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Topological susceptibility and the second normalized cumulant in the chiral perturbation theory of QCD

Yao-Yuan Mao*[1], Ting-Wai Chiu[1,2] (for the TWQCD Collaboration)

[1] *Department of Physics, and Center for Theoretical Sciences, National Taiwan University, Taipei 10617, Taiwan*
[2] *Center for Quantum Science and Engineering, National Taiwan University, Taipei 10617, Taiwan*

We derive the topological susceptibility to the one-loop order in chiral perturbation theory (ChPT), for an arbitrary number of flavors. This formula provides a viable way for lattice QCD to determine the low-energy constants, F_π, L_6, L_7, L_8 and the chiral condensate Σ. Moreover, we derive the second normalized cumulant c_4 at the tree level of ChPT, and point out that the ratio $c_4/\chi_t = -1/4$ for $N_f = 2$ in the isospin limit ($m_u = m_d$), which agrees with recent results from unquenched lattice QCD, and rules out the instanton gas/liquid model which gives $c_4/\chi_t = -1$.

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*Speaker.

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1. Introduction

In Quantum Chromodynamics (QCD), the topological susceptibility (χ_t) is the most crucial quantity to measure the topological charge fluctuation of the QCD vacuum, which plays an important role in breaking the $U_A(1)$ symmetry. Theoretically, χ_t is defined as

$$\chi_t = \int d^4x \langle \rho(x)\rho(0)\rangle , \quad \rho(x) = \frac{1}{32\pi^2}\varepsilon_{\mu\nu\lambda\sigma}\mathrm{tr}[F_{\mu\nu}(x)F_{\lambda\sigma}(x)]. \tag{1.1}$$

Using the Chiral Perturbation Theory (ChPT), Leutwyler and Smilga [1] obtained the following relations in the chiral limit

$$\chi_t = \Sigma\left(\frac{1}{m_u}+\frac{1}{m_d}\right)^{-1}, \quad (N_f = 2), \tag{1.2}$$

$$\chi_t = \Sigma\left(\frac{1}{m_u}+\frac{1}{m_d}+\frac{1}{m_s}\right)^{-1}, \quad (N_f = 3), \tag{1.3}$$

where m_u, m_d, and m_s are the quark masses, and Σ is the chiral condensate. This implies that in the chiral limit ($m_u \to 0$) the topological susceptibility is suppressed due to internal quark loops. Most importantly, (1.2) and (1.3) provide a viable way to extract Σ from χ_t in the chiral limit.

From (1.1), one obtains

$$\chi_t = \frac{\langle Q_t^2\rangle}{\Omega}, \quad Q_t \equiv \int d^4x\rho(x), \tag{1.4}$$

where Ω is the volume of the system, and Q_t is the topological charge (which is an integer for QCD). Thus, one can determine χ_t by counting the number of gauge configurations for each topological sector. Furthermore, we can also obtain the second normalized cumulant

$$c_4 = -\frac{1}{\Omega}\left[\langle Q_t^4\rangle - 3\langle Q_t^2\rangle^2\right], \tag{1.5}$$

which is related to the leading anomalous contribution to the $\eta' - \eta'$ scattering amplitude in QCD, as well as the dependence of the vacuum energy on the vacuum angle θ.

Recently, the topological susceptibility and the second normalized cumulant have been measured in unquenched lattice QCD with exact chiral symmetry, for $N_f = 2$ and $N_f = 2+1$ lattice QCD with overlap fermion in a fixed topology [2, 3, 4], and $N_f = 2+1$ lattice QCD with domain-wall fermion [5]. The results of topological susceptibility turn out in good agreement with the Leutwyler-Smilga relation in the chiral limit, with the values of the chiral condensate as follows.

