 2a_4\left(u^{n-3}\partial_u - u^{n-4}\right)\right) \eta(u,x,t) \\
& - \left(3a_6 u^{n-1}\partial_{uux} + 2a_5 u^{n-2}\partial_{ux} + a_4 \frac{(\partial_x a_6)}{a_6} u^{n-3} - (\partial_x a_4) u^{n-3} + a_3 u^{n-1}\partial_{uu}\right) \theta_1(u,x,t) \\
& - \left(2a_2 u^{n-2}\partial_u\right) \theta_1(u,x,t) - \left(a_4\left(\frac{(\partial_t a_6)}{a_6} - \frac{(\partial_t a_4)}{a_4}\right) u^{n-3}\right) \theta_2(u,x,t) = 0 \ , \\
eq_{18} \ : \ & \left(3a_6 u^{n-1}\partial_{uu} + a_5\left(u^{n-2}\partial_u - u^{n-3}\right)\right) \eta(u,x,t) \\
& - \left(9a_6 u^{n-1}\partial_{ux} + a_5 \frac{(\partial_x a_6)}{a_6} u^{n-2} - (\partial_x a_5) u^{n-2} - a_3 u^{n-1}\partial_u\right) \theta_1(u,x,t) \\
& - \left(a_5\left(\frac{(\partial_t a_6)}{a_6} - \frac{(\partial_t a_5)}{a_5}\right) u^{n-2}\right) \theta_2(u,x,t) = 0 \ , \\
eq_{19} \ : \ & \left(a_6 u^{n-1}\partial_{xxx} + a_3 u^{n-1}\partial_{xx} + a_1 u^{m-1}\partial_x + a_0\left(u^m \partial_u + u^{m-1}(m-n+1)\right) + \partial_t\right) \eta(u,x,t) \\
& - \left(a_0 u^m \left(\frac{(\partial_x a_6)}{a_6} - \frac{(\partial_x a_0)}{a_0} - 3\partial_x\right)\right) \theta_1(u,x,t) \\
& - \left(a_0\left(\frac{(\partial_t a_6)}{a_6} - \frac{(\partial_t a_0)}{a_0}\right) u^m\right) \theta_2(u,x,t) = 0 \ , \\
eq_{20} \ : \ & \left(3a_6 u^{n-1}\partial_{ux} + a_5 u^{n-2}\partial_x\right) \eta(u,x,t) \\
& - \left(3a_6 u^{n-1}\partial_{xx} + a_3 u^{n-1}\left(\frac{(\partial_x a_6)}{a_6} - \frac{(\partial_x a_3)}{a_3} - \partial_x\right)\right) \theta_1(u,x,t) \\
& - \left(a_5\left(\frac{(\partial_t a_6)}{a_6} - \frac{(\partial_t a_3)}{a_3}\right) u^{n-1}\right) \theta_2(u,x,t) = 0 \ . \qquad (2.6)
\end{aligned}
$$

The substitution of the dominant derivative from Eq. (1.1) into the determining system of infinitesimal symmetry transformations results in a set of equations in the coefficients a_i, $i = 0, \ldots, 6$.

3. The $K(m,n)$ equations with variable coefficients

In order to find the $K(m,n)$ generalized equations, we impose that Eq. (2.3) admits subalgebras of ℓ_{KdV} as symmetry Lie algebras. Recalling that ℓ_{KdV} is spanned by the generators X_i, $i = 1,2,3,4$, we have the following cases.

Symmetry $\{\mathbf{X}_4\}$: By imposing this symmetry generator to Eq. (2.3), the resulting system of equations in the coefficients a_i, $i = 0, \ldots, 6$, is

