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Exact beta function and glueball spectrum in large-N Yang-Mills theory

Marco Bochicchio*
INFN Sezione di Roma
Dipartimento di Fisica, Universita' di Roma 'La Sapienza'
Piazzale Aldo Moro 2 , 00185 Roma
E-mail: marco.bochicchio@roma1.infn.it

In the pure large-N Yang-Mills theory there is a quasi-*BPS* sector that is exactly solvable at large N. It follows an exact beta function and the glueball spectrum in this sector. The main technical tool is a new holomorphic loop equation for quasi-*BPS* Wilson loops, that occurs as a non-supersymmetric analogue of Dijkgraaf-Vafa holomorphic loop equation for the glueball superpotential of $\mathcal{N}=1$ $SUSY$ gauge theories. The new holomorphic loop equation is localized, i.e. reduced to a critical equation, by a deformation of the loop that is a vanishing boundary in homology, somehow in analogy with Witten's cohomological localization by a coboundary deformation in $SUSY$ gauge theories.

European Physical Society Europhysics Conference on High Energy Physics, EPS-HEP 2009,
July 16 - 22 2009
Krakow, Poland

*Speaker.

 http://pos.sissa.it/

PoS(EPS-HEP 2009)075

1. Introduction

The pure $SU(N)$ Yang-Mills (YM) theory simplifies considerably in the large-N limit. For example, to the leading large-N order, the expectation value of a product of normalized local gauge invariant operators factorizes:

$$< \frac{1}{N}\sum_{\alpha\beta} TrF^2_{\alpha\beta}(x_1)...\frac{1}{N}\sum_{\alpha\beta} TrF^2_{\alpha\beta}(x_k) >=< \frac{1}{N}\sum_{\alpha\beta} TrF^2_{\alpha\beta}(x_1) > ... < \frac{1}{N}\sum_{\alpha\beta} TrF^2_{\alpha\beta}(x_k) > . \tag{1.1}$$

Thus, to this order, the only information that survives in this correlator is the value of the condensate $< \sum_{\alpha\beta} \frac{1}{N} TrF^2_{\alpha\beta}(x) >$, that for a suitable regularization must be proportional to the appropriate power of the renormalization group invariant scale, Λ_{QCD}. In turn Λ_{QCD} encodes the information on the beta function of the large-N theory. Another remarkable simplification, that occurs to the next to leading $\frac{1}{N}$ order, is that, because of confinement, the connected two point-functions of local gauge invariant operators must be saturated by a sum of pure poles. For example, for the scalar glueball propagator the equation must hold:

$$\int < \frac{1}{N}\sum_{\alpha\beta} TrF^2_{\alpha\beta}(x)\sum_{\alpha\beta}\frac{1}{N} TrF^2_{\alpha\beta}(0) >_{conn} \; e^{ipx} d^4x = \sum_r \frac{Z_r}{p^2+M_r^2}. \tag{1.2}$$

The sum of pure poles is constrained by the perturbative operator product expansion. It must agree asymptotically for large momentum with the "anomalous dimension" of the glueball propagator as computed by perturbation theory plus perhaps the sum over the condensates that occur in the operator product expansion [1]. Indeed the scalar glueball propagator behaves in perturbation theory at large momentum, within two-loop accuracy, up to contact terms, i.e. polynomials in the momentum squared p^2, and up to a sum over condensates, as:

$$\gamma_G g^4(p) p^4 \log(\frac{p^2}{\mu^2}), \tag{1.3}$$

where γ_G is a numerical factor [1]. The factors of g, the renormalized 't Hooft coupling at momentum p, occur because of the canonical normalization of the action in perturbation theory and to this order they contain the information about the one-loop beta function. The extra logarithm in the two-loop computation is due to an "anomalous dimension".