$$\begin{aligned}
\Sigma^{\overline{\mathrm{MS}}}(2\ \mathrm{GeV}) &= [259(7)(8)\ \mathrm{MeV}]^3, \quad (N_f = 2), \qquad \mathrm{Ref.}[2,\ 3],\\
\Sigma^{\overline{\mathrm{MS}}}(2\ \mathrm{GeV}) &= [258(8)(7)\ \mathrm{MeV}]^3, \quad (N_f = 2+1), \quad \mathrm{Ref.}[4],\\
\Sigma^{\overline{\mathrm{MS}}}(2\ \mathrm{GeV}) &= [259(6)(9)\ \mathrm{MeV}]^3, \quad (N_f = 2+1), \quad \mathrm{Ref.}[5].
\end{aligned}$$

These results assure that lattice QCD with exact chiral symmetry is the proper framework to tackle the strong interaction physics with topologically non-trivial vacuum fluctuations. Obviously, the next task for unquenched lattice QCD with exact chiral symmetry is to determine the second normalized cumulant c_4 to a good precision, and to address the question how the vacuum energy

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depends on the vacuum angle θ and related problems. Theoretically, it is interesting to obtain an analytic expression of c_4 in ChPT, as well as to extend the Leutwyler-Smilga relation to the one-loop order of ChPT.

Recently, we have derived the topological susceptibility χ_t to the one-loop order in ChPT, for an arbitrary number of flavors, as well as the second normalized cumulant c_4 at the tree level of ChPT [6]. In this talk, we outline our derivations and point out the salient features of our results.

2. Topological susceptibility and c_4 at the tree level of ChPT

The leading terms of the effective chiral lagrangian for QCD with N_f flavor at $\theta = 0$ [7] are the kinetic term and the symmetry breaking term,

$$\mathscr{L}^{(2)} = \mathscr{L}^{(2)}_{\text{eff}} + \mathscr{L}^{(2)}_{\text{s.b.}} = \frac{F_\pi^2}{4}\text{Tr}(\partial_\mu U \partial^\mu U^\dagger) + \frac{\Sigma}{2}\text{Tr}(\mathscr{M}U^\dagger + U\mathscr{M}^\dagger), \tag{2.1}$$

where $U(x) = \exp\{2i\phi^a(x)t^a/F_\pi\}$ is a group element of $SU(N_f)$, $\mathscr{M}$ is the quark mass matrix, F_π is the pion decay constant, and $\Sigma = \langle\bar\psi\psi\rangle_{vac}$ is the chiral condensate of the QCD vacuum. It is well known that the physical vacuum angle on which all physical quantities depend is $\theta_{\text{phys}} = \theta + \arg\det(\mathscr{M})$ rather than θ. Also, the θ-dependence of $Z_{N_f}(\theta)$ always enters through the combinations $\mathscr{M}e^{i\theta/N_f}$ and $\mathscr{M}^\dagger e^{-i\theta/N_f}$. For small quark masses ($L \ll m_\pi^{-1}$), the unitary matrix U does not depend on x_μ. Thus the kinetic term in the leading-order chiral lagrangian can be dropped, the partition function becomes

$$Z_{N_f}(\theta) = \int dU \exp\left\{\Omega\,\Sigma\,\text{Re}\left[\text{Tr}(\mathscr{M}e^{i\theta/N_f}U^\dagger)\right]\right\}, \tag{2.2}$$

where $\Omega = L^3T$ is the space-time volume. If we consider a sufficiently large volume Ω satisfying $m_j\Sigma\Omega \gg 1$, then the group integral in the partition function (2.2) is largely due to the U which minimizes the minus exponent of the integrand. So we have the vacuum energy density,

$$\varepsilon_{\text{vac}}(\mathscr{M},\theta) = -\frac{1}{\Omega}\log Z_{N_f}(\theta) = \varepsilon_0 - \Sigma\min_U\left\{-\text{Re}\left[\text{Tr}(\mathscr{M}e^{i\theta/N_f}U^\dagger)\right]\right\}, \tag{2.3}$$

where ε_0 corresponds to the normalization factor of the partition function.

