# A New Dispersive Analysis of $\eta \to 3\pi$

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We present a new dispersive analysis of the isospin breaking decay $\eta \to 3\pi$. The resulting representation of the decay amplitude allows us to determine the quark mass double ratio $Q$ and we find as a preliminary result $Q = 22.3 \pm 0.4$. Finally, we discuss a number of improvements that we intend to implement in the future.

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## 1. Introduction

The decay $\eta \to 3\pi$ is forbidden by isospin symmetry, as Bose statistics does not allow three pions to form a state with vanishing total isospin and total angular momentum. Neglecting electromagnetic contributions that are strongly suppressed [1], the decay amplitude $A(s,t,u)$ is proportional to $(m_d - m_u)$ or, alternatively, to the quark mass double ratio

$$\frac{1}{Q^2} = \frac{m_d^2 - m_u^2}{m_s^2 - \hat{m}^2}, \tag{1.1}$$

where $\hat{m} = (m_u + m_d)/2$. As $\Gamma \propto |A|^2 \propto Q^{-4}$, we can get $Q$ by comparing a theoretical result for the amplitude with a measurement of the decay width $\Gamma$. Alternatively, $Q$ can also be calculated from a ratio of meson masses [2, 3].

As is well known, the chiral perturbation theory series converges rather slowly: At tree-level, the decay width is $\Gamma = 66$ eV [4, 5], while at one-loop it is already $\Gamma = 160$ eV [6], which is still quite far away from the experimental value $\Gamma = 295$ eV [7]. Of course, the theoretical values need an input for $(m_d - m_u)$, which is calculated from meson masses:

$$(m_d - m_u)B_0 = (m_{K^0}^2 - m_{K^+}^2)_{QCD} = m_{K^0}^2 - m_{K^+}^2 - m_{\pi^0}^2 + m_{\pi^+}^2, \tag{1.2}$$

where the second equality relies on Dashen's theorem, stating that at leading order in the low energy expansion the electromagnetic contribution to the mass differences $m_{K^0}^2 - m_{K^+}^2$ and $m_{\pi^+}^2 - m_{\pi^0}^2$ is the same [8]. The slow convergence of the chiral series is mainly due to final state rescattering of the pions, which can be very well treated by means of dispersion relations.

The idea to use dispersion techniques to calculate the $\eta \to 3\pi$ amplitude is, in fact, not new: it has been done already more than 10 years ago by Kambor, Wiesendanger and Wyler [9] and by Anisovich and Leutwyler [10]. The methods used in these works differ in technical aspects, but lead to compatible results. We follow the approach as presented in ref. [10]. There has been a lot of activity in this area since then and therefore, we think it is worth to take a fresh look at this problem. There has been considerable improvement concerning the $\pi\pi$ phase shifts [11 – 14], which are the most important physical input in the dispersion relations. Furthermore, there are a number of new measurements of this decay by the KLOE [15, 16], MAMI [17, 18] and WASA [19, 20] collaborations, which will be useful for the determination of the subtraction polynomials, and also a full two-loop calculation in chiral perturbation theory [21].

## 2. The Method

We only give a very brief account of the method that we are using and refer to ref. [10] for a more detailed description. For the charged decay $\eta \to \pi^0\pi^+\pi^-$, we define the Mandelstam variables as $s = (p_{\pi^+} + p_{\pi^-})^2$, $t = (p_{\pi^-} + p_{\pi^0})^2$ and $u = (p_{\pi^0} + p_{\pi^+})^2$. They are related by $s + t + u = m_\eta^2 + m_{\pi^0}^2 + 2m_{\pi^+}^2 \equiv 3s_0$. At the order we are working the amplitudes for the charged and the neutral channel are related by

$$A_{\text{neutral}}(s,t,u) = A_{\text{charged}}(s,t,u) + A_{\text{charged}}(t,u,s) + A_{\text{charged}}(u,s,t). \tag{2.1}$$

In the following we will restrict ourselves to the charged channel, as we could get the neutral channel easily by the above relation. We are working in the isospin limit where $m_{\pi^0} = m_{\pi^+}$. It is not a priori clear, what value for the pion mass in the isospin limit should be used and we choose $m_\pi = m_{\pi^0}$.

