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Podolsky's Electromagnetic Theory on the Null-Plane

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We have analyzed the null-plane canonical structure of Podolsky's electromagnetic theory. As a theory that contains higher order derivatives in the Lagrangian function, it was necessary to redefine the canonical momenta related to the field variables. We were able to find a set of first and second-class constraints, and also to derive the field equations of the system.

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1. Introduction

The generalized electromagnetic theory of Podolsky was developed in the 1940's [1, 2] as a generalization of Maxwell's electromagnetism. Besides the fact that the quantum electrodynamics, based on Maxwell's theory, is the most successful theory of the modern physics, it suffers from problems of divergences as the infinite self energy of the punctual electron and a divergent vacuum polarization current, difficulties that come from the fact that the classical electrodynamics presents a $\mathbf{r}^{-1}$ singularity in the electrostatic potential.

Podolsky's theory adds a higher order derivative term in Maxwell's Lagrangian, which maintains the most important features of the classical electromagnetism, such as the invariance under $U(1)$ and Poincaré, and also gives linear field equations. In fact, it is shown in [3] that Podolsky's Lagrangian density,

$$\mathscr{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{a^2}{2}\partial_\lambda F^{\mu\lambda}\partial^\gamma F_{\mu\gamma}\,, \qquad F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu\,, \tag{1.1}$$

with field equations

$$\left(1+a^2\Box\right)\Box A_\mu - \partial_\mu\left(1+a^2\Box\right)\partial^\nu A_\nu = 0, \tag{1.2}$$

is the only possible generalization of the electromagnetic field that preserves these qualities. Moreover, the theory is free from the problem of electron's infinite energy [4], since the electrostatic potential becomes Yukawa's type. If these features are not enough to credentiate Podolsky's theory as a viable effective theory, at least in the infrared sector, the theory also predicts massive photon modes, whose mass is proportional to a^{-1}, which allows experimentation.

In this paper we intend to analyze the canonical structure of Podolsky's electromagnetic theory on the null-plane. Dirac [5] was the first to notice that the usual quantization programme, which requires the choice of the "time axis", the $t = x^0$ axis, as the evolution parameter of the theory, also called the "instant-form" dynamics, was not the only way of Hamiltonian dynamics. The null-plane dynamics is taken choosing $x^+ \equiv 1/\sqrt{2}\left(x^0+x^3\right)$ as the evolution parameter, which implies quantization over a hyper-surface of constant "time" x^+, called the null-plane. Theories analyzed on the null-plane always have second-class constraints, resulting in Dirac's brackets defined on these surfaces.

In the study of the Podolsky's theory we will follow the procedure outlined in [6], which brings a systematic analysis of the canonical structure of Podolsky's field in instant-form, as well as the references [7, 8, 9], in which we can find specific applications.

2. Podolsky's canonical analysis on the null-plane

The coordinates of the light-cone are the best coordinate system to analyze the null-plane dynamics. They are given by

$$x^+ \equiv \frac{1}{\sqrt{2}}\left(x^0+x^3\right), \quad x^- \equiv \frac{1}{\sqrt{2}}\left(x^0-x^3\right), \quad x^1 = x^1, \quad x^2 = x^2. \tag{2.1}$$

This transformation implies the metric

$$\eta_{\mu\nu} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -\mathbf{I} \end{pmatrix}, \tag{2.2}$$

where $\mathbf{I}$ is the 2×2 identity matrix. The coordinate x^+ is chosen to be the "time" parameter of the system. The dynamics is given by a set of kinematical generators, which generates infinitesimal transformations on a given null-plane of constant x^+, and a set of dynamical generators, which results in transformations from a null-plane to another of "later" value of x^+. Special care is needed for the initial-boundary conditions of the system, since the surfaces of constant x^+ are not Cauchy surfaces. Null-planes are actually Characteristics surfaces and a systematic study of boundary conditions can be found in refs. [10, 11, 12, 13, 14].