Without loss of generality, the unitary matrix U can be taken to be diagonal with elements $e^{i\alpha_j}$, where $\sum_{j=1}^{N_f}\alpha_j = 0$. We can also choose the mass matrix to be diagonal $\mathscr{M} = \text{diag}(m_1,\ldots,m_{N_f})$. Then the vacuum energy density can be written as

$$\varepsilon_0 - \Sigma\min_\phi\left\{-\sum_{j=1}^{N_f} m_j\cos\phi_j\right\}, \quad \sum_{j=1}^{N_f}\phi_j = \theta, \tag{2.4}$$

where $\phi_j = \theta/N_f - \alpha_j$, and $\sum_j\phi_j = \theta$.

Now we solve the minimization problem. For the purpose of obtaining the topological susceptibility and the second normalized culmulant, we can consider the limit of small θ (and ϕ_j's) because $U = \mathbf{1}$ gives the minimal vacuum energy at $\theta = 0$. To the order of θ^4, we still have the exact result of χ_t and c_4 (at the tree level). Expanding $\cos\phi \simeq 1 - \frac{1}{2}\phi^2 + \frac{1}{24}\phi^4$ and introducing the Lagrange multiplier λ to incorporate the constraint $\sum_i \phi_i = \theta$, we can solve the minimization problem and get ϕ_i to the order of θ^3,

$$\phi_i = \frac{\bar{m}}{m_i}\theta + \frac{\theta^3}{6}\left[\left(\frac{\bar{m}}{m_i}\right)^3 - \left(\frac{\bar{m}}{m_i}\right)\sum_{j=1}^{N_f}\left(\frac{\bar{m}}{m_j}\right)^3\right] + \mathscr{O}(\theta^5).$$

where $\bar{m} \equiv \left(\sum_{i=1}^{N_f} m_i^{-1}\right)^{-1}$ is the "reduced mass" of the N_f quark flavors. Keeping to the order of θ^4, the vacuum energy density is

$$\varepsilon_{\text{vac}}(\theta) = \varepsilon_0 + \Sigma\left(\sum_{j=1}^{N_f}\frac{1}{m_j}\right)^{-1}\frac{\theta^2}{2} - \Sigma\sum_{i=1}^{N_f} m_i^{-3}\left(\sum_{j=1}^{N_f}\frac{1}{m_j}\right)^{-4}\frac{\theta^4}{24} + \mathscr{O}(\theta^6).$$

It immediately follows that the topological susceptibility and the second normalized culmulant are

$$\chi_t = \left.\frac{\partial^2 \varepsilon_{\text{vac}}}{\partial\theta^2}\right|_{\theta=0} = \Sigma\left(\sum_{j=1}^{N_f}\frac{1}{m_j}\right)^{-1}, \tag{2.5}$$

$$c_4 = \left.\frac{\partial^4 \varepsilon_{\text{vac}}}{\partial\theta^4}\right|_{\theta=0} = -\Sigma\sum_{i=1}^{N_f} m_i^{-3}\left(\sum_{j=1}^{N_f}\frac{1}{m_j}\right)^{-4}. \tag{2.6}$$

3. Topological susceptibility to the one-loop order of ChPT

To the one-loop order of ChPT, one has to include $\mathscr{L}^{(4)}$ [7] at the tree level as well as the one-loop contributions of $\mathscr{L}^{(2)}$. In 1984, Gasser and Leutwyler [7] considered the low-energy expansion, where both p and $\mathscr{M}$ are assumed to be small but $\mathscr{M}/p^2$ can have a finite value, such that the value of M_π^2/p^2 can be fixed. In this case, the external sources $a_\mu(x)$ and $p(x)$ can be counted as order of Φ, and $v_\mu(x)$ and $s(x)-\mathscr{M}$ as order of Φ^2. Gasser and Leutwyler showed that at the one-loop order, the chiral effective action can be written as $W = W_t + W_u + W_A + \mathscr{O}(\Phi^6)$, where W_t denotes the sum of tree diagrams and tadpole contributions (of order Φ^2), W_u the unitarity correction (of order Φ^3), and W_A the anomaly contribution (of order Φ^4). Because the θ dependence enters the Lagrangian only through $\mathscr{M}$, we can count χ_t as order of Φ^2, thus for the evaluation of topological susceptibility to the one-loop order, and it suffices to consider W_t only.

