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A method to measure the Kbar-N scattering length in lattice QCD

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We propose a method to determine the isoscalar $\bar{K}N$ scattering length on the lattice. Our method represents the generalization of Lüscher's approach in the presence of inelastic channels (complex scattering length). In addition, the proposed approach allows one to find the position of the S-matrix pole corresponding the the $\Lambda(1405)$ resonance.

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1. The antikaon-nucleon scattering amplitude is of fundamental importance in nuclear, particle and astrophysics, see e.g. Refs. [1, 2, 3, 4]. In particular, the $\bar{K}N$ system at threshold provides an interesting testing ground of the chiral dynamics of QCD with strange quarks due to the $\Lambda(1405)$ resonance just below the scattering threshold. In fact, experimental information on the K^-p scattering length from scattering data and kaonic hydrogen level shifts is contradictory, as first stressed in [5] and further elaborated on in Refs. [6, 7, 8]. A clarification is expected from the upcomimg SIDDHARTA experiment at DAΦNE, that intends to remeasure kaonic hydrogen with unprecedented accuracy and is expected to give further constraints to the isoscalar and isovector kaon-nucleon scattering lengths from the first measurement of the energy spectrum of kaonic deuterium.

On the theoretical side, effective field theory methods in various disguises are employed to pin down the $\bar{K}N$ scattering length. Scattering information is usually analyzed in terms of unitarized versions of chiral perturbation theory (see, e.g. [6, 7, 8, 9]), while the alalysis of kaonic hydrogen data is firmly rooted in non-relativistic bound-state effective field theory (for a recent review, see [10]). Unitarization of course introduces some unwanted model-dependence and while the extraction of the $\bar{K}N$ scattering length from kaonic hydrogen data is devoid of that model-dependence it rests on the availablity of precise kaonic atom data. But here the existing data from DEAR [11] and KEK [12] are conflicting. It would therefore be very helpful to have another tool at hand that would allow one to determine this fundamental quantity.

We want to argue here that lattice QCD provides such a framework. As first shown by Lüscher, finite volume simulations of the energy levels of two-particle states can give access to scattering information [13, 14]. The $1/L$ expansion (here, L is the size of a box with volume $L\times L\times L$) of two-particle state energy levels (well separated from bound states or resonances in the given channel) takes a generic form wich can be used to extract the desired scattering length (see e.g. [15]).

However, for the extraction of the $\bar{K}N$ scattering length, a generalization of this scheme is needed since there is a strong channel coupling between $\bar{K}N$ and $\Sigma\pi$, the latter channel having its threshold just about 100 MeV below the opening of the $\bar{K}N$ one. In addition, the appearance of the $\Lambda(1405)$ just between these two thresholds further complicates the picture. All these features can be captured by a two-channel Lippmann-Schwinger equation. As we will show in the following, a suitable formulation of this equation in the finite volume allows for an unambigous extraction of the complex-valued isoscalar K^-N scattering length. Note also that our method is similar to the approach adopted in Ref. [17], where the problem was treated within potential scattering theory. In this paper, we use instead the language of the non-relativistic effective field theory (NR EFT), which enables one to systematically address the effects of particle creation/annihillation and relativistic corrections.

2. First we consider a two-channel Lippmann-Schwinger (LS) equation in NR EFT in the infinite volume. Note that we are using a covariant version of the NR EFT, considered in Refs. [18]. The channel number 1 refers to $\bar{K}N$ and 2 to $\Sigma\pi$ with total isospin $I=0$. The resonance $\Lambda(1405)$ is located between two thresholds, on the second Riemann sheet, close to the real axis (these thresholds are defined by $s_t=(m_N+M_K)^2$ and $s_t'=(m_\Sigma+M_\pi)^2$) [1]. Consider first energies above

[1] In the following, we work in the isospin limit and thus do not resolve the further splitting of these thresholds.