From the action built with the Lagrangian density (1.1), we are able to write the energy-momentum tensor which is the conserved current due to Poincaré invariance [6]. In light-cone coordinates the dynamical generator of x^+ evolution is given by the Hamiltonian

$$H_c \equiv \int d^3x T_{+-}(x) = \int d^3x \left[p^\mu \partial_+ A_\mu + \pi^\mu \partial_+\partial_+ A_\mu - \mathscr{L}\right], \qquad d^3x \equiv dx^- d^2\mathbf{x}, \tag{2.3}$$

where the conjugate momenta for the Podolsky's field are

$$p^\mu = F^{\mu+} - a^2\left(\eta^{\mu-}\partial_-\partial_\lambda F^{+\lambda} + \eta^{\mu i}\partial_i\partial_\lambda F^{+\lambda} - 2\partial_-\partial_\lambda F^{\mu\lambda}\right), \tag{2.4}$$

$$\pi^\mu = a^2\eta^{\mu+}\partial_\lambda F^{+\lambda}. \tag{2.5}$$

The Hessian matrix of this system is just $W^{\mu\nu} = -a^2\eta^{\mu+}\delta^\nu_-\eta^{++} = 0$, so the system has constraints. Following Ostrogradski's method [15] we consider A_μ and ∂_+A_μ as independent variables. Therefore, using the notation $\bar{A}_\mu \equiv \partial_+A_\mu$, being A_μ and $\bar{A}_\mu$ independent fields, primary constraints are given by

$$\phi_1 = \pi^+ \approx 0, \tag{2.6}$$

$$\phi_2^i = \pi^i \approx 0, \tag{2.7}$$

$$\phi_3 = p^+ - \partial_-\pi^- \approx 0, \tag{2.8}$$

$$\phi_4^i = p^i - \partial_i\pi^- + F_{i-} + 2a^2\partial_-\left[\partial_i\bar{A}_- - 2\partial_-\bar{A}_i + \partial_i\partial_-A_+ - \partial_jF_{ij}\right] \approx 0. \tag{2.9}$$

The canonical Hamiltonian density can be expressed by

$$\begin{aligned}\mathscr{H}_c =\ & p^\mu\bar{A}_\mu + \pi^-\left(\partial_-\bar{A}_+ - \partial^i\bar{A}_i + \partial^i\partial_iA_+\right) - \frac{1}{2}\left(\bar{A}_- - \partial_-A_+\right)^2 - \left(\bar{A}_i - \partial_iA_+\right)F_{-i}\\ & + \frac{1}{4}F_{ij}F^{ij} + \frac{a^2}{2}\left(\partial_i\bar{A}_- - 2\partial_-\bar{A}_i + \partial_i\partial_-A_+ - \partial_jF_{ij}\right)^2,\end{aligned} \tag{2.10}$$

which will be used to define the primary Hamiltonian

$$H_P \equiv H_c + \int d^3x u^a(x)\phi_a(x), \qquad \{a\} = \{1,2,3,4\}. \tag{2.11}$$

To proceed with the calculus of the consistency conditions we use the primary Hamiltonian as generator of the x^+ evolution and define the fundamental equal x^+ Poisson brackets with the expressions

$$\left\{A_\mu(x), p^\nu(y)\right\}_{x^+=y^+} = \left\{\bar{A}_\mu(x), \pi^\nu(y)\right\}_{x^+=y^+} = \delta^\nu_\mu\delta^3(x-y), \tag{2.12}$$

where $\delta^3(x-y) \equiv \delta(x^- - y^-)\delta^2(\mathbf{x}-\mathbf{y})$. We verify that the condition $\dot{\phi}_1 \approx 0$ gives just the constraint $\phi_3 \approx 0$, which is already satisfied. The consistency for the remaining constraints gives equations for some Lagrange multipliers. The conditions for ϕ_2^i and ϕ_3,