Moreover, Gasser and Leutwyler [7] showed that the pole terms due to the one-loop contributions of $\mathscr{L}^{(2)}$ can be absorbed by the low-energy coupling constants of $\mathscr{L}^{(4)}$, and W_t is given by [7]

$$\begin{aligned} W_t &= \sum_P \int d^4x \frac{F_\pi^2}{2}\left\{\frac{1}{N_f} - \frac{M_P^2}{16\pi^2 F_\pi^2}\ln\frac{M_P^2}{\mu_{sub}^2}\right\}\sigma_{PP}^{\Delta} \\ &\quad + \sum_P \int d^4x \frac{F_\pi^2}{2}\left\{\frac{N_f}{N_f^2-1} - \frac{M_P^2}{16\pi^2 F_\pi^2}\ln\frac{M_P^2}{\mu_{sub}^2}\right\}\sigma_{PP}^{\chi} + \int d^4x \mathscr{L}^{r(4)}, \end{aligned} \tag{3.1}$$

where M_P^2's are the squared meson masses, σ_{PP}^{Δ} corresponds to the kinetic term which can be dropped in the limit of small quark masses, σ_{PP}^{χ} corresponds to the symmetry breaking term,

$$\sigma_{PP}^{\chi} = \frac{1}{8}\text{Tr}\left(\left\{\lambda_P, \lambda_P^\dagger\right\}(\chi^\dagger U + U^\dagger \chi)\right) - M_P^2\,, \tag{3.2}$$

and $\mathscr{L}^{r(4)}$ is just $\mathscr{L}^{(4)}$ with renormalized low-energy coupling constants,

$$\begin{aligned}\mathscr{L}^{r(4)} = & L_1^r\left\{\mathrm{Tr}[D_\mu U(D^\mu U)^\dagger]\right\}^2 + L_2^r\mathrm{Tr}\left[D_\mu U(D_\nu U)^\dagger\right]\mathrm{Tr}\left[D^\mu U(D^\nu U)^\dagger\right] \\ & +L_3^r\mathrm{Tr}\left[D_\mu U(D^\mu U)^\dagger D_\nu U(D^\nu U)^\dagger\right] + L_4^r\mathrm{Tr}\left[D_\mu U(D^\mu U)^\dagger\right]\mathrm{Tr}\left(\chi U^\dagger + U\chi^\dagger\right) \\ & +L_5^r\mathrm{Tr}\left[D_\mu U(D^\mu U)^\dagger(\chi U^\dagger + U\chi^\dagger)\right] + L_6^r\left[\mathrm{Tr}\left(\chi U^\dagger + U\chi^\dagger\right)\right]^2 \\ & +L_7^r\left[\mathrm{Tr}\left(\chi U^\dagger - U\chi^\dagger\right)\right]^2 + L_8^r\mathrm{Tr}\left(U\chi^\dagger U\chi^\dagger + \chi U^\dagger\chi U^\dagger\right) \\ & -iL_9^r\mathrm{Tr}\left[F^R_{\mu\nu}D^\mu U(D^\nu U)^\dagger + F^L_{\mu\nu}(D^\mu U)^\dagger D^\nu U\right] + L_{10}^r\mathrm{Tr}\left(UF^L_{\mu\nu}U^\dagger F_R^{\mu\nu}\right) \\ & +H_1^r\mathrm{Tr}\left(F^R_{\mu\nu}F_R^{\mu\nu} + F^L_{\mu\nu}F_L^{\mu\nu}\right) + H_2^r\mathrm{Tr}\left(\chi\chi^\dagger\right). \end{aligned} \tag{3.3}$$

Here $\chi = 2(\Sigma/F_\pi^2)\mathscr{M} \equiv 2B_0\mathscr{M}$, λ_P's are the generators of $SU(N)$ in the physical basis, $\{L_i^r(\mu_{sub}), i = 1, \cdots, 10\}$ are renormalized low-energy coupling constants, and the last two contact terms (with couplings $H_1^r(\mu_{sub})$ and $H_2^r(\mu_{sub})$) are the counter terms required for renormalization of the one-loop diagrams.