$\bar{K}N$ threshold, $s > (m_N + M_K)^2$. The coupled-channel LS equation for the T-matrix elements $T_{ij}(s)$ in dimensionally regularized NR EFT reads (we only consider S-waves here)

$$\begin{aligned} T_{11} &= H_{11} + H_{11}\, iq_1 T_{11} + H_{12}\, iq_2 T_{21} \,, \\ T_{21} &= H_{21} + H_{21}\, iq_1 T_{11} + H_{22}\, iq_2 T_{21} \,, \end{aligned} \tag{1}$$

with $q_1 = \lambda^{1/2}(s, m_N^2, M_K^2)/(2\sqrt{s})$, $q_2 = \lambda^{1/2}(s, m_\Sigma^2, M_\pi^2)/(2\sqrt{s})$ and $\lambda(x,y,z)$ stands for the Källen function. Furthermore, the $H_{ij}(s)$ denote the driving potential in the corresponding channel. Continuation of the center-of-mass momentum q_1 below threshold $(m_\Sigma + M_\pi)^2 < s < (m_N + M_K)^2$ is obtained via

$$iq_1 \to -\kappa_1 = -\frac{(-\lambda(s, M_K^2, m_N^2))^{1/2}}{2\sqrt{s}} \tag{2}$$

The resonance corresponds to a pole on the second Riemann sheet in the complex s-plane, its position can de determined from the secular equation

$$\Delta(s) = 1 + \kappa_1^R H_{11} - \kappa_2^R H_{22} - \kappa_1^R \kappa_2^R \left(H_{11} H_{22} - H_{12}^2\right) \tag{3}$$

with $\kappa_1^R = -(-\lambda(s_R, m_N^2, M_K^2))^{1/2}/(2\sqrt{s_R})$ and $\kappa_2^R = (-\lambda(s_R, m_\Sigma^2, M_\pi^2)^{1/2})/(2\sqrt{s_R})$. The energy and width of the resonance are then given by $\sqrt{s_R} = E_R - i\Gamma_R/2$.

The $\bar{K}N$ scattering length is related to the amplitude T_{11} at $s = s_t = (m_N + M_\pi)^2$ via

$$a_{11} \equiv T_{11}(s_t) = H_{11}(s_t) + \frac{iq_2(s_t)\,(H_{12}(s_t))^2}{1 - iq_2(s_t)\,H_{22}(s_t)} \,. \tag{4}$$

Thus, to pin down its complex value, we need to determine the three real quantities H_{11}, H_{12}, H_{22} at $s = s_t$ appearing in Eq. (4)

3. We now consider the same problem in a finite volume. The rotational symmetry is broken to a cubic symmetry so that the infinite volume version of the LS equation Eq. (1) takes the form (we consider only S-waves here, neglecting the small mixing to higher partial waves. The mixing can be easily included at latter stage, see e.g., Ref. [19].),

$$\begin{aligned} T_{11} &= H_{11} - \frac{2}{\sqrt{\pi}L} Z_{00}(1;k_1^2)\, H_{11} T_{11} - \frac{2}{\sqrt{\pi}L} Z_{00}(1;k_2^2)\, H_{12} T_{21} \,, \\ T_{21} &= H_{21} - \frac{2}{\sqrt{\pi}L} Z_{00}(1;k_1^2)\, H_{21} T_{11} - \frac{2}{\sqrt{\pi}L} Z_{00}(1;k_2^2)\, H_{22} T_{21} \,, \end{aligned} \tag{5}$$

with

$$\begin{aligned} k_1^2 &= \left(\frac{L}{2\pi}\right)^2 \frac{\lambda(s, M_K^2, m_N^2)}{4s} \,, \\ k_2^2 &= \left(\frac{L}{2\pi}\right)^2 \frac{\lambda(s, M_\pi^2, m_\Sigma^2)}{4s} \,, \\ Z_{00}(1;k^2) &= \frac{1}{\sqrt{4\pi}} \lim_{r\to 1} \sum_{\vec{n}\in R^3} \frac{1}{(\vec{n}^{\,2} - k^2)^r} \,. \end{aligned} \tag{6}$$

Here, we have neglected the terms that vanish exponentially at a large L. The secular equation that determines the spectrum can be brought into the form

$$1-\frac{2}{\sqrt{\pi}L}Z_{00}(1;k_2^2)F(s,L)=0\,,$$
$$F(s,L)=\left[H_{22}-\frac{2}{\sqrt{\pi}L}Z_{00}(1;k_1^2)\,(H_{11}H_{22}-H_{12}^2)\right]\left[1-\frac{2}{\sqrt{\pi}L}Z_{00}(1;k_1^2)H_{11}\right]^{-1} \tag{7}$$

This is rewritten as

$$\delta(s,L) = -\phi(k_2)+n\pi\,,\quad n=0,1,2,\ldots$$
$$\phi(k_2) = -\arctan\frac{\pi^{3/2}k_2}{Z_{00}(1;k_2^2)}\,, \tag{8}$$

with

$$\tan\delta(s,L)=q_2(s)F(s,L)\,. \tag{9}$$

$\delta(s,L)$ is called the *pseudophase*. It is a function of the energy $\sqrt{s}$ and the level index n, $\delta_n(s)=\delta(s,L_n(s))$.