$$\begin{aligned} \dot{\phi}_2^i &= -\phi_4^i + 4a^2\partial_-\partial_- u_i^4 \approx 0, \\ \dot{\phi}_3 &= \partial_- p^- + \partial_i p^i + 4a^2\partial_i\partial_-\partial_- u_i^4 \approx 0, \end{aligned}$$

give equations for the same parameters u_i^4. These equations must be consistent to each other, which result on a secondary constraint

$$\chi \equiv \partial_- p^- + \partial_i p^i \approx 0. \tag{2.13}$$

For this secondary constraint, $\dot{\chi} = 0$, and no more constraints can be found.

It happens that χ and ϕ_1 are first-class constraints, while ϕ_2^i , ϕ_3 and ϕ_4^i are second-class ones. However, constructing the matrix of the second-class constraints we found that it is singular of rank four, which indicates that there must be a first-class constraint, and its construction is made from the corresponding eigenvector which gives a linear combination of second-class constraints. The combination happens to be just $\Sigma_2 \equiv \phi_3 - \partial_i\phi_2^i$ and it is independent of χ and ϕ_1. Therefore, we have the renamed set of first-class constraints

$$\begin{aligned} \Sigma_1 &\equiv \pi^+ \approx 0, && (2.14) \\ \Sigma_2 &\equiv p^+ - \partial_-\pi^- - \partial_k\pi^k \approx 0, && (2.15) \\ \Sigma_3 &\equiv \partial_- p^- + \partial_i p^i \approx 0, && (2.16) \end{aligned}$$

and a set of irreducible second-class constraints

$$\begin{aligned} \Phi_1^i &\equiv \pi^i \approx 0, && (2.17) \\ \Phi_2^i &\equiv p^i - \partial_i\pi^- + F_{i-} + 2a^2\partial_-\left[\partial_i\bar{A}_- - 2\partial_-\bar{A}_i + \partial_i\partial_- A_+ - \partial_j F_{ij}\right] \approx 0. && (2.18) \end{aligned}$$

The second-class constraints do not appear in the instant-form dynamics [6] for this theory: they are a common effect of the null-plane dynamics.

Here we are in position to write the total Hamiltonian

$$H_T \equiv H_c + \int d^3x u^a(x)\Sigma_a(x) + \int d^3x \lambda_i^I(x)\Phi_I^i(x). \tag{2.19}$$

We have the set of first-class constraints $\Sigma_a \approx 0$, and Dirac's conjecture states that they are generators of symmetries of the action. As a consequence of this fact we have that field equations must be independent of the second-class constraints parameters, λ_i^I, as they can be eliminated by introduction of Dirac's brackets, under appropriate initial-boundary conditions. The fact that first-class constraints are generators of canonical transformations is a natural consequence of the Hamilton-Jacobi formalism applied to constrained systems [16, 17, 18].

Now let us calculate the canonical equations of the system for the variables A_μ, $\bar{A}_\mu$, p^μ and π^μ. For A_μ we have the equations

$$\partial_+ A_\mu = \bar{A}_\mu + \delta_\mu^+ u^2 - \delta_\mu^- \partial_- u^3 - \delta_\mu^i\left[\partial_i u^3 - \lambda_i^2\right], \tag{2.20}$$

which just means that the canonical variable $\bar{A}_\mu$ is defined as $\partial_+ A_\mu$ plus a linear combination of the still arbitrary Lagrange multipliers. The equations for $\bar{A}_\mu$ give

$$\partial_+\bar{A}_\mu \approx \delta_\mu^+ u^1 + \delta_\mu^- \left[\partial_-\bar{A}_+ + \partial_i\bar{A}_i - \partial_i\partial_i A_+ + \partial_- u^2 + \partial_i\lambda_i^2\right] + \delta_\mu^i \left[\partial_i u^2 + \lambda_i^1\right]. \tag{2.21}$$