To say it in a nutshell, in this talk we report the computation of a large-N exact beta function [2] and of the glueball propagator in a certain quasi-BPS sector, to be defined in the next sections, of the large-N pure YM theory to the leading non-trivial $\frac{1}{N}$ order. We find that the glueball propagator, in the case of Wilsonian normalization of the action, in the quasi-BPS sector of the large-N theory reduced to two dimensions by the large-N non-commutative Eguchi-Kawai (EK) reduction (for a review see [3] p.6) is:

$$\int < \frac{1}{N} Tr(\mu^2)(x_+x_-)\frac{1}{N} Tr(\bar{\mu}^2)(0) >_{conn} e^{i(p_+x_-+p_-x_+)} d^2x = \sum_{k=1}^{\infty} \frac{k^2\Lambda_W^6}{\alpha' p_+p_- + (k\delta-\gamma)\Lambda_W^2}, \tag{1.4}$$

for some non-vanishing dimensionless coefficients α', δ, γ that admit an expansion in powers of $\frac{1}{\sqrt{k}}$ and whose precise value will be computed elsewhere. Here μ, the complex field that occurs in the quasi-BPS sector, is a linear combination of the anti-self-dual (ASD) components of the gauge curvature, $\mu = F^-_{01} + iF^-_{02}$. $\bar{\mu}$ is its Hermitean conjugate and x_-, x_+ are light-cone coordinates. Λ_W is the renormalization group invariant scale in the Wilsonian scheme. It is not hard to see that, setting $k^2\Lambda_W^4 = \frac{1}{\delta^2}[((k\delta-\gamma)\Lambda_W^2 + \alpha' p_+ p_-)((k\delta-\gamma)\Lambda_W^2 - \alpha' p_+ p_-) + (\alpha' p_+ p_-)^2 + (k2\gamma\delta - \gamma^2)\Lambda_W^4]$, after some simple algebra Eq.(1.4) can be written as a logarithmic divergent sum that reproduces the correct logarithmic behavior of perturbation theory:

$$(\frac{\alpha'}{\delta})^2 \sum_{k=1}^{\infty} \frac{(p_+p_-)^2}{\alpha' p_+ p_- \Lambda_W^{-2} + (k\delta - \gamma)} + \ldots, \tag{1.5}$$

up to a divergent sum of condensates, proportional to a power of Λ_W, and up to a divergent sum of contact terms. Eq.(1.4) is rescaled by a power of the renormalized canonical coupling constant in the case of canonical normalization of the action, thus accounting of the perturbative dependence on powers of g. It is a result of [2] that the canonical coupling renormalizes according to the following exact large-N beta function:

$$\frac{\partial g}{\partial \log \Lambda} = \frac{-\beta_0 g^3 + \frac{\beta_J}{4} g^3 \frac{\partial \log Z}{\partial \log \Lambda}}{1 - \beta_J g^2}, \tag{1.6}$$

with:

$$\beta_0 = \frac{1}{(4\pi)^2}\frac{11}{3}, \beta_J = \frac{4}{(4\pi)^2}, \tag{1.7}$$

where g is the 't Hooft canonical coupling constant and $\frac{\partial \log Z}{\partial \log \Lambda}$ is computed to all orders in the 't Hooft Wilsonian coupling constant, g_W, by:

$$\frac{\partial \log Z}{\partial \log \Lambda} = \frac{\frac{1}{(4\pi)^2}\frac{10}{3} g_W^2}{1 + c g_W^2}, \tag{1.8}$$

with c a scheme dependent arbitrary constant. At the same time, the beta function for the 't Hooft Wilsonian coupling is exactly one loop:

$$\frac{\partial g_W}{\partial \log \Lambda} = -\beta_0 g_W^3. \tag{1.9}$$

As a check, once the result for $\frac{\partial \log Z}{\partial \log \Lambda}$ to the lowest order in the canonical coupling,

$$\frac{\partial \log Z}{\partial \log \Lambda} = \frac{1}{(4\pi)^2}\frac{10}{3} g^2 + \ldots, \tag{1.10}$$

is inserted in Eq.(1.6), it implies the correct value of the first and second perturbative coefficients of the beta function:

$$\frac{\partial g}{\partial \log \Lambda} = -\beta_0 g^3 + (\frac{\beta_J}{4}\frac{1}{(4\pi)^2}\frac{10}{3} - \beta_0\beta_J) g^5 + \ldots = -\frac{1}{(4\pi)^2}\frac{11}{3} g^3 - \frac{1}{(4\pi)^4}\frac{34}{3} g^5 + \ldots, \tag{1.11}$$

which are known to be universal, i.e. scheme independent. It has been argued in [2] that there is a scheme, that corresponds to a certain choice of c, in which the canonical coupling coincides with a certain definition of the physical effective charge in the inter-quark potential.

The plan of this talk is as follows. The next section on localization summarizes briefly some of the concepts that are necessary to understand the rationale behind the computation of the beta function and of the glueball spectrum. This section is very sketchy for reasons of space. See [2] for more details. In the last section we describe in the same sketchy way the computation of the glueball spectrum and we outline further developments, leaving the details for a forthcoming paper.

