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# Minimally doubled fermions at one-loop level

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Single fermionic degrees of freedom together with standard chiral symmetry at finite lattice spacing, correct continuum limit and local interactions only are precluded by the Nielsen-Ninomiya no-go theorem. The class of minimally doubled fermion actions exhibits exactly two chiral modes. Recent interest in these actions has been sparked by the investigation of fermionic actions defined on "hyperdiamond" lattices.
Due to the necessity of breaking hypercubic symmetry explicitly, radiative corrections generate operator mixings with relevant and marginal operators that should vanish in continuum QCD. These cannot be avoided and must be taken into account in particular by a peculiar wave-function renormalisation and additive momentum renormalisation.
Renormalisation properties at one-loop level of the self-energy, local bilinears and conserved vector and axial-vector currents are presented for Boriçi-Creutz and Karsten-Wilczek actions. Distinct differences and similarities between both actions are elucidated.

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## 1. Introduction

For many years since the early days of lattice QCD, chiral symmetry was regarded as incompatible with lattice regularisation. The Nielsen-Ninomiya no-go theorem forbids the existence of a single chiral mode, which has a correct continuum limit as well as local interactions only.

Minimally doubled fermions represent a class of actions, which exactly satisfy the minimal requirements of the no-go theorem. A prominent representative of the minimally doubled fermion class is the Boriçi-Creutz Dirac operator [1, 2]:

$$D_{BC}(k) = i\sum_{\mu} \tilde{k}_\mu \gamma_\mu - \frac{i}{2} a \sum_{\mu} \hat{k}_\mu^2 \gamma'_\mu + m_0. \tag{1.1}$$

The trigonometric functions of the lattice momenta are defined as usual and the second set of gamma matrices is defined by a relation, which breaks hypercubic symmetry:

$$\tilde{k}_\mu \equiv \frac{1}{a}\sin(ak_\mu), \quad \hat{k}_\mu \equiv \tfrac{2}{a}\sin(a\tfrac{k_\mu}{2}), \tag{1.2}$$

$$\gamma'_\mu \equiv \Gamma\gamma_\mu\Gamma = \Gamma - \gamma_\mu, \; 2\Gamma \equiv \sum_\mu \gamma_\mu = \sum_\mu \gamma'_\mu. \tag{1.3}$$

This Dirac operator posesses two zeros. Their nature is made transparent, when the Boriçi-Creutz term is cast into another form:

$$-\frac{i}{2} a \sum_{\mu} \hat{k}_\mu^2 \gamma'_\mu = i\frac{1}{a}\sum_{\mu} \cos(ak_\mu)\gamma'_\mu - 2i\frac{1}{a}\Gamma. \tag{1.4}$$

The cosine functions are reduced to unity at $k = (0,0,0,0)$. Application of (1.3) clearly shows that both parts of the Boriçi-Creutz term on the right hand side of (1.4) compensate at this point. On the other hand, the first sum in (1.1) evaluated at $k = (\frac{\pi}{2a}, \frac{\pi}{2a}, \frac{\pi}{2a}, \frac{\pi}{2a})$ compensates the second half of the Boriçi-Creutz term, while the cosine functions in its first half vanish. Other zeros do not exist. Both are situated on the hypercubic main diagonal, which is the symmetry breaking axis. A combined symmetry transformation in all components

$$k_\mu \rightarrow \frac{\pi}{2a} - k_\mu, \quad \begin{pmatrix} \gamma_\mu \\ \gamma'_\mu \end{pmatrix} \rightarrow \begin{pmatrix} \gamma'_\mu \\ \gamma_\mu \end{pmatrix} \tag{1.5}$$

does not change the Boriçi-Creutz Dirac operator, but changes the sign of the chirality matrix: $\gamma'_5 = \Gamma\gamma_5\Gamma = -\gamma_5$. It corresponds to an exchange of the poles which have opposite chirality.

## 2. Perturbation theory for Boriçi-Creutz fermions

### 2.1 Propagators and vertices

Our recent study [3] of Boriçi-Creutz fermions proved the occurence of operator mixings due to one-loop effects. Effects of this sort had been conjectured [4] previously. Here, we revisit the properties of Boriçi-Creutz fermions and compare them later on to Karsten-Wilczek fermions.

