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Chiral expansions of the π^0 decay amplitude

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We consider the anomaly dominated process of a neutral pion decaying into two photons and present the computation of the amplitude up to NNLO in the two-flavour chiral expansion. We show that the result is renormalizable and that chiral logarithms are present at this order, unlike the case at lower orders. Their influence turns out to be very small for physical pion masses. We also discuss the matching between the two-flavour and the three-flavour expansions at one loop. This allows to make an estimate of the chiral coupling constants involved and then a phenomenological prediction. The uncertainties are discussed and the result is compared to previous ones and to a new experimental result which was presented at this conference.

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1. Introduction

The development of methods to perform the chiral expansion at NNLO [1, 2] has made it possible to make predictions of unprecedented precision in hadronic physics (see e.g. H. Leutwyler's talk at this conference). Furthermore, as was also shown at this conference, progress in lattice QCD is providing new determinations of chiral coupling constants as well as new insight into convergence properties of the chiral expansion as a function of the number of light flavours, N_f and the size of the light quark masses

I will present here our results on the structure of the amplitude for neutral pion decay into two photons, calculated to NNLO in the $N_f = 2$ chiral expansion. This observable is one of the key probes of the spontaneous breaking of chiral symmetry in QCD and of the anomaly phenomenon. It is expected that results from lattice QCD should become available in the near future [3] which could be matched to our NNLO formula. Furthermore, at this conference, the results from the PRIMEX experiment measuring the π^0 lifetime with 3% precision have been officially presented. In the second part of this talk I will review the phenomenological determination of the relevant chiral coupling constants and the theoretical prediction for the π^0 lifetime.

2. Amplitude at LO and NLO

At leading order of the chiral expansion the $\pi^0 \to \gamma\gamma$ decay amplitude

$$\mathscr{T}(\pi^0 \to \gamma(k_1)\gamma(k_2)) = e^2 \varepsilon(e_1^*, k_1, e_2^*, k_2)\, T_{\pi\gamma\gamma} \tag{2.1}$$

can be deduced directly from the anomalous Ward identity in the $N_f = 2$ chiral limit

$$\partial^\mu A_\mu^3 = -\frac{\alpha}{8\pi}\varepsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta} \tag{2.2}$$

(in exactly the same way as the Nambu-Goldberger-Treiman [4] relation), and one gets

$$T^{LO}_{\pi^0\gamma\gamma} = \frac{1}{4\pi^2 F} \tag{2.3}$$

where $F = \lim_{m_u=m_d=0} F_\pi$. Alternatively, this result can be recovered from a tree diagram of the Wess-Zumino $O(p^4)$ chiral Lagrangian [5].

According to the Weinberg rules [6] the NLO corrections are obtained by a) computing the one-loop diagrams with one vertex from the WZ Lagrangian, which was performed in refs. [7, 8] and b) adding the contributions of the tree diagrams from the $O(p^6)$ chiral Lagrangian in the WZ sector. The classification of a minimal set of independent terms of this Lagrangian was achieved in [9, 10] who found that there are 13 independent terms for $N_f = 2$ (with couplings c_i^W) and 23 terms for $N_f = 3$ (with couplings C_i^W). Collecting these results, the amplitude at NLO is given by replacing F by $F_\pi^{(4)} = F\left(1+\frac{M^2}{16\pi^2F^2}\bar{l}_4\right)$ in the LO term (2.3) and by adding to it

$$T^{NLO}_{\pi^0\gamma\gamma} = \frac{16}{3F}\left(M^2[-4c_3^{Wr} - 4c_7^{Wr} + c_{11}^{Wr}] + \frac{4}{3}B(m_d - m_u)[5c_3^{Wr} + c_7^{Wr} + 2c_8^{Wr}]\right)\,. \tag{2.4}$$

with $M^2 = B(m_d + m_u)$, which can be replaced by m_π^2 at this order. There are two remarkable things about this result. Firstly, isospin breaking and isospin conserving corrections appear, a priori, on the same footing unlike many pionic observables which depend quadratically on $m_d - m_u$ (it will actually turn out that isospin breaking corrections are dominant). Secondly, once expressed in terms of F_π, the amplitude is exactly polynomial in the quark masses.