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2. Localization and holomorphic loop equation

Extending the finite dimensional theory of Duistermaat-Heckman [4], Witten [5] introduced localization in the infinite dimensional realm of quantum field theory. The basic idea of Witten's localization is that the integral of the exponential of a closed form, $QS_{SUSY} = 0$, the action, S_{SUSY}, of a supersymmetric ($SUSY$) gauge theory in most of the applications, can be deformed by a coboundary, $Q\alpha$, without changing its value:

$$\int \exp(-S_{SUSY} - tQ\alpha), \tag{2.1}$$

because $Q^2 = 0$ and $\int Q\alpha = 0$. Taking the limit $t \to +\infty$, the integral localizes on the set of critical points of the coboundary. Thus Witten's localization in quantum field theory is a cohomology theory in which certain functional integrals are viewed as cohomology classes and they are computed choosing suitable representatives. In most of the applications in quantum field theory the cohomology in question is generated by a $BRST$ operator, Q, that in turn is a twisted super-charge of a $SUSY$ gauge theory. In this case often the $SUSY$ functional integral reduces to the evaluation of a sum of finite dimensional integrals over the moduli space of instantons. A remarkable example is the partition function of the $\mathcal{N} = 2$ $SUSY$ YM theory, whose logarithm is the prepotential. The prepotential has been found independently on localization arguments by means of the Seiberg-Witten solution. Later Nekrasov [6] has reproduced the Seiberg-Witten solution using cohomological localization. A remarkable feature of Nekrasov work is that also the finite-dimensional integrals over the instantons moduli space can be evaluated by localization methods provided a compactification of the moduli space of instantons is chosen. The compactification is absolutely necessary to assure that the integral of a coboundary vanishes, i.e. to define the integrals over instantons moduli as cohomology classes. The choice of the compactification introduces to some extent a certain arbitrariness, that is fixed by physical or mathematical arguments [6]. As a result, according to Nekrasov, in the $\mathcal{N} = 2$ $SUSY$ case, the functional integral reduces to a sum of Abelian instantons with no moduli, that can be performed exactly [6].

Since cohomology is dual to homology, we may wonder as to whether we can compute functional integrals by homological rather than cohomological deformations. Were the answer be affirmative, we could get localization without supersymmetry.

The natural arena for homological localization in gauge theory, as opposed to cohomological localization, is the loop equation [7].

In general the loop equation is the sum of a classical equation of motion and of a quantum term, that involves the contour integral along the loop. By homological localization of the loop equation we mean a homological deformation of the loop for which the quantum term vanishes and after which the loop equation is reduced to a critical equation for an effective action [2]. Hence the needed homological deformation has to satisfy the following two properties.

It has to leave the expectation value of the loop invariant.

It has to imply the vanishing of the quantum term in the loop equation, i.e. of the term that contains the contour integral along the loop.

In this homology framework there is a very natural analogue of the operation of adding a coboundary in cohomology, that is based on the zig-zag symmetry of Wilson loops. The zig-zag symmetry is the invariance of a Wilson loop by the addition of a backtracking arc ending with a cusp. This deformation is a "vanishing boundary" in singular homology. In a regularized version the arc is the boundary of a tiny strip. While this symmetry holds classically in most of the cases, quantum mechanically the renormalization process may spoil it, unless a certain fine tuning of the renormalization scheme occurs. The reason is that in general Wilson loops have perimeter and cuspidal divergences. The perimeter divergence is linear in the cut-off scale. The cuspidal divergence is logarithmic, with a coefficient that in turn is divergent for backtracking cusps. To maintain the zig-zag symmetry the renormalization scheme must be fine tuned in such a way that the cuspidal divergence cancels that part of the perimeter divergence that is associated to the extra perimeter due to the addition of backtracking cusps. However, in $SUSY$ theories with extended $SUSY$ there are (locally-BPS) Wilson loops that have no perimeter divergence [8]. Remarkably this property does not follow directly from the extended $SUSY$ but only from rotational invariance of the parent theory with minimal $SUSY$, whose dimensional reduction is the daughter theory with extended $SUSY$ [8]. This opens the way to find Wilson loops with similar properties in the pure large-N YM theory. In this case the quasi-BPS Wilson loops of the daughter theory, the large-N non-commutative EK reduced theory, share the non-renormalization properties with their supersymmetric cousins because of the four-dimensional rotational invariance [2] of the parent large-N YM theory. The quasi-BPS Wilson loops are defined as follows [2]:

$$Tr\Psi(C_{ww}) = TrP\exp i\int_{C_{ww}}(A_z+D_u)dz+(A_{\bar{z}}+D_{\bar{u}})d\bar{z}, \tag{2.2}$$

where $D_u = \partial_u + iA_u$ is the covariant derivative of the non-commutative large-N reduced theory along the non-commutative direction u. The plane $z, \bar{z}$ is instead commutative. The loop, C_{ww}, starts and ends at the marked point, w. The trace in Eq.(2.2) is over the tensor product of the $U(N)$ Lie algebra and of the infinite dimensional Fock space that defines the Hilbert space representation of the non-commutative plane $u, \bar{u}$ [2, 3, 9]. The limit of infinite non-commutativity is understood, being equivalent to the large-N limit of the commutative gauge theory [3]. The curvature, $F(B)$, of the connection, $B = B_z dz + B_{\bar{z}} d\bar{z} = (A_z + D_u)dz + (A_{\bar{z}} + D_{\bar{u}})d\bar{z}$, that occurs in the quasi-BPS Wilson loops, is the field μ that arises in the glueball propagator in the quasi-BPS sector, $F(B) = \mu = F^-_{01} + iF^-_{02}$. We stress that the homological localization holds only in the quasi-BPS sector because it is only in this sector of the large-N YM theory that the zig-zag symmetry is satisfied without fine tuning of the renormalization scheme. As a consequence only the glueball propagator for the field μ can be obtained by homological localization.

The second property, the vanishing of the quantum term in the loop equation, is harder to obtain. In fact this property can be satisfied only if a scheme can be found in which not only the Wilson loop but also the quantum term in the loop equation can be regularized in a manifestly zig-zag invariant way. This is not possible in the usual Makeenko-Migdal loop equation, not even for quasi-BPS Wilson loops [2]. It turns out that it is necessary to write the loop equation in new variables, changing variables from the connection to the ASD part of the curvature, introducing in the functional integral the appropriate resolution of identity:

$$\begin{aligned} 1 &= \int \delta(F^-_{\alpha\beta} - \mu^-_{\alpha\beta})D\mu^-_{\alpha\beta} \\ &= \int \delta(F(B)-\mu)|_{\mu=g_-p_+g_-^{-1}}D\mu\,\delta(F^-_{02}-\mu^-_{02})\delta(F^-_{03}-\mu^-_{03})D\mu^-_{02}D\mu^-_{03}, \end{aligned} \tag{2.3}$$

where the integral over $\mu = gp'_+g^{-1} = g_-p_+g_-^{-1}$ is on an orbit of the unitary group with measure $D\mu = \Delta(\mu)Dp_+Dg_-$, with g unitary, p'_+ and p_+ upper triangular and $g_- = 1+n_-$ with n_- nilpotent and lower triangular. $\Delta(\mu)$ is the Vandermonde determinant of the eigenvalues of μ. The resolution of identity that occurs in the right hand side of the second equality contains an integral over the non-Hermitean field, μ, and should be interpreted in the sense of holomorphic matrix models [10]. This deformation of the Hermitean integral over μ^-_{01} to the non-Hermitean complex integral over μ is absolutely necessary to write the new holomorphic loop equation by means of a further change of variables, defined by the choice of a holomorphic gauge in which $B'_{\bar{z}} = 0$ and $F(B'_z) = \mu'$. This allows us to reduce the quantum term in the loop equation to the evaluation of a residue that can be regularized in a manifestly zig-zag invariant way (see below).

In the eighties the change of variables from the gauge connection to the *ASD* part of the gauge curvature has been known as the Nicolai map [11]. It has been invented by Nicolai for *SUSY* gauge theories because, for example, it leads to remarkable cancellations between the Jacobian of the Nicolai map and the fermion determinant in the $\mathcal{N} = 1$ *SUSY YM* theory. However, it makes sense also in the pure *YM* theory [2], although it does not lead to unexpected cancellations in this theory. The further change of variables to a holomorphic gauge is a new key feature of our approach to the large-*N YM* theory. It is based on the idea that quasi-*BPS* Wilson loops, being holomorphic functionals of μ', behave as the chiral (i.e. holomorphic) super-fields of a $\mathcal{N} = 1$ *SUSY* gauge theory. In fact the new holomorphic loop equation resembles for the cognoscenti the holomorphic loop equation that occurs in the Dijkgraaf-Vafa theory of the glueball superpotential in $\mathcal{N} = 1$ *SUSY* gauge theories [12, 9, 10]. The partition function thus becomes:

$$Z = \int \exp(-\frac{N8\pi^2}{g^2}Q - \frac{N}{4g^2}\sum_{\alpha\neq\beta}\int Tr(\mu^{-2}_{\alpha\beta})d^4x)\delta(F^-_{02}-\mu^-_{02})\delta(F^-_{03}-\mu^-_{03})$$
$$\delta(F(B)-\mu)|_{\mu=g_-p_+g_-^{-1}}DBD\bar{B}\frac{D\mu}{D\mu'}|_{\mu'=G\lambda G^{-1}}D\mu' D\mu^-_{02}D\mu^-_{03}, \tag{2.4}$$

where the integral over $\mu' = G\lambda G^{-1}$ is on an orbit of the complexification of the gauge group with measure $D\mu' = \Delta(\mu)^2 D\lambda DG$ and with λ the diagonal matrix of the eigenvalues of μ. We can write the partition function in the new form:

$$Z = \int \exp(-\frac{N8\pi^2}{g^2}Q - \frac{N}{4g^2}\sum_{\alpha\neq\beta}\int Tr(\mu^{-2}_{\alpha\beta})d^4x)$$
$$Det'^{-\frac{1}{2}}(-\Delta_A\delta_{\alpha\beta} + D_\alpha D_\beta + iad_{\mu^-_{\alpha\beta}})\frac{DB^{g_-}{}'}{Dg_-}\Delta^{-1}(\mu)\frac{D(p_+,g_-)}{D(\lambda,G)}D\mu' D\mu^-_{02}D\mu^-_{03}$$
$$= \int \exp(-\Gamma)D\mu' D\mu^-_{02}D\mu^-_{03}, \tag{2.5}$$

where the integral over the gauge connection of the delta functions has been now explicitly performed and, by an abuse of notation, the connection A in the determinants denotes the solution of the equation $F^-_{\alpha\beta} - \mu^-_{\alpha\beta} = 0$. The $'$ superscript in the first two determinants requires projecting away the zero modes due to gauge invariance and possibly to moduli, since gauge-fixing is not yet implied though it is understood. The determinant of zero modes associated to the (holomorphic) moduli is $\frac{DB^{g_-}{}'}{Dg_-}$, where B^{g_-} is the gauge transform of B by the singular gauge transformation for which $F(B^{g_-}) = p_+$ [1]. The new holomorphic loop equation follows [2] :

$$< Tr(\frac{\delta\Gamma}{\delta\mu(z)'}\Psi'(C_{zz})) >= \frac{1}{2\pi}\int_{C_{zz}} \frac{dw}{z-w} < Tr\Psi'(C_{zw}) >, \quad (2.6)$$

where Ψ' is the holonomy of B in the gauge $B'_{\bar{z}} = 0$. Γ is determined by the holomorphic loop equation up to an anti-holomorphic functional of μ. The residue is regularized in a gauge invariant way by analytic continuation from Euclidean to hyperbolic signature and Γ is defined globally on the double cover of the conformal compactification of space-time with hyperbolic signature [2, 13]:

$$\frac{dw}{z-w} \to \frac{dy_+}{x_+ - y_+ + i\varepsilon}. \quad (2.7)$$

In a lattice regularization the integral over the ASD curvature would live over plaquettes that are dual, in the plane over which the loop lies, to points. These points become the cusps that are the end points, p, of the backtracking strings, b_p, that perform the deformation of the loop, C. Adding the backtracking strings implies the homological localization of the holomorphic loop equation:

$$< Tr(\frac{\delta\Gamma([b_p])}{\delta\mu'(z_p)}\Psi'(C \cup [b_p])) >= 0. \quad (2.8)$$

The regularized residue vanishes at the backtracking cusps [2] because of its manifest zig-zag symmetry. The lattice regularization of the Nicolai map is obtained identifying the cusps with parabolic singularities of the reduced EK theory. These point-like parabolic singularities of the partial large-N EK reduction are daughters of codimension-two singularities of the four-dimensional parent gauge theory. The following lattice functional integral is a discretization, corresponding to the lattice of parabolic singularities, of the resolution of identity that defines the Nicolai map in Eq.(2.3):

$$1 = \int \delta(F^-_{\alpha\beta}(A) - \sum_p \mu^-_{\alpha\beta}(p)\delta^{(2)}(z - z_p(u,\bar{u}))) \prod_p D\mu^-_{\alpha\beta}(p). \quad (2.9)$$