The propagator is obtained from the inversion of the Dirac operator [3, 5]:

$$S_{BC} = \frac{-i\sum\limits_\mu \tilde{k}_\mu\gamma_\mu + \frac{i}{2}a\sum\limits_\mu \hat{k}_\mu^2\gamma'_\mu + m_0}{\sum\limits_\mu \hat{k}_\mu^2 + a\sum\limits_\mu \tilde{k}_\mu\left(\hat{k}_\mu^2 - \frac{1}{2}\sum\limits_\nu \hat{k}_\nu^2\right) + m_0^2}. \tag{2.1}$$

The violation of hypercubic symmetry is obvious, as the denominator cannot be cast into a form with definite behaviour under reversal of any direction.

The weak coupling expansion of the gauge field $U_\mu(x) = e^{iag_0 A_\mu(x+\frac{a}{2}e_\mu)}$ is performed in the usual manner [3, 5]. Quark vertices with one or two gluons are denoted by $V^1$ and $V^2$:

$$V^1_\mu(p_1,p_2) \;=\; -ig_0\Big(\gamma_\mu \cos\frac{(p_1+p_2)_\mu}{2} - \gamma'_\mu \sin\frac{(p_1+p_2)_\mu}{2}\Big), \tag{2.2}$$

$$V^2_\mu(p_1,p_2) \;=\; i\frac{a}{2}g_0^2\Big(\gamma_\mu \sin\frac{(p_1+p_2)_\mu}{2} + \gamma'_\mu \cos\frac{(p_1+p_2)_\mu}{2}\Big). \tag{2.3}$$

These vertices can be derived from Wilson fermion vertices by the replacement $rf_\mu \rightarrow -i\gamma'_\mu f_\mu$. Due to the subtle difference that the Dirac structure of $-i\gamma'_\mu f_\mu$ is different for each $\mu$, even the evaluation of simple diagrams is very complex and generates vast numbers of terms.

### 2.2 Self-energy

Two diagrams[1] add up to the self-energy at one-loop level. Due to the Dirac structure of the n-point functions, computation of the lattice integrals requires evaluation of every possible combination of indices of Dirac matrices and momenta.

The tadpole diagram's contribution,

$$g_0^2 C_F \frac{Z_0}{2}\Big(1-\frac{1}{4}(1-\alpha)\Big)\Big(i\not{p} + 2i\frac{1}{a}\Gamma\Big), \tag{2.4}$$

with $Z_0 = 24.466100/(16\pi^2)$, contains a power-divergent part. The possibility that this power-divergence might be canceled by the sunset diagram is not realised [4] . Nevertheless, gauge invariance requires at least a cancellation of the part proportional to $(1-\alpha)$.

Evaluation of the sunset diagram yields

$$i\not{p}\cdot\tfrac{g_0^2 C_F}{16\pi^2}\;\Big(\log a^2p^2 - 5.42642 + (1-\alpha)\big(-\log a^2p^2 + 7.850272\big)\Big) \tag{2.5}$$

$$+m_0\cdot\tfrac{g_0^2 C_F}{16\pi^2}\;\Big(4\log a^2p^2 - 29.48729 + (1-\alpha)\big(-\log a^2p^2 + 5.792010\big)\Big) \tag{2.6}$$

$$+i\Gamma\Pi\cdot\tfrac{g_0^2 C_F}{16\pi^2}\cdot\;1.52766 \tag{2.7}$$

$$+i\tfrac{1}{a}\Gamma\cdot\tfrac{g_0^2 C_F}{16\pi^2}\;\Big(5.07558 + 6.11653(1-\alpha)\Big). \tag{2.8}$$

Herein, we used the definition $\Pi \equiv \sum_\mu p_\mu$. Thus, the structure in (2.7) is proportional to the momentum projection on the hypercubic symmetry-breaking axis. It can be cast into a more transparent form by using anticommutation relations for Dirac matrices including $\Gamma$:

$$\Gamma\Pi = \frac{1}{2}\{\Gamma,\{\Gamma,\not{p}\}\} = \not{p} + \not{p}', \tag{2.9}$$

with $\not{p}' \equiv \sum_\mu p_\mu \gamma'_\mu$.

Furthermore, (2.8) cancels the power-divergent part of (2.4) that is forbidden by gauge invariance. The further power-divergences, however, not only fail to cancel, but amplify each other.

[1] For a list of the diagrams see Fig. 1 in [3].