The dependence of the pseudophase on s and L (or, equivalently, on the level index n) is very different from that of the usual scattering phase. Namely, the elastic phase extracted from the lattice data by using Lüscher's formula is independent of the volume modulo terms that exponentially vanish at a large L. Further, the energies where the phase passes through $\pi/2$ lie close to the real resonance locations. In contrast with this, the pseudophase contains terms which are only power suppressed at a large L. Moreover, it contains the tower of resonances which are not related to the dynamics of the system in the infinite volume and merely reflect the existence of the discrete energy levels in the "shielded" channel.

Measuring the pseudophase on the lattice can be used to determine the $\bar{K}N$ scattering length. It can be directly seen from the expression of the pseudophase, which depends on real functions H_{11},H_{12},H_{22}. Extracting these from the data, we then find the scattering length by using Eq. (4). Note that in the expression for the scattering length we need $H_{ij}(s)$ evaluated at threshold $s=s_t$. We shall however demonstrate below that replacing $H_{ij}(s)$ by $H_{ij}(s_t)$ in certain observables, related to the pseudophase, introduces very small correction, since the effective range term proportional to $(s-s_t)$ is suppressed by L^{-3} as compared to the leading order result. To be specific, we consider the following three observables:

1. For some chosen value of n, we measure the value of the pseudophase $\delta(s_t;L(s_t))\doteq\delta_t$ at threshold $s_t=(m_N+M_K)^2$ and $E_t=\sqrt{s_t}$ (see Fig. 1 where a specific representation of the pseudophase based on a two-channel K -matrix model de- scribed below, is shown). On the other hand, we may express δ_t through H_{ij} at $s=s_t$ in the following way. At threshold, Z_{00} is singular,

$$Z_{00}(1;k^2)=-\frac{1}{\sqrt{4\pi}}\frac{1}{k^2}+\mathscr{O}(1)\,, \tag{10}$$

so that

$$F(s,L)|_{s\to s_t} = H_{22}(s_t)-H_{12}^2(s_t)/H_{11}(s_t) \tag{11}$$
$$\tan\delta(s_t;L(s_t)) = q_2(s_t)\left(H_{22}(s_t)-H_{12}^2(s_t)/H_{11}(s_t)\right)\doteq q_2(s_t)I(s_t)\,. \tag{12}$$

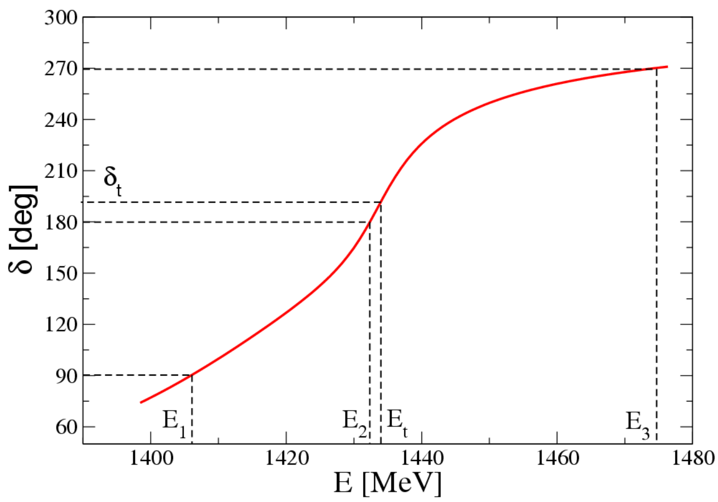

Figure 1: The pseudophase δ. The energy E_1 at which the pseudophase passes through $\pi/2$ corresponds to the $\Lambda(1405)$.

Thus, measuring δ_t, we may extract the combination $H_{22}-H_{12}^2/H_{11}$.