For small quark masses ($L \ll m_\pi^{-1}$), the unitary matrix U does not depend on x_μ, thus the term involving σ^Δ_{PP} in (3.1) can be dropped. Only the term with σ^χ_{PP} in (3.1), and the sixth, seventh, and eighth terms in $\mathscr{L}^{r(4)}$ (3.3) are relevant to the partition function.

Now we follow the same procedure as that in deriving the tree-level formula. First, we replace $\mathscr{M}$ with $\mathscr{M}e^{i\theta/N_f}$. Then we take U and $\mathscr{M}$ to be diagonal, defining $\phi_j = \theta/N_f - \alpha_j$, and $\sum_j \phi_j = \theta$, similar to Eq. (2.4). Next we consider a sufficiently large volume $m_j\Omega\Sigma \gg 1$, such that we can use saddle-point approximation to evaluate the partition function. Also we use small θ (small ϕ_j's) approximation and keep terms up to the order of ϕ_j^2. Then to obtain the vacuum energy density amounts to the minimization problem,

$$\begin{aligned}\varepsilon_{vac} = & \varepsilon_0 - \min_\phi\left[\frac{\Sigma}{2}\sum_{j=1}^{N_f} m_j\phi_j^2 - \frac{\Sigma}{8F_\pi^2}\sum_P\sum_{j=1}^{N_f}\left\{\lambda_P, \lambda_P^\dagger\right\}_{jj} m_j\phi_j^2\frac{M_P^2}{16\pi^2}\ln\frac{M_P^2}{\mu_{sub}^2}\right. \\ & \left. +16B_0^2L_6^r\sum_{i=1}^{N_f}m_i\sum_{j=1}^{N_f}m_j\phi_j^2 + 16B_0^2L_7^r\left(\sum_{j=1}^{N_f}m_j\phi_j\right)^2 + 16B_0^2L_8^r\sum_{j=1}^{N_f}m_j^2\phi_j^2\right], \end{aligned} \tag{3.4}$$

with the constraint $\sum_j \phi_j = \theta$. We introduce the Lagrange multiplier λ to incorporate this constraint in finding the minimum. For simplicity, we define

$$\begin{aligned} A_j \equiv & \frac{\Sigma}{2}m_j - \frac{\Sigma}{8F_\pi^2}\sum_P\left\{\lambda_P, \lambda_P^\dagger\right\}_{jj} m_j\frac{M_P^2}{16\pi^2}\ln\frac{M_P^2}{\mu_{sub}^2} + 16B_0^2\left(L_6^r m_j\sum_{i=1}^{N_f}m_i + L_8^r m_j^2\right), \\ B_j \equiv & 4B_0(L_7^r)^{1/2}m_j. \end{aligned}$$

Then the minimization problem amounts to solving the equation

$$\frac{\partial}{\partial\phi_i}\left[\sum_{j=1}^{N_f}A_j\phi_j^2 + \left(\sum_{j=1}^{N_f}B_j\phi_j\right)^2 - \lambda\left(\sum_{j=1}^{N_f}\phi_j - \theta\right)\right] = 0. \tag{3.5}$$

Defining $(\mathbf{T})_{ij} \equiv 2A_i\delta_{ij} + 2B_iB_j$, (3.5) becomes $\sum_{j=1}^{N_f}(\mathbf{T})_{ij}\phi_j = \lambda$, which is a set of linear equations. Thus we can solve ϕ_i's and obtain λ from this set of equations and the constraint. Finally we obtain