3. Chiral structure of π^0 decay at NNLO

3.1 Double chiral log's

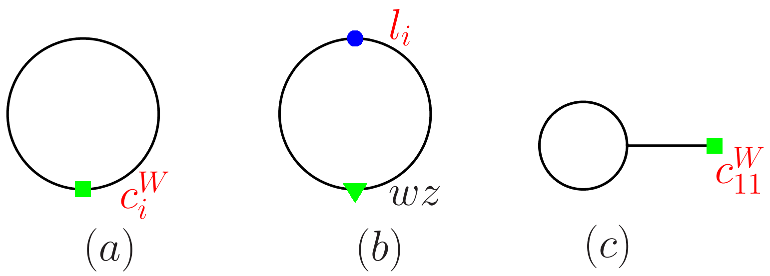

Figure 1: *One-loop graphs to the generating functional relevant for the computation of the double logarithms*

In order to go one step further in the chiral expansion one must collect a) all two-loop diagrams having vertices from LO Lagrangians, b) all one loop diagrams with one vertex from an NLO Lagrangian (i.e. involving either a coupling l_i or c_i^W) and c) the tree diagrams from the $O(p^8)$ Lagrangian in the WZ sector. From a phenomenological point of view the most interesting contributions, a priori, are those involving squared logarithms of the pion mass (so-called double log's) as they are enhanced by the smallness of the pion mass. Such terms could be of comparable size to the NLO corrections since the latter are not enhanced by chiral logarithms. Based on arguments [6] on the cancellation of non-local divergences from two-loop graphs with those of one-loop graphs one can deduce the double log terms from computing only one-loop graphs. The relevant ones in our case are shown in fig. 1. These are generating functional graphs which can be computed using methods of ref. [2] which gives

$$T^{log^2}_{\pi^0\gamma\gamma} = \frac{1}{4\pi^2 F_\pi}\left(-\frac{1}{6}\right)\left(\frac{m_\pi^2}{16\pi^2 F_\pi^2} L_\pi\right)^2 \quad \text{with } L_\pi = \log\frac{m_\pi^2}{F_\pi^2} \tag{3.1}$$

The peculiarity of the NNLO double logarithms is that they depend only on m_π and F_π. The one-loop graphs generate, in addition, single log as well as polynomial terms proportional to NLO coupling constants. These will be listed below

3.2 NNLO: complete chiral structure

In order to get a more complete description of the amplitude we have computed the two-loop graphs which are displayed in fig. 2. As a result, we have verified the cancellation of the non-local