$$\begin{aligned}
eq_1 &: \frac{1}{2}u^m\left(a_0\left(\frac{(\partial_x a_6)}{a_6}x + 3\frac{(\partial_t a_6)}{a_6}t + 2\left(m - n - \frac{3}{2}\right)\right) - (\partial_x a_0)x - 3(\partial_t a_0)t\right) = 0\ , \\
eq_2 &: \frac{1}{2}u^{m-1}\left(a_1\left(\frac{(\partial_x a_6)}{a_6}x + 3\frac{(\partial_t a_6)}{a_6}t + 2(m - n - 1)\right) - (\partial_x a_1)x - 3(\partial_t a_1)t\right) = 0\ , \\
eq_3 &: \frac{1}{2}u^{n-2}\left(a_2\left(\frac{(\partial_x a_6)}{a_6}x + 3\frac{(\partial_t a_6)}{a_6}t - 1\right) - (\partial_x a_2)x - 3(\partial_t a_2)t\right) = 0\ , \\
eq_4 &: \frac{1}{2}u^{n-1}\left(a_3\left(\frac{(\partial_x a_6)}{a_6}x + 3\frac{(\partial_t a_6)}{a_6}t - 1\right) - (\partial_x a_3)x - 3(\partial_t a_3)t\right) = 0\ , \\
eq_5 &: \frac{1}{2}u^{n-3}\left(a_4\left(\frac{(\partial_x a_6)}{a_6}x + 3\frac{(\partial_t a_6)}{a_6}t\right) - (\partial_x a_4)x - 3(\partial_t a_4)t\right) = 0\ , \\
eq_6 &: \frac{1}{2}u^{n-2}\left(a_5\left(\frac{(\partial_x a_6)}{a_6}x + 3\frac{(\partial_t a_6)}{a_6}t\right) - (\partial_x a_5)x - 3(\partial_t a_5)t\right) = 0\ , \\
eq_7 &: 2\left(\frac{(\partial_x a_6)}{a_6}x + 3\frac{(\partial_t a_6)}{a_6}t\right) - n + 1 = 0\ .
\end{aligned} \tag{3.1}$$

Taking the general solution of the system given by Eqs. (3.1) into Eqs. (2.4) implies that

$$u_t + [fu^m + g(u^n)_{xx}]_x = 0\ , \tag{3.2}$$

with

$$f(x,t) = \tfrac{c_1 t x^{2m-7}}{m} \quad \text{and} \quad g(x,t) = \tfrac{c_2 t x^{2n-5}}{n}\ , \tag{3.3}$$

being c_1 and c_2 arbitrary constants. The choice of symmetries $\{\mathbf{X}_1, \mathbf{X}_4\}$, $\{\mathbf{X}_2, \mathbf{X}_4\}$ and $\{\mathbf{X}_3, \mathbf{X}_4\}$ leads to the same result as $\{\mathbf{X}_4\}$.

Symmetry $\{\mathbf{X}_3\}$: Following the prescription described, we have another class of $K(m,n)$ equations with variable coefficients

$$f(u,x,t) = \tfrac{c_1 t}{m} e^{(1-m)x/2tu} + \tfrac{2u^{2-m}}{m(m-1)} \quad \text{and} \quad g(u,x,t) = \tfrac{c_2 t}{n} e^{(1-n)x/2tu}\ , \tag{3.4}$$

with c_1 and c_2 as arbitrary constants.

Symmetries $\{\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3, \mathbf{X}_4\}$: Finally, replacing separately the coefficients of each generator X_i, $i = 1, 2, 3, 4$, into the system of determining Eqs. (2.6), we have obtained

$$f(u,x,t) = \tfrac{c_1 x t}{m} e^{(1-m)x/2tu} \quad \text{and} \quad g(u,x,t) = \tfrac{c_2 t}{n} e^{(1-n)x/2tu}\ , \tag{3.5}$$

with arbitrary constants c_1 and c_2.

4. General Remarks

In this work we have used the known symmetry algebra of the classical KdV equation, ℓ_{KdV}, to find and classify $K(m,n)$ equations with space- and time-dependent coefficients. This programme required an intensive use of computer algebra, and we have used the package SADE (*Symmetry Analysis of Differential Equations*) [11] for solving the determining equations. These equations with variable coefficients are particularly useful to understand the nonlinear mechanism that underlies processes described by the $K(m,n)$ equations. For this class of nonlinear KdV-type equations, a study of symmetry invariant solutions is in progress.

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