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The full one-loop expression for the self energy is

$$\Sigma(p,m_0) = i\not{p}\Sigma_1(p) + m_0\Sigma_2(p) + c_1(g_0^2)i(\not{p}+\not{p}') + c_2(g_0^2)i\frac{1}{a}\Gamma, \tag{2.10}$$

with

$$\Sigma_1(p) = \ 1 + \frac{g_0^2 C_F}{16\pi^2}\Big(\log a^2p^2 + 6.80663 + (1-\alpha)\big(-\log a^2p^2 + 4.792010\big)\Big), \tag{2.11}$$

$$\Sigma_2(p) = \ 1 + \frac{g_0^2 C_F}{16\pi^2}\Big(4\log a^2p^2 - 29.48729 + (1-\alpha)\big(-\log a^2p^2 + 5.792010\big)\Big), \tag{2.12}$$

$$c_1(g_0^2) = \ 1.52766 \cdot \frac{g_0^2 C_F}{16\pi^2}, \tag{2.13}$$

$$c_2(g_0^2) = \ 29.54170 \cdot \frac{C g_0^2 C_F}{16\pi^2}. \tag{2.14}$$

Obviously, (2.9) must enter into the wave-function renormalisation in a non-trivial way. Therefore, $Z_\psi = Z_\psi(\Sigma_1(p), c_1(g_0^2))$.

### 2.3 Local bilinears

The renormalisation of local bilinears is a straigthforward procedure. As chiral symmetry demands, scalar and pseudoscalar densities as well as local vector and axial-vector currents have the same renormalisation factors. Without taking the wave-function renormalisation into account, the proper renormalisation factors read

$$\Lambda_S(p) = \frac{C_f g_0^2}{16\pi^2}\Big(-4\log a^2p^2 + 29.48729 + (1-\alpha)\big(\log a^2p^2 - 5.792010\big)\Big), \tag{2.15}$$

$$\Lambda_V(p) = \frac{C_f g_0^2}{16\pi^2}\Big(-\log a^2p^2 + 9.54612 + (1-\alpha)\big(\log a^2p^2 - 4.792010\big)\Big), \tag{2.16}$$

$$\Lambda_T(p) = \frac{C_f g_0^2}{16\pi^2}\Big(2.16548 + (1-\alpha)\big(\log a^2p^2 - 3.792010\big)\Big). \tag{2.17}$$

The local vector and axial-vector currents suffer from an additional operator mixing besides the wave-function renormalisation:

$$\overline{\psi}\gamma_\mu\psi \rightarrow \overline{\psi}^R\gamma_\mu\left(1 + Z_\psi + \Lambda_V(p)\right)\psi^R + c^{vtx}(g_0^2)\overline{\psi}^R\Gamma\psi^R, \tag{2.18}$$

with $c^{vtx}(g_0^2) = -0.10037 \cdot \frac{g_0^2 C_F}{16\pi^2}$. Since each coordinate axis has a non-vanishing projection on the hypercubic main diagonal, the symmetry breaking operator mixes with each of the four components. The nature of this mixing can be visualised by applying (1.3): $\Gamma = \gamma_\mu + \gamma'_\mu$.

## 3. One-loop properties

### 3.1 Momentum renormalisation

Due to the fact that it is proportional to a Dirac gamma matrix, the power-divergence in the self energy is unlike its counterpart in the Wilson case. The mass is protected from additive renormalisation as chiral symmetry is unbroken. Instead, the four-momentum is subject to renormalisation:



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$$\check{p}_\mu = p_\mu - \frac{c_2(g_0^2)}{2a} \quad \rightarrow \quad \not{p} = \check{\not{p}} + \frac{c_2(g_0^2)}{a}\Gamma. \tag{3.1}$$

The conjecture that quark velocities had to renormalised [2] is thus verified. Since neither pole lies at $(0,0,0,0)$ any more, the definition of a quark rest frame becomes non-trivial. Besides that issue, the relevant quantity for periodic boundaries is $c_2(g_0^2)$ modulo $2\pi$.

### 3.2 Conserved currents

After the Boriçi-Creutz action is cast into coordinate space,

$$S_{BC} = a^4 \sum_x \Big( \sum_\mu \tfrac{1}{2a} \Big( \overline{\psi}(x)(\gamma_\mu + i\gamma'_\mu) U_\mu(x) \psi(x + a\, e_\mu) - \overline{\psi}(x + a\, e_\mu)(\gamma_\mu - i\gamma'_\mu) U^\dagger_\mu(x) \psi(x) \Big) + \overline{\psi}(x) \big( m_0 - 2i \tfrac{1}{a} \Gamma \big) \psi(x) \Big), \tag{3.2}$$

conserved vector and axial-vector currents can be derived by application of the Ward identities [6]. The transformations