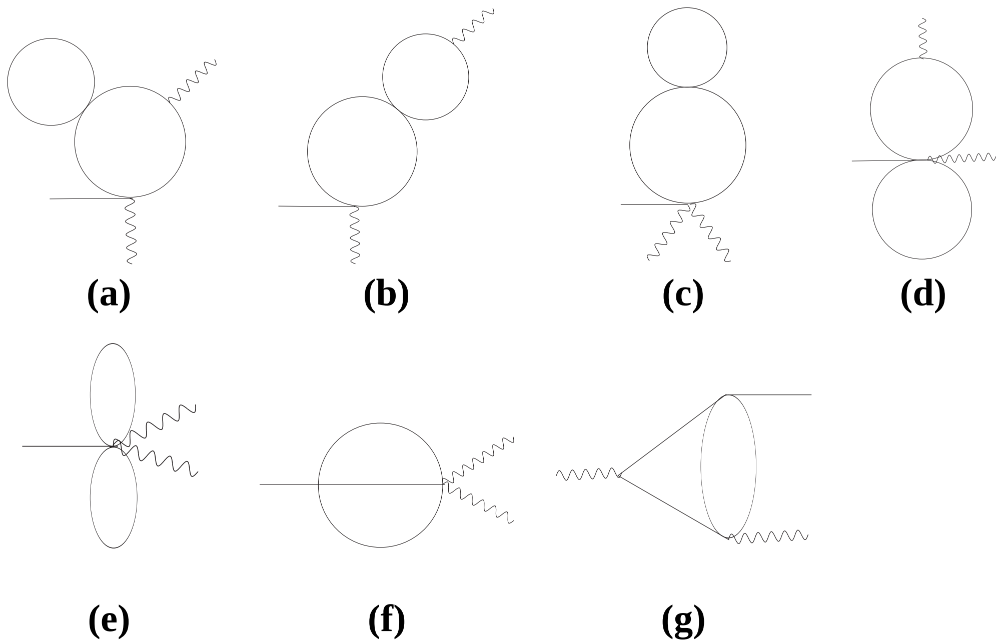

Figure 2: *Two-loop Feynman graphs (one-particle irreducible) contributions to the* $\pi^0 \to 2\gamma$ *amplitude.*

divergences (thus the renormalizability in this particular case). In addition, this allows us to obtain a complete expression of the single logarithmic terms as well as the complete contributions from one and two loop graphs to the terms which are polynomial in the quark masses. The result is shown in eq. (3.2)

$$
\begin{aligned}
F_\pi T^{NNLO}_{\pi^0\gamma\gamma} &= -\frac{M^4}{24\pi^2F^4}\left(\frac{1}{16\pi^2}L_\pi\right)^2 \\
&+\frac{M^4}{16\pi^2F^4}L_\pi\left[\frac{3}{256\pi^4}+\frac{32F^2}{3}\left(2c_2^{Wr}+4c_3^{Wr}+2c_6^{Wr}+4c_7^{Wr}-c_{11}^{Wr}\right)\right] \\
&+\frac{32M^2B(m_d-m_u)}{48\pi^2F^4}L_\pi\left[-6c_2^{Wr}-11c_3^{Wr}+6c_4^{Wr}-12c_5^{Wr}-c_7^{Wr}-2c_8^{Wr}\right] \\
&+\frac{M^4}{F^4}\lambda_+ +\frac{M^2B(m_d-m_u)}{F^4}\lambda_- +\frac{B^2(m_d-m_u)^2}{F^4}\lambda_{--}\ ,
\end{aligned}
\tag{3.2}
$$

The constants λ_+, λ_-, λ_{--} can be expressed as follows in terms of renormalized chiral coupling constants,

$$
\begin{aligned}
\lambda_+ &= \frac{1}{\pi^2}\left[-\frac{2}{3}d_+^{Wr}(\mu)-8c_6^r-\frac{1}{4}(l_4^r)^2+\frac{1}{512\pi^4}\left(-\frac{983}{288}-\frac{4}{3}\zeta(3)+3\sqrt{3}\,\mathrm{Cl}_2(\pi/3)\right)\right] \\
&\quad+\frac{16}{3}F^2\left[8l_3^r(c_3^{Wr}+c_7^{Wr})+l_4^r(-4c_3^{Wr}-4c_7^{Wr}+c_{11}^{Wr})\right] \\
\lambda_- &= \frac{64}{9}\left[d_-^{Wr}(\mu)+F^2l_4^r(5c_3^{Wr}+c_7^{Wr}+2c_8^{Wr})\right] \\
\lambda_{--} &= d_{--}^{Wr}(\mu)-128F^2l_7(c_3^{Wr}+c_7^{Wr})\ .
\end{aligned}
\tag{3.3}
$$

where we have denoted the three combinations of renormalized chiral couplings from the $O(p^8)$ Lagrangian in the anomalous sector by d_+^{Wr}, d_-^{Wr} and d_{--}^{Wr}. The scale dependence of these combination can be deduced from eq. (3.2). The complete expression of the decay amplitude at NNLO is obtained by adding to eq. (3.2) the contribution from eq. (2.3) in which F is replaced by $F_\pi^{(6)}$ as well as the contribution from eq. (2.4) in which M^2 and F are replaced by $(m_\pi^2)^{(4)}$ and $F_\pi^{(4)}$ respectively.