The equation for $\bar{A}_+$ is just $\partial_+\bar{A}_+ \approx u^1$, which is expected since $\bar{A}_+$ is a degenerate variable. The expression for $\bar{A}_-$ can be written, using (2.20), as

$$\partial_\mu F^{-\mu} \approx -\left[\partial^i\partial_i + \partial^+\partial_+\right] u^3. \tag{2.22}$$

The Hamiltonian equations for the momenta p^μ are given, with (2.20) and $\pi^- = +a^2\partial_\lambda F^{+\lambda}$, by

$$\begin{aligned}
&\partial_+ p^+ \approx \partial_\lambda F^{\lambda+} - a^2\partial_i\partial_-\partial_\lambda F^{\lambda i} - a^2\partial_i\partial_i\partial_\lambda F^{\lambda+} + \left(1 + a^2\partial_i\partial_i\right)\partial_-\partial_- u^3,\\
&\partial_+ p^- \approx \partial_i F^{i-} + \partial_i\partial_i u^3,\\
&\partial_+ p^{\,i} \approx \partial_- F^{-i} + \partial_j F^{ji} - a^2\partial_\mu\partial^\mu\partial_j F^{ij} - \partial_-\partial_i u^3.
\end{aligned}$$

The equations for π^μ are, using the fact that π^+ and π^i are weakly zero,

$$\begin{aligned}
&p^+ \approx a^2\partial_-\partial_\lambda F^{+\lambda},\\
&p^- \approx F^{-+} + a^2\partial_-\partial_\lambda F^{-\lambda} + \partial_- u^3 - a^2\partial_-\partial_i\partial_i u^3,\\
&p^{\,i} \approx F^{i+} - a^2\left(\partial^i\partial_\lambda F^{+\lambda} - 2\partial_-\partial_\lambda F^{i\lambda}\right) + 2a^2\partial_-\partial_-\partial_i u^3.
\end{aligned}$$

The last equations reproduce the definition of the canonical momenta p with some combination of the Lagrange multipliers. If we use these equations on the earlier equations for $\partial_+ p^\mu$, and also using (2.22), we have

$$\left(1 + a^2\Box\right)\partial_\lambda F^{\lambda+} + \left(1 + a^2\partial_i\partial_i\right)\partial_-\partial_- u^3 \approx 0, \tag{2.23}$$

$$\left(1 + a^2\Box\right)\partial_\lambda F^{\lambda-} + a^2\partial_+\partial_-\partial_i\partial_i u^3 \approx 0, \tag{2.24}$$

$$\left(1 + a^2\Box\right)\partial_\lambda F^{\lambda i} - \left(1 + 2a^2\partial_+\partial_-\right)\partial_-\partial_i u^3 \approx 0. \tag{2.25}$$

These equations are compatible with the Lagrangian field equations (1.2) only if suitable gauge conditions are chosen in order to eliminate the Lagrange multiplier u^3.

3. Remarks

We have analyzed the canonical structure of Podolsky's electrodynamics on the null-plane. We have observed the appearance of a set of first-class constraints, as well as a set of second-class ones. The first-class constraints are responsible for the $U(1)$ invariance of the Action, in agreement with the fact that gauge invariance is independent of the chosen dynamics. The form of this set is analogous to the set found in instant-form [6], which is also expected. The appearance of second-class constraints, however, is a common effect of the null-plane dynamics [10, 7, 8, 9]. Because of them, we need less degrees of freedom to uniquely describe the system on the null-plane.

In a posterior, more complete work, we intend to analyze proper gauge conditions to fix the first-class constraints, as done in ref. [6] in usual coordinates. To evaluate the physical degrees of freedom it is necessary to choose proper gauge conditions for the theory, which is a subject that needs closer inspection. We also intend to make a complete analysis of the initial-boundary problem related to the Podolsky's theory, since the construction of unique Dirac's brackets due to the second-class constraints is dependent of a proper choice of these conditions.

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