$$\begin{pmatrix} \psi(x) \\ \overline{\psi}(x) \end{pmatrix} \rightarrow \begin{pmatrix} (1 + i\alpha_V)\psi(x) \\ \overline{\psi}(x)(1 - i\alpha_V) \end{pmatrix}, \quad \begin{pmatrix} \psi(x) \\ \overline{\psi}(x) \end{pmatrix} \rightarrow \begin{pmatrix} (1 + i\alpha_A\gamma_5)\psi(x) \\ \overline{\psi}(x)(1 + i\alpha_A\gamma_5) \end{pmatrix} \tag{3.3}$$

yield conserved point-split currents [2]:

$$V^c_\mu(x) = \tfrac{1}{2} \Big( \overline{\psi}(x)(\gamma_\mu + i\gamma'_\mu) U_\mu(x) \psi(x + a\, e_\mu) + \overline{\psi}(x + a\, e_\mu)(\gamma_\mu - i\gamma'_\mu) U^\dagger_\mu(x) \psi(x) \Big), \tag{3.4}$$

$$A^c_\mu(x) = \tfrac{1}{2} \Big( \overline{\psi}(x)(\gamma_\mu + i\gamma'_\mu)\gamma_5 U_\mu(x) \psi(x + a\, e_\mu) + \overline{\psi}(x + a\, e_\mu)(\gamma_\mu - i\gamma'_\mu)\gamma_5 U^\dagger_\mu(x) \psi(x) \Big). \tag{3.5}$$

Four diagrams[3] contribute to their renormalisation: vertex diagram, operator tadpole and two sails. In the case of the vector current, the proper current renormalisation amounts to

$$\begin{aligned} \Lambda_{V^c}(p) &= \frac{g_0^2 C_F}{16\pi^2} \Big( -\log a^2 p^2 - 6.80664 + (1-\alpha)\big(\log a^2 p^2 - 4.792010\big) \Big), \\ c_{V^c}(g_0^2) &= -1.52766 \cdot \frac{g_0^2 C_F}{16\pi^2}. \end{aligned} \tag{3.6}$$

The full expression for the renormalisation of the conserved vector current is

$$Z_{V^c} \overline{\psi} \gamma_\mu \psi = \Big( (1 + Z_\psi) \overline{\psi}^R \gamma_\mu \psi^R + \overline{\psi}^R (\Lambda_{V^c}(p)\gamma_\mu + c_{V^c}(g_0^2)\Gamma) \psi^R \Big). \tag{3.7}$$

It is not straightforward to proof that $Z_{V^c}$ is unity. $\Gamma = \gamma_\mu + \gamma'_\mu$, which is obtained from (1.3), reveals the same structure as (2.9). Furthermore, the renormalisation factors inside the wave-function renormalisation ($\Sigma_1(p)$, $c_1(g_0^2)$) have signs exactly opposite to those in the proper current renormalisation ($\Lambda_{V^c}(p)$, $c_{V^c}(g_0^2)$). It seems reasonable to assume that these structures cancel, even though an algebraic proof does not exist yet.

[2]Obviously, the axial-vector current is conserved only in the chiral limit.

[3]The diagrams are listed in Fig. 1 in [3].

## 4. Karsten-Wilczek fermions

A full study of the one-loop properties of Karsten-Wilczek fermions [7] is in preparation [8]. The Karsten-Wilczek Dirac operator,

$$S_{KW}(k) = i\sum_{\mu}\tilde{k}_{\mu}\gamma_{\mu} + \frac{i}{2}\lambda a\sum_{\mu}\hat{k}_{\mu}^{2}\gamma_4(1-\delta_{\mu 4}) + m_0, \tag{4.1}$$

has two zeros located at $k=(0,0,0,0)$ and $k=(0,0,0,\frac{\pi}{a})$. It contains the Wilczek parameter $\lambda$, which is constrained to $|\lambda| \geq \frac{1}{2}$. The cancellation of the additional zeros could not be achieved otherwise. A second set of gamma matrices can be defined:

$$\gamma_{\mu}' \equiv \gamma_4\gamma_{\mu}\gamma_4 = (2\delta_{\mu 4}-1)\gamma_{\mu}. \tag{4.2}$$

Due to the Kronecker symbol, the Karsten-Wilczek term includes only the three spatial directions. Dependence on spatial momenta and on the energy shows severe symmetry breaking effects. The vertices can be obtained from those of the Wilson case by $rf_{\mu} \to i\lambda\gamma_4(1-\delta_{\mu 4})f_{\mu}$. This interaction does not change the temporal vertices at all!