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the vacuum energy density

$$\varepsilon_{\rm vac}(\theta) = \varepsilon_0 + \frac{\theta^2}{2}\left[\sum_{i,j=1}^{N_f} (\mathbf{T}^{-1})_{ij}\right]^{-1}. \tag{3.6}$$

To simplify the expression, we rewrite the matrix $\mathbf{T}$ as

$$(\mathbf{T})_{ij} \equiv 2A_i\delta_{ij} + 2B_iB_j = \Sigma(\mathscr{M} + \mathbf{T}')_{ij}. \tag{3.7}$$

Since $\mathscr{M}^{-1/2}\mathbf{T}'\mathscr{M}^{-1/2}$ is real and symmetric, and each eigenvalue is much less than one in the chiral limit, we can use the Taylor expansion

$$(\mathbb{1} + \mathscr{M}^{-1/2}\mathbf{T}'\mathscr{M}^{-1/2})^{-1} \simeq \mathbb{1} - \mathscr{M}^{-1/2}\mathbf{T}'\mathscr{M}^{-1/2} + \mathscr{O}(m^2),$$

and obtain the topological susceptibility

$$\begin{aligned}\chi_t &= \left.\frac{\partial^2\varepsilon_{\rm vac}}{\partial\theta^2}\right|_{\theta=0} = \left[\sum_{i,j=1}^{N_f} (\mathbf{T}^{-1})_{ij}\right]^{-1} \\ &\simeq \Sigma\bar{m}\left\{1 - \frac{1}{4F_\pi^2}\sum_P\sum_{j=1}^{N_f}\left\{\lambda_P,\lambda_P^\dagger\right\}_{jj}\left(\frac{\bar{m}}{m_j}\right)\frac{M_P^2}{16\pi^2}\ln\frac{M_P^2}{\mu_{sub}^2} + K_6\sum_{i=1}^{N_f} m_i + N_f(N_fK_7+K_8)\bar{m}\right\}, \end{aligned} \tag{3.8}$$

where

$$K_i \equiv \frac{32B_0^2L_i^r(\mu_{sub})}{\Sigma} = 32\left(\frac{\Sigma}{F_\pi^4}\right)L_i^r(\mu_{sub}), \qquad \bar{m} \equiv \left(\sum_{i=1}^{N_f} m_i^{-1}\right)^{-1},$$

and all terms proportional to K_i^2 or K_iK_j have been dropped. Equation (3.8) is the main result we have derived in [6].

For $N_f = 2$, there are three mesons, π^+, π^0, and π^-. If we take their masses to be the same, we obtain

$$\chi_t = \Sigma\left(\frac{1}{m_u}+\frac{1}{m_d}\right)^{-1}\left[1 - \frac{3}{2F_\pi^2}\frac{M_\pi^2}{16\pi^2}\ln\frac{M_\pi^2}{\mu_{sub}^2} + K_6(m_u+m_d) + 2(2K_7+K_8)\frac{m_um_d}{m_u+m_d}\right]. \tag{3.9}$$

Next we turn to the case $N_f = 3$. Taking the eight pseudoscalar mesons with non-degenerate masses, we obtain

$$\begin{aligned}\chi_t \;=\; \Sigma\bar{m}\Bigg\{1 - \frac{1}{2F_\pi^2}\Bigg[&\sum_{i\neq j}\left(\frac{\bar{m}}{m_i}+\frac{\bar{m}}{m_j}\right)\frac{B_0(m_i+m_j)}{16\pi^2}\ln\frac{B_0(m_i+m_j)}{\mu_{sub}^2} \\ &+\left(\frac{\bar{m}}{m_u}+\frac{\bar{m}}{m_d}\right)\frac{M_{\pi^0}^2}{16\pi^2}\ln\frac{M_{\pi^0}^2}{\mu_{sub}^2} + \frac{1}{3}\left(\frac{\bar{m}}{m_u}+\frac{\bar{m}}{m_d}+4\frac{\bar{m}}{m_s}\right)\frac{M_\eta^2}{16\pi^2}\ln\frac{M_\eta^2}{\mu_{sub}^2}\Bigg] \\ &+K_6(m_u+m_d+m_s) + 3(3K_7+K_8)\bar{m}\Bigg\}, \end{aligned} \tag{3.10}$$