Codimension-two singularities of this kind have been introduced in [14] in the pure YM theory as an "elliptic fibration of parabolic bundles" and later in [15] in the $\mathcal{N} = 4$ $SUSY$ YM theory, for the study of the geometric Langlands correspondence, under the name of "surface operators". In fact they have been studied originally at classical level in [16] as singular instantons. It turns out that when a codimension-two surface is non-commutative, as in our case, the YM action of the corresponding non-commutative reduced EK model is rescaled by a power of the inverse cut-off ([3] p.6 and [9] p.21) that cancels precisely [2] the quadratic divergence that occurs evaluating the classical YM action on surface operators. The effective action, $\Gamma([b_p])$, is defined on a Mandelstam graph, that is a conformal transformation of the half-plane, obtained drawing backtracking strings ending with pairs of cusps. We may think that it is a change of the conformal structure that generates the cusps. Since the quasi-BPS loops are diagonally embedded in space-time [3], this

[1] This gauge transformation is in fact non-singular on the Mandelstam graph introduced later to obtain localization.

[2] The holomorphic loop equation is written in linear form since it is assumed that the loop C_{zz} is simple, i.e. it has no self-intersections.

[3] This can be seen by the fact that the connection B contains terms of the form $D_u dz$, implying implicitly that $dz = du$ on the Riemann surface over which the loop lies.

two-dimensional conformal transformation lifts to a conformal rescaling of the four-dimensional metric and thus acts by the renormalization group (*RG*) by adding a conformal anomaly to the effective action, that amounts to a local counterterm, i.e. to change of the subtraction point, $\Gamma([b_p]) = \Gamma([p]) + ConformalAnomaly([b_p])$. Therefore there is a symmetry of the *RG* flow that generates the homological deformation of the loop by a vanishing boundary, i.e. by backtracking strings. This is the analogue of the action being a closed form in cohomology, since in the last case there is a symmetry of the action (i.e. the twisted supersymmetry) that generates the coboundary. The effective action is a functional defined on a lattice of surface operators. The beta function is obtained extracting the divergences of the effective action. An important point is that a regularization exists for which the loop expansion of the first functional determinant in Eq.(2.5) satisfies the usual power counting as in the background-field computation of the beta function. This regularization of the effective action is a point-splitting regularization of the propagator in the background of the lattice of surface operators. A typical example is the following one-loop logarithmic contribution to the beta function in Euclidean configuration space:

$$\frac{1}{(4\pi^2)^2}\sum_{p\neq p'}\int d^2u d^2v \frac{NTr(\mu_p \bar{\mu}_{p'})}{(|z_p - z_{p'}|^2 + |u-v|^2)^2}, \tag{2.10}$$

where the sum over p, p' runs over the planar lattice of the parabolic divisors of the surface operators. The logarithmic divergence arises for a p-independent μ_p. Had the contribution with $p = p'$ been included, there would appear a quadratic divergence, thus spoiling the usual power counting. This lattice point-spitting regularization [4] is followed by Epstein-Glaser renormalization in Euclidean configuration space (see [17] for references) and it is a possible starting point of a new constructive approach to large-N YM theory.

3. The glueball spectrum

It turns out that the beta function is saturated by the Z_N non-Abelian vortices of the *EK* reduction [2]: $[D_z, D_{\bar{z}}] - [D_u, D_{\bar{u}}] = \sum_p \mu_p \delta^{(2)}(x - x_p) - H1$, $[D_{\bar{z}}, D_u] = 0$, $[D_z, D_{\bar{u}}] = 0$. Here H is the (vanishing small) inverse of the parameter of non-commutativity. For Z_N vortices of charge k, $N-k$ eigenvalues in a $SU(N)$ orbit of the residue of the curvature, μ_p, are equal to $\frac{2\pi k}{N}$ and k eigenvalues are equal to $\frac{2\pi(k-N)}{N}$. The complex dimension of the local moduli space at each point, p, which coincides with the complex dimension of the $SU(N)$ orbit, is $k(N-k)$. From Eq.(2.5) it follows that the effective action in the μ-$\bar{\mu}$-sector [5] is given by:

$$\Gamma = \frac{N8\pi^2}{g^2}Q + \int Tr(\frac{N}{g^2}\mu\bar{\mu} + \log|\Delta(\mu)|^2)d^2x - \log Det'^{-\frac{1}{2}}(-\Delta_A \delta_{\alpha\beta} + D_\alpha D_\beta + iad_{\mu^-_{\alpha\beta}}). \tag{3.1}$$