4. Phenomenology

What phenomenological predictions are we now able to make? The preceding formulas show that chiral log's are present at NNLO. However, the contribution from the double chiral log turns out to be very small numerically $\simeq -2\,10^{-4}$ relative to the current algebra amplitude (taking a value of the scale $\mu = m_\eta$). We expect then that the quark mass contributions to the π^0 decay amplitude are essentially given by the NLO part (2.4), which should be of the order of a few percent, with only a very small extra contribution from the chiral logarithms at NNLO. It is therefore important to be able to estimate the combinations of coupling constants which enter into these NLO terms. This could become possible in the future by matching with lattice QCD results. At present, phenomenological predictions rely on performing a three flavours rather than a two flavours expansion [11, 12, 13, 14, 15, 16] (sometimes, a large N_c expansion is also performed). Starting from our $N_f = 2$ expressions, this amounts to estimate the c_i^{Wr} couplings based on expanding them as a function of the strange quark mass. A technically convenient method for performing such an expansion has recently been devised [17]. Matching the generating functionals at one loop one obtains the first two terms i.e. $O(1/m_s)$ (which is induced by $\pi^0-\eta$ mixing) and $O(m_s^0)$, of the m_s expansion. The two combinations of interest to us get expressed as follows

$$\begin{aligned}
-4c_3^{Wr} - 4c_7^{Wr} + c_{11}^{Wr} &= -4C_7^{Wr} + O(m_s)\\
5c_3^{Wr} + c_7^{Wr} + 2c_8^{Wr} &= \frac{9}{1024\pi^2}\frac{1}{m_sB} - \frac{9}{32\pi^2}\frac{1}{F_0^2}\left[3L_7^r + L_8^r + \frac{5}{3}\log\frac{m_sB}{\mu^2} + \frac{2}{3}\log\frac{4}{3}\right]\\
&\quad +6(C_7^{Wr} + 3C_8^{Wr}) + O(m_s)
\end{aligned} \tag{4.1}$$

The two $N_f = 3$ couplings C_7^{Wr}, C_8^{Wr} can be estimated by first using a chiral sum rule which shows that $C_7^{Wr} \ll C_8^{Wr}$ (assuming no unexpectedly large contributions from large momenta) and an experimental input on the $\eta \to \gamma\gamma$ decay amplitude. This gives $C_8^{Wr} = (0.58 \pm 0.20)\,10^{-3}$ GeV^{-2}. Alternatively, one may estimate C_8^{Wr} from a sum rule and large N_c approximations, obtaining a compatible result.

One can include some terms from the NNLO formula (3.2) on the basis of a modified three flavour chiral counting in which m_u, m_d are counted as $O(p^2)$ as usual but m_u/m_s, m_d/m_s are counted as $O(p)$ rather than $O(1)$. In this sense our estimates of the chiral couplings allow us to include the corrections: $O(p)$, $O(p^2)$ from (2.4) as well as the log enhanced terms $O(p^3 \log p)$ and $O(p^4 \log^2 p)$ from (3.2). One can also include the $O(e^2)$ electromagnetic contribution which was discussed in [13]. The numerical values of these various contributions are shown in table 1. The entries in the table have been computed using updated values for F_π and for the quark mass ratios, see ref. [18] for more details. Finally, one obtains for the π^0 decay amplitude

$$\Gamma_{\pi^0\to\gamma\gamma} = (8.09 \pm 0.11)\ \text{eV}\ . \tag{4.2}$$

CA	$O(p)$	$O(p^2)$	$O(e^2)$	$O(p^3 \log p)$	$O(p^4 \log^2 p)$
7.76	0.09	0.29	-0.05	0.005	-0.004

Table 1: *Current algebra contribution to the $\pi^0 \to 2\gamma$ decay width (in eV) and corrections of various chiral orders using the modified $SU(3)$ counting.*

This theoretical prediction is in very good agreement with the one performed in ref. [15] which combines chiral $SU(3)$ and large N_c expansions. The uncertainty in the prediction is dominated by the unknown higher order terms in the m_s expansion and by the not so precisely known isospin breaking mass difference $m_d - m_u$. Within the errors this result agrees with the experimental measurement which was presented by A. Bernstein at this conference.

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