where $\bar{m} = \left(m_u^{-1}+m_d^{-1}+m_s^{-1}\right)^{-1}$, and $B_0 = \Sigma/F_\pi^2$.

4. Concluding remark

We have derived the topological susceptibility to the one-loop order in ChPT, in the limit $m\Sigma\Omega \gg 1$, for $N_f = 2$ [Eq. (3.9)], $N_f = 3$ [Eq. (3.10)], and an arbitrary number of flavors N_f [Eq. (3.8)] respectively.

For $N_f = 3$, since the mass of the strange quark is much heavier than the masses of u and d quarks, it seems reasonable just to incorporate the one-loop corrections due to the u and d quarks. Then, for $N_f = 2+1$ (u and d quarks to the one-loop order, and s quark at the tree level), the topological susceptibility becomes

$$\chi_t = \Sigma\left\{ \left(\frac{1}{m_u} + \frac{1}{m_d}\right) \left[1 + \frac{3}{2F_\pi^2}\frac{M_\pi^2}{16\pi^2} \ln\frac{M_\pi^2}{\mu_{sub}^2} - K_6(m_u+m_d) - 2(2K_7+K_8)\frac{m_u m_d}{m_u+m_d}\right] + \frac{1}{m_s}\right\}^{-1}. \tag{4.1}$$

This supplements (3.10) for the case $N_f = 2+1$.

In view of the one-loop results of χ_t, [Eqs. (3.9), (3.10), and (4.1)], it would be interesting to see whether the χ_t measured in lattice QCD with exact chiral symmetry would agree with the prediction of ChPT. Most importantly, these one-loop formulas provide a viable way to determine the low-energy constants F_π, L_6, L_7 and L_8, in addition to the chiral condensate Σ which has already been determined [3, 5, 4] using the formula of χ_t at the tree level (2.5). At this point, we note that the finite volume effect on χ_t (to one-loop order in ChPT) has been recently studied in [8].

Finally, we turn to the second normalized cumulant c_4. At this moment, we only have a formula of c_4 (2.6) at the tree level. For $N_f = 2$, the ratio $c_4/\chi_t = -1/4$ in the isospin limit ($m_u = m_d$) seems to rule out the instanton gas/liquid model which predicts that $c_4/\chi_t = -1$. Obviously, it would be interesting to derive a formula of c_4 for the next (non-vanishing) order in ChPT.

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References

[1] H. Leutwyler and A. Smilga, Phys. Rev. D **46**, 5607 (1992).

[2] T. W. Chiu *et al.* [JLQCD and TWQCD Collaborations], PoS **LAT2007**, 068 (2007)

[3] S. Aoki *et al.* [JLQCD and TWQCD Collaborations], Phys. Lett. B **665**, 294 (2008)

[4] T. W. Chiu *et al.* [JLQCD and TWQCD Collaborations], PoS **LAT2008**, 072 (2008) [arXiv:0810.0085 [hep-lat]].

[5] T. W. Chiu, T. H. Hsieh and P. K. Tseng [TWQCD Collaboration], Phys. Lett. B **671**, 135 (2009) [arXiv:0810.3406 [hep-lat]].

[6] Y. Y. Mao and T. W. Chiu [TWQCD Collaboration], Phys. Rev. D **80**, 034502 (2009) [arXiv:0903.2146 [hep-lat]].

[7] J. Gasser and H. Leutwyler, Nucl. Phys. B **250**, 465 (1985).

[8] S. Aoki and H. Fukaya, arXiv:0906.4852 [hep-lat].