The logarithm of the product of the determinant of zero modes and of the determinant for the choice of the holomorphic gauge has been omitted, since it does not contribute to the μ-$\bar{\mu}$ propagator, because it is the sum of a holomorphic and an anti-holomorphic functional of μ. The logarithm of

[4] This regularization has been found during joint work with Arthur Jaffe.

[5] This is the effective action whose critical equation is equivalent to the loop equation for both the connection B and its Hermitean conjugate $\bar{B}$. This explains the occurrence of the square of the modulus of the Vandermonde determinant in Eq.(3.1). Equivalently the effective action in the μ-$\bar{\mu}$-sector can be obtained by holomorphic/anti-holomorphic fusion [18], on the double cover of the conformal compactification of space-time with hyperbolic signature [13], inserting the anti-holomorphic resolution of identity on the image of a hemisphere by the antipodal map.

the Vandermonde determinant is instead a holomorphic functional everywhere but at coinciding eigenvalues and therefore it must be included in the effective action in the μ-$\bar{\mu}$-sector. As an aside we notice that in the $\mathcal{N}=1$ $SUSY\ YM$ theory the Jacobian of the Nicolai map would cancel the gluinos determinant in the light-cone gauge [11], while the Vandermonde determinant would be absent, since by ordinary cohomological localization due to the tautological Parisi-Sourlas supersymmetry associated to the Nicolai map [6] the partition function (with some insertions of gluinos operators to be non-vanishing) would be localized on instantons, for which $\mu^{-}_{\alpha\beta}=0$, as opposed to vortices. In this case only the topological term (i.e. the second Chern class, Q, not to be confused with the $BRST$ charge) and the logarithm of the (super-)determinant of zero modes due to the instantons moduli, $SDet\,\omega$, would survive in the effective action. Of course localization on instantons reproduces the $NSVZ$ beta function [19] since the only source of divergences are the instantons zero modes.

Since in this talk we do not compute the precise value of the coefficients that occur in the glueball propagator, but only its general structure, we do not bother about the precise normalization of each term that occurs in the effective action. Working in the Wilsonian scheme the beta function is exactly one-loop. We want to extract the glueball spectrum from the effective action. The easiest part to compute is the glueball potential, that originates the glueball masses (squared). After introducing the density of the vortices:

$$\rho = N_v^{-1}\sum_p \delta^{(2)}(z-z_p), \tag{3.2}$$

the lowest order contribution to the renormalized glueball potential in the loop expansion of the effective action [7] , up to normalization of each term, reads:

$$\int \rho^2(x) N Tr(\mu\bar{\mu}) d^4x + \int \rho^2(x)\log|\Delta(\mu)|^2 d^4x. \tag{3.3}$$

The subtraction point in each sector labelled by the charge, k, of the vortices lattice is defined in such a way that the vortices condensate, i.e. the RG invariant part of the gluon condensate, $\rho^2(x)Tr(\mu\bar{\mu})(x)$, is the same in each sector. This condition is required by general principles, because all the Z_N vortices must "condense at once". This implies that $\rho^2(x)\frac{k(N-k)}{N}$ is k-independent and proportional to Λ_W^4, from which remarkable consequences follow. Expanding the glueball potential to the second order in $\mu,\bar{\mu}$ asymptotically for large k, the first term is of order of $\frac{1}{k}$ while the second one is of order of 1, and both occur with multiplicity k. This occurs because vortices of charge k have eigenvalues with multiplicity k and $N-k$. The contribution of the second term is a distribution in color space of the form $\delta^{(2)}(\mu_i-\mu_j)$, that arises taking the second derivative of the logarithm of the modulus of the Vandermonde determinant. Its support has non-void intersection with the vortices eigenvalues precisely because the vortices have an eigenvalues spectrum that is partially degenerated [8]. As we stressed in the previous section, the quantum term in the loop equation is regularized in a gauge invariant way by analytic continuation from Euclidean to hyperbolic

[6] M. Bochicchio, to appear.