Unitarity allows us to relate the imaginary part of the decay amplitude $A_n \doteq A_{\eta \to n}$, where the final state $n$ is some three pion state, to the $\pi\pi$ scattering amplitude and $A_n$ itself as

$$\text{Im} \, A_n = \frac{1}{2} \sum_{n'} \left\{ (2\pi)^4 \delta^4(p_n - p_{n'}) T^*_{nn'} \right\} A_{n'} \,. \tag{2.2}$$

This is a linear constraint for the amplitude and we can thus extract a normalisation factor

$$A(s,t,u) = -\frac{1}{Q^2} \frac{m_K^2(m_K^2 - m_\pi^2)}{3\sqrt{3} m_\pi^2 F_\pi^2} M(s,t,u) \,. \tag{2.3}$$

With this choice, the tree-level expression for $M(s,t,u)$ is normalised to 1 at the centre of the Dalitz plot where $s = t = u = s_0$, and the one-loop amplitude only depends on measurable quantities. In particular it does not depend on $Q$.

If the discontinuities of $D$- and higher waves are neglected, we can decompose the amplitude into isospin components in the same way as was done for the $\pi\pi$ scattering amplitude in ref. [22]:

$$M(s,t,u) = M_0(s) + (s-u)M_1(t) + (s-t)M_1(u) + M_2(t) + M_2(u) - \frac{2}{3}M_2(s). \tag{2.4}$$

As only the S- and P-waves have discontinuities up to two-loop order in chiral perturbation theory, the decomposition is exact up to that order.

Inserting this decomposition in eq. (2.2), we get for the discontinuities of the isospin amplitudes

$$\text{disc} \, M_I(s) \doteq \frac{M_I(s+i\varepsilon) - M_I(s-i\varepsilon)}{2i} = \left\{ M_I(s) + \hat{M}_I(s) \right\} e^{-i\delta_I(s)} \sin \delta_I(s), \tag{2.5}$$

where $\delta_I(s)$ are the S- and P-wave $\pi\pi$ scattering phase shifts and $I = 0,1,2$. The inhomogeneities $\hat{M}_I(s)$ consist of angular averages over the $M_I$:

$$\hat{M}_0(s) = \frac{2}{3} \langle M_0 \rangle + 2(s - s_0)\langle M_1 \rangle + \frac{20}{9} \langle M_2 \rangle + \frac{2}{3} \kappa \langle z M_1 \rangle, \tag{2.6a}$$

$$\hat{M}_1(s) = \frac{1}{\kappa} \left\{ 3\langle z M_0 \rangle + \frac{9}{2}(s - s_0)\langle z M_1 \rangle - 5\langle z M_2 \rangle + \frac{3}{2} \kappa \langle z^2 M_1 \rangle \right\}, \tag{2.6b}$$

$$\hat{M}_2(s) = \langle M_0 \rangle - \frac{3}{2}(s - s_0)\langle M_1 \rangle + \frac{1}{3} \langle M_2 \rangle - \frac{1}{2} \kappa \langle z M_1 \rangle, \tag{2.6c}$$

where

$$\langle z^n f \rangle(s) = \frac{1}{2} \int_{-1}^{1} dz \, z^n f \left( \frac{3s_0 - s + z\kappa(s)}{2} \right), \tag{2.7}$$

and

$$\kappa(s) = \sqrt{\frac{s - 4m_\pi^2}{s}} \sqrt{\{(m_\eta + m_\pi)^2 - s\} \{(m_\eta - m_\pi)^2 - s\}} \,. \tag{2.8}$$

The integration variable in eq. (2.7) is $z = \cos\theta$, where $\theta$ is the scattering angle.

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These expressions for the discontinuities enable us to write down a set of dispersion integrals, coming from Cauchy representations of the functions $M_I(s)/\Omega_I(s)$ [10]:

$$M_0(s) = \Omega_0(s) \left\{ \alpha_0 + \beta_0 s + \gamma_0 s^2 + \frac{s^2}{\pi} \int_{4m_\pi^2}^\infty \frac{ds'}{s'^2} \frac{\sin \delta_0(s') \hat{M}_0(s')}{|\Omega_0(s')|(s'-s-i\varepsilon)} \right\}, \tag{2.9a}$$

$$M_1(s) = \Omega_1(s) \left\{ \beta_1 s + \frac{s}{\pi} \int_{4m_\pi^2}^\infty \frac{ds'}{s'} \frac{\sin \delta_1(s') \hat{M}_1(s')}{|\Omega_1(s')|(s'-s-i\varepsilon)} \right\}, \tag{2.9b}$$

$$M_2(s) = \Omega_2(s) \frac{s^2}{\pi} \int_{4m_\pi^2}^\infty \frac{ds'}{s'^2} \frac{\sin \delta_2(s') \hat{M}_2(s')}{|\Omega_2(s')|(s'-s-i\varepsilon)}. \tag{2.9c}$$