2. Suppose that $\tan\delta(s;L(s))$ is infinite at $s=s_3=E_3^2$ and $L=L_3=L(s_3)$. This occurs at the energy where the denominator of Eq. (7) vanishes

$$1-\frac{2}{\sqrt{\pi}L}Z_{00}(1;k_1^2(s_3))H_{11}(s_3)=0\,. \tag{13}$$

We solve this equation by expanding both $H_{11}(s)$ and $Z_{00}(1;k_1^2(s))$ in Taylor series in the vicinity of $s=s_t$. Substituting these expansions into Eq. (13), we obtain

$$q_1^2(s_3)=-\frac{4\pi H_{11}(s_t)}{L_3^3}\left(1+c_1\frac{H_{11}(s_t)}{L_3}+c_2\frac{H_{11}(s_t)^2}{L_3^2}+\mathscr{O}\left(\frac{1}{L_3^3}\right)\right)\,, \tag{14}$$

where c_1 and c_2 are known real numbers from the expansion of Z_{00}. This means that measuring the value of s, where the $\tan\delta(s;L(s))$ becomes infinite, we may extract H_{11} at $s=s_t$. Note that the effective-range term, which contains $H_{11}'(s_t)$, contributes first at $\mathscr{O}(L^{-6})$.

3. Similarly, suppose that $\tan\delta(s,L)=0$ at $s=s_2=E_2^2$ and $L=L_2=L(s_2)$ (see Fig. 1). Using the same technique as just described, we obtain:

$$q_1^2(s_2)=-\frac{4\pi G(s_t)}{L_2^3}\left(1+c_1\frac{G(s_t)}{L_2}+c_2\frac{G(s_t)^2}{L_2^2}+\mathscr{O}\left(\frac{1}{L_2^3}\right)\right)\,, \tag{15}$$

where

$$G(s_t)=H_{11}(s_t)-H_{12}^2(s_t)/H_{22}(s_t)\,. \tag{16}$$

Therefore, measuring s_2, we get $H_{11}-H_{12}^2/H_{22}$.

Using finally Eq. (4), we can express the scattering length in terms of the three quantities H_{11}, I and G, all taken at $s=s_t$

$$a_{11}=H_{11}(s_t)+\frac{iq_2(s_t)I(s_t)H_{11}(s_t)\left(H_{11}(s_t)/G(s_t)-1\right)}{1-iq_2(s_t)I(s_t)H_{11}(s_t)/G(s_t)}\,. \tag{17}$$

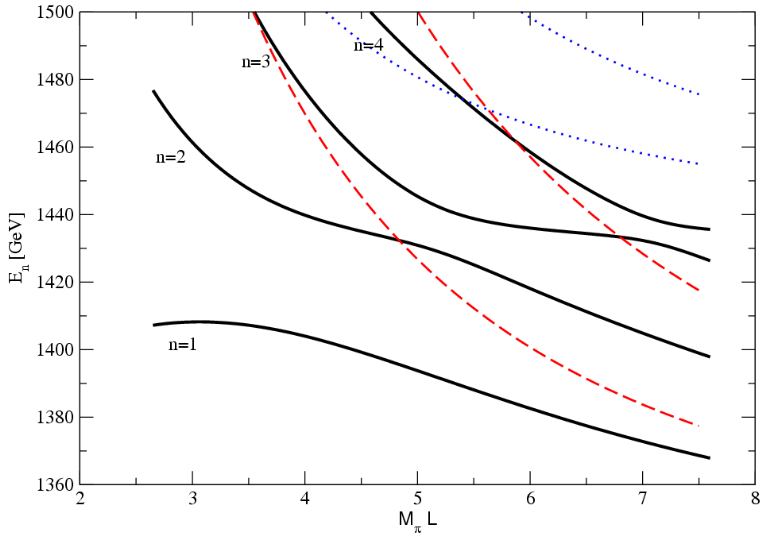

Figure 2: Energy levels for the two-channel model with an explicit $\Lambda(1405)$ resonance in the finite volume. The avoided level crossing which is observed at the energies between 1430 MeV and 1440 MeV is not related to the physical resonance in the infinite volume but reflects the presence of the $\bar{K}N$ threshold. For comparison, we plot the energy levels levels for the non-interacting two-particle systems $\pi\Sigma$ (dashed lines) and $\bar{K}N$ (dotted lines).