The self-energy is computed (for $\lambda = 1$) in the same way as in the Boriçi-Creutz case:

$$\Sigma(p,m_0) = i\not{p}\Sigma_1(p) + m_0\Sigma_2(p) + c_1(g_0^2)i\cdot\frac{1}{2}(\not{p}+\not{p}') + c_2(g_0^2)i\frac{1}{a}\gamma_4, \tag{4.3}$$

with $\not{p}' = \sum_{\mu} p_{\mu}\gamma_{\mu}'$, where $\gamma_{\mu}'$ is defined in (4.2), $p_4\gamma_4 \equiv \frac{1}{2}(\not{p}+\not{p}')$ and

$$\Sigma_1(p) = 1 + \frac{g_0^2 C_F}{16\pi^2}\Big(\log a^2p^2 + 9.2409 + (1-\alpha)\big(-\log a^2p^2 + 4.792010\big)\Big), \tag{4.4}$$

$$\Sigma_2(p) = 1 + \frac{g_0^2 C_F}{16\pi^2}\Big(4\log a^2p^2 - 24.3688 + (1-\alpha)\big(-\log a^2p^2 + 5.792010\big)\Big), \tag{4.5}$$

$$c_1(g_0^2) = -0.1255\cdot\frac{g_0^2 C_F}{16\pi^2}, \tag{4.6}$$

$$c_2(g_0^2) = -29.5323\cdot\frac{g_0^2 C_F}{16\pi^2}. \tag{4.7}$$

The power-divergence in (4.3) concerns only the fourth direction. Otherwise, momentum renormalisation is handled analogously: $\check{p}_4 = p_4 - \frac{c_2(g_0^2)}{a}$. We note in passing, that the definition of a rest frame is not problematic here. The magnitude of $c_2(g_0^2)$ in (4.7) is nearly the same as in the Boriçi-Creutz case (2.14), but the sign has changed.

The local bilinears (for $\lambda = 1$) show similarities to the previous case. Exact chiral symmetry is maintained. The corrections to the proper vertices without wave-function renormalisation read

$$\Lambda_S(p) = \frac{g_0^2 C_F}{16\pi^2}\Big(-4\log a^2p^2 + 24.36875 + (1-\alpha)\big(\log a^2p^2 - 5.792010\big)\Big), \tag{4.8}$$

$$\Lambda_V(p) = \frac{g_0^2 C_F}{16\pi^2}\Big(-\log a^2p^2 + 10.44610 + (1-\alpha)\big(\log a^2p^2 - 4.792010\big)\Big), \tag{4.9}$$

$$\Lambda_T(p) = \frac{g_0^2 C_F}{16\pi^2}\Big(4.17551 + (1-\alpha)\big(\log a^2p^2 - 3.792010\big)\Big). \tag{4.10}$$

Vector and axial currents suffer further operator mixing beyond the wave-function renormalisation

$$\overline{\psi}\,\gamma_\mu\,\psi \rightarrow \overline{\psi}^R\gamma_\mu\left(1+Z_\psi+\Lambda_V(p)\right)\psi^R+c^{vtx}(g_0^2)\overline{\psi}^R\gamma_4\psi^R, \tag{4.11}$$

with $c^{vtx}(g_0^2)=-2.88914$ and $2\gamma_4=2\delta_{\mu 4}\gamma_\mu=\gamma_\mu+\gamma'_\mu$. This mixing can be taken into account more simply than in the Boriçi-Creutz case by choosing the spatial components' renormalisation factors different from the temporal one.

Conserved currents are defined for Karsten-Wilczek fermions as well. The vector current reads

$$\begin{aligned} V^c_\mu(x) = \ & \tfrac{1}{2}\Big(\overline{\psi}(x)(\gamma_\mu - i\gamma_4(1-\delta_{\mu 4}))U_\mu(x)\psi(x+a\,e_\mu) \\ & +\overline{\psi}(x+a\,e_\mu)(\gamma_\mu + i\gamma_4(1-\delta_{\mu 4}))U^\dagger_\mu(x)\psi(x)\Big). \end{aligned} \tag{4.12}$$

The axial current $A^c_\mu$ is obtained by inserting $\gamma_5$ behind the Dirac matrices. We have verified for $\lambda=1$ that their one-loop renormalisation factors are unity under analogous assumptions.

## 5. Conclusions

We have demonstrated that the one-loop diagrams of Boriçi-Creutz fermions and Karsten-Wilczek fermions contain both similar structures.

Mixings with marginal operators can be cast into forms which are clearly equivalent. Conserved point-split currents can be defined and their renormalisation factors are unity.

Whereas in the former case all components of the four-momentum are subject to additive renormalisation, the latter case requires only renormalisation of the fourth component. The definition of a rest frame is problematic only in the former case. Thus we expect that the implementation of Karsten-Wilczek fermions poses less obstacles than Boriçi-Creutz fermions.

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