$\Omega_I(s)$ are the so called Omnès functions [23], which are the solutions of eq. (2.5) for $\hat{M}_I(s) = 0$ and are given by

$$\Omega_I(s) = \exp \left\{ \frac{s}{\pi} \int_{4m_\pi^2}^\infty \frac{\delta_I(s')}{s'(s'-s)} ds' \right\}. \tag{2.10}$$

Eqs. (2.9) contain four parameters $\alpha_0$, $\beta_0$, $\gamma_0$ and $\beta_1$. These subtraction constants are not determined by the dispersion relations and have to be fixed otherwise. Actually, there are three more of these constants in the equations for $M_1$ and $M_2$, but they can be eliminated because the decomposition given in eq. (2.4) is not unique. The $M_I$ can be shifted by a polynomial without changing the total amplitude $M(s,t,u)$, and the polynomial can be chosen such that three subtraction constants vanish. The remaining four are determined by a matching to the one-loop result from chiral perturbation theory.

We solve the integral equations numerically by an iterative procedure: We set the $M_I(s)$ to the tree-level result from chiral perturbation theory and calculate the $\hat{M}_I(s)$ by eqs. (2.6) and then get a new result for the $M_I(s)$ from eqs. (2.9). The subtraction constants $\alpha_0$ and $\beta_0$ can now be calculated by a matching to the one-loop amplitude at the Adler zero, as the amplitude there is protected by the chiral SU(2) $\times$ SU(2) symmetry and does not change much after the one-loop level. While these two constants have to be determined in every iteration step, the other two, $\gamma_0$ and $\beta_1$, can be set to their final value right from the start. Once the four constants are fixed, we add the subtraction polynomial and then repeat the same procedure until the subtraction constants $\alpha_0$ and $\beta_0$ converge.

Special care has to be taken when evaluating the angular averages $\hat{M}_I(s)$ in eqs. (2.6), as the starting and end points of the integration path, which are functions of $s$, can take complex values. In the first iteration step, the functions $M_I(s)$ are analytic everywhere and the integral does not depend on the path. However, the situation changes afterwards: the $M_I(s)$ have a cut along the real axis and the integration path has to be deformed in such a way that it does not cross the cut [24].

## 3. Preliminary Results

In the future, we will incorporate a number of improvements to the method described in the
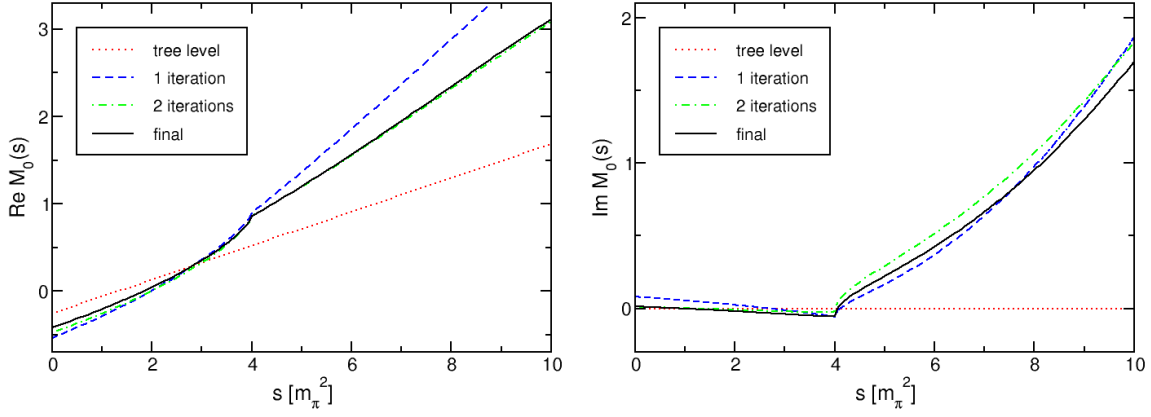
**Figure 1:** The function $M_0(s)$ and its change during the iterations. The left panel shows the real part, the right one the imaginary part. The red dotted line is the tree-level $\chi$PT result used as initial configuration, the blue dashed and the green dash-dotted lines are the results after one and two iterations, respectively. The solid black line is the final result after convergence of the subtraction constants is reached. In the real part, it is almost impossible to distinguish the result after two iterations from the final result.