This is the central result of this letter.

Finally, note that in the analysis of the lattice data it may be more convenient to directly fit the explicit expression of the pseudophase given in Eq. (7) to the measured values on the lattice around $s = s_t$, replacing $H_{ij}(s)$ by $H_{ij}(s_t)$ and considering $H_{ij}(s_t)$ $(ij = 11, 12, 22)$ as three independent fitting parameters. From the above discussion one may expect that such a fit will lead to the precise determination of these parameters. The effective range terms can be neglected since their contribution is suppressed by three powers of L. In this case, one does not need to measure the pseudophase in the whole interval between s_2 and s_3.

4. Given the parameters H_{ij} determined from fitting to the pseudophase, the position of the pole on the second Riemann sheet of the complex variable s, which corresponds to the $\Lambda(1405)$-resonance, can be determined from the secular equation (3). We expect that replacing $H_{ij}(s)$ by $H_{ij}(s_t)$ allows one to locate the pole position at a reasonable accuracy.

5. In order to demonstrate the above-described proposal in practice, we have investigated a coupled-channel model with an explicit $\Lambda(1405)$ resonance located at $\mathrm{Re}\sqrt{s_R} = 1406\,\mathrm{MeV}$ and $-2\,\mathrm{Im}\sqrt{s_R} = 50\,\mathrm{MeV}$. Effective range terms are neglected. [2] The resulting first four energy levels as a function of $M_\pi L$ are shown in Fig. 2. We notice that the lowest level ($n = 1$) does only show a moderate volume dependence in the interval considered, quite in contrast to the excited ones with $n \geq 2$. For $M_\pi L \simeq 2 \ldots 3$ the ground state level flattens around $E = 1406$ MeV that corresponds to the $\Lambda(1405)$. It is clear that, for this reason, the lowest level can not be used for the extraction of the $\bar{K}N$ scattering length. The excited levels show a more complicated behavior in this interval of L. At the first

[2]The matrix elements H_{ij} are taken equal to $H_{11} = -1.47573$ fm$, H_{12} = 0.91581$ fm$, H_{22} = -0.34159$ fm. This corresponds to $a_{11} = a_0(K^-N) = (-1.26 + i\,0.70)\,\mathrm{fm}$.

glance, these levels exhibit the so-called avoided level crossing somewhere between 1430 MeV and 1440 MeV. In the elastic case, such a behavior of the energy levels signalizes the presence of a narrow resonance near this energy. However, this is not the case here. The peculiar behavior of the excited energy levels is caused by the opening of the $\bar{K}N$ threshold. At higher energies, the picture repeats – an avoided level crossing emerges, if the $\bar{K}N$ system has a discrete eigenvalue at this energy in a finite volume. If the volume changes, the avoided level crossing moves (in difference to the avoided level crossing corresponding to the "true" resonance). For $L \to \infty$ the bifurcation lines accumulate at threshold $s = s_t$. In this limit, the scattering amplitude is not analytic at $s = s_t$ (unitary cusp).

In Fig. 1 gives the pseudophase derived from the second ($n = 2$) energy level. It shows the expected behavior. First, it crosses $\pi/2$ at $\sqrt{s} = E_1$, very close to the mass of the $\Lambda(1405)$. Then, it passes π at $\sqrt{s} = \sqrt{s_2} = E_2$, close to the threshold where its value is $\delta_t > \pi$. At E_2, the tangent of the pseudophase vanishes at since $q_1^2(s_2) < 0$, we can conclude that $G(s_t) > 0$, cf. Eq. (15). Finally, the value of $3\pi/2$ is reached at $\sqrt{s} = \sqrt{s_3} = E_3$. Here, $q_1^2(s_3) > 0$ and consequently $H_{11}(s_t) < 0$. This can be deduced from Eq. (15) after the substitutions $s_2 \to s_3$ and $L_2 \to L_3$.

6. In this letter, we have generalized Lüscher's algorithm for the extraction of the scattering length from the finite-volume energy spectrum measured on the lattice. The modified algorithm applies to the case when the scattering length is complex due to the presence of the open channel(s) below threshold. In the case of the $\bar{K}N$ scattering with total isospin $I = 0$, the scattering length can be determined by measuring the volume dependence of the first excited level around the threshold energy.

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