[7] The loop expansion is in fact an expansion in powers of the vortices density, ρ, that scales as $\frac{1}{\sqrt{k}}$, see below.

[8] The $\delta^{(2)}(0)$ in color space is regularized as prescribed by the large-N EK reduction as $\frac{N}{(2\pi)^2}$ (see [3] p.6 and [9] p.21).

signature [13]. This has remarkable consequences too. Indeed, after that the renormalization described in the previous section is performed in Euclidean space, the kinetic term for the vortices eigenvalues arises from the finite part of the Jacobian of the Nicolai map by analytic continuation to hyperbolic signature. The basic idea is that, after this analytic continuation, the finite parts of the Jacobian can be computed as residues in coordinate space by the Cauchy formula. The first term that leads to two derivatives in the expansion of the Jacobian of the Nicolai map in powers of $\mu, \bar{\mu}$ is of order two. The typical order-two diagram (Eq.(2.10)), after analytic continuation to hyperbolic signature, leads to [9]:

$$\frac{N}{k}\int Tr(\delta\mu(x)\delta\bar{\mu}(y))\frac{1}{((x_+-y_+)(x_--y_-)+(u_+-v_+)(u_--v_-))^2}|_{x=u,y=v}d^2xd^2y \tag{3.4}$$

and therefore, applying the Cauchy formula, to a kinetic term of the form:

$$\frac{N}{k}\int Tr(\delta\mu(x)\partial_+\partial_-\delta\bar{\mu}(x))d^2x. \tag{3.5}$$

The final result for the glueball propagator in the quasi-*BPS* sector in the Wilsonian scheme, asymptotically for large k, is therefore of the form:

$$\sum_{k=1}^{\infty}\frac{k\Lambda_W^6}{\frac{1}{k}\alpha' p_+p_- + (\delta-\frac{1}{k}\gamma)\Lambda_W^2}. \tag{3.6}$$

In the canonical scheme the field μ is rescaled by a power of g and this matches the rescaling in perturbation theory. The quasi-*BPS* glueball masses squared are asymptotically linear in the charge of the vortices lattice, k, rather than in the angular momentum, J. From a qualitative point of view the existence of the magnetic quantum number, k, matches quite well the spectrum found by the numerical lattice computation for $SU(8)$ [20]. An interpretation of the numerical data in terms of Regge trajectories, labeled by J, seems instead more complicated [20]. Morally this calls for a stringy interpretation in which the string theory dual to pure *YM* that is the most simple describes the open strings fluctuations of the magnetic-vortices sheets of the condensate (i.e. of the codimension-two singularities in the language of Langlands duality), rather than the fluctuations of the dual confined electric closed string fluxes. This stringy magnetic description follows by 't Hooft picture of the *YM* vacuum as a dual superconductor. In fact this magnetic string theory actually exists in the quasi-*BPS* sector, by means of a dual cohomological twistorial string theory [21], that provides conjecturally the cohomology theory dual to the homological localization. We point out that, conjecturally, the electric string theory, dual to the magnetic one considered in this talk, could be obtained by a solution of the loop equation by means of the geometric Langlands duality in the sense of Beilinson-Drinfeld as suggested in [22] (see par.(6)). It is an interesting open problem how to extend the homological approach to localization to large-N $\mathcal{N}=1$ *SUSY YM* and to large-N *QCD*. In particular in the $\mathcal{N}=1$ *SUSY YM* case the cohomological and homological approaches should coexist, since they apply to different observables, the gluino condensate and the *BPS* Wilson loops respectively.

[9] The restriction to the diagonal $x=u, y=v$ of the fluctuations $\delta\mu$ arises from the diagonal embedding of the quasi-*BPS* Wilson loops. It can be implemented in Eq.(2.9) adding fluctuations of the form $\sum_p \delta\mu^-_{\alpha\beta}(p)\delta^{(2)}(z-z_p(u,\bar{u}))$ with $z_p(u,\bar{u})=u, \bar{z}_p(u,\bar{u})=\bar{u}$. In the stringy version [21] which is referred to at the end of this section the diagonal embedding is substituted by a Lagrangian one.

Acknowledgments

We thank the organizers of the HEP-EPS 2009 conference, and in particular Antonio Polosa, for the fruitful atmosphere, the interesting conversations and for inviting us to give this talk.

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