To have a smooth representation of $M_0(s)$ near $s = 4m_\pi^2$, we have to ensure that the threshold of the phase shifts agrees with the lower limit of integration. As we know the phase shifts only for $m_\pi = m_{\pi^+}$, we also used the charged pion mass to create these figures.

last section. The results we present here are therefore only preliminary. The planned extensions are discussed in the next section.

Fig. 1 demonstrates how the function $M_0(s)$ develops during the iteration procedure. While for the real part the curve does not change considerably after the second iteration step, the imaginary part shows that it is worthwhile going beyond that, as there is still an obvious difference between the result after two iteration steps and the final result.

From the functions $M_I(s)$ we can then calculate the amplitude and the Dalitz plot. We use the dimensionless standard Dalitz plot variables defined as

$$X = \frac{\sqrt{3}}{2m_\eta Q_\eta}(u - t), \qquad Y = \frac{3}{2m_\eta Q_\eta}\left[(m_\eta - m_{\pi^0})^2 - s\right] - 1, \qquad (3.1)$$

where

$$Q_\eta = m_\eta - 2m_{\pi^+} - m_{\pi^0}. \qquad (3.2)$$

Fig. 2 shows our result for the Dalitz plot, normalised to 1 for $X = Y = 0$. It agrees with the polynomial representation provided by KLOE [15] at the 10 % level.

With the experimental decay width $\Gamma = 295 \pm 20$ eV [7], we get for the quark mass double ratio $Q = 22.3 \pm 0.4$. This is in good agreement with other results presented in fig. 3, which range from 20.7 to 24.3. The error is only due to the experimental uncertainty in the decay width, as we do not have an estimate for the theoretical error yet.

As a last illustration, fig. 4 shows the development of $Q$ over several iteration steps. Already after two iteration steps, the result for $Q$ is well within the experimental error band of the final result. However, going beyond that will not only shift the central value within the error bounds, but also reduce the theoretical error, which is not included in this figure.
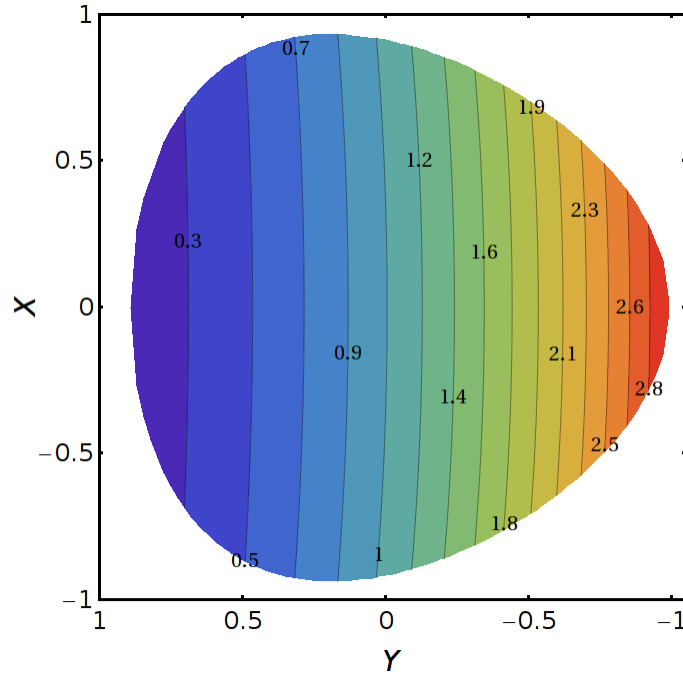
**Figure 2:** The Dalitz plot for $\eta \to \pi^0\pi^+\pi^-$ normalised to 1 for $X = Y = 0$. The dimensionless Dalitz plot variables $X$ and $Y$ are defined in the text.
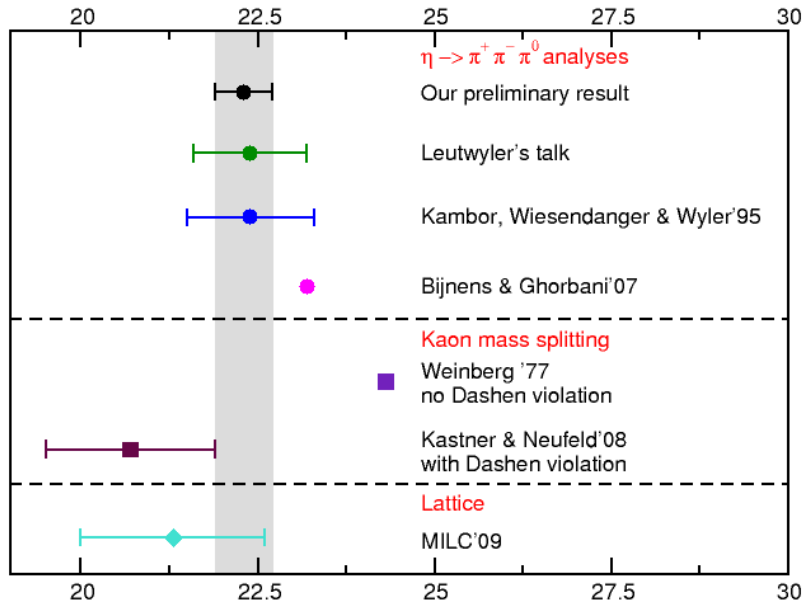


**Figure 3:** A selection of results for $Q$. Our result is $22.3 \pm 0.4$ and is indicated by the grey band. The error is only due to the experimental uncertainty on the decay width. The other results are taken from Leutwyler's talk at this conference [25], from a dispersive analysis in ref. [9], from a two-loop calculation in $\chi$PT [21], from Weinberg's quark mass ratios [2] and from an analysis including Dashen violation [3]. The last value we calculated from the MILC quark mass ratios presented by Heller at this conference [26].
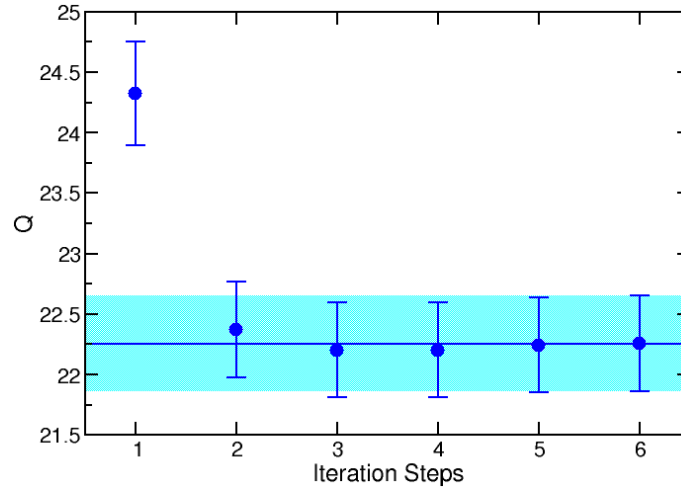
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**Figure 4:** The development of $Q$ over several iteration steps. The blue line is the final result after many iterations, the blue band is the corresponding error coming only from the uncertainty of the decay width.

## 4. Conclusion and Outlook

We have presented a new dispersive analysis of the decay $\eta \to 3\pi$ and argued that there have been several important developments in this and related fields that make a new analysis worthwhile. A number of preliminary results have been given, in particular we get for the quark mass double ratio $Q = 22.3 \pm 0.4$, which agrees well with a number of results from other works. We stress again that this number is preliminary and that the error given is only due to the uncertainty in the decay width and does not contain any estimate for the theoretical error.

The method as presented here has a number of shortcomings that we will have to address in the future. Besides solving these problems, we will also include additional features that have not been part of earlier dispersive treatments of this decay and that the precision reached in experiments nowadays has rendered mandatory.

As mentioned above, we calculate the decay amplitude in the isospin limit and use the mass of the neutral pion. However, it is not a priori clear, what value for the pion mass should be used and indeed the final result depends on this choice: setting the pion mass to $m_{\pi^+}$ shifts $Q$ from 22.3 down to 21.0. The pion mass enters the dispersion integrals in eq. (2.9) directly, as it is contained in the integration limits and in $\kappa(s)$, and also via the phase shifts. While it is easy to use the physical masses for the former, it is not for the latter. We would need different phase shifts for scatterings of pions with different flavours, which are not available at the moment. Indeed the above-mentioned shift in $Q$ is obtained without altering the pion mass in the phase shifts.

A number of other contributions have not been included yet: electromagnetic interactions [27, 28], inelasticity and the imaginary parts of D- and higher partial waves. Future extensions of the algorithm will include estimates of these effects.

The subtraction constants have been determined so far by a matching to the one-loop result from $\chi$PT. We plan to make use of experimental data instead and calculate the subtraction constants by fitting our Dalitz plot to a measurement.

Of course we can and will also extract a number of other interesting parameters than just $Q$, e.g. the quark mass ratio $R$, the branching ratio $r = \Gamma_{3\pi^0}/\Gamma_{\pi^0\pi^+\pi^-}$ or the quadratic slope parameter $\alpha$.

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