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Derivation of Lüscher's finite size formula for $N\pi$ and NN system

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I present derivation of Lüscher's finite size formula for the elastic $N\pi$ and the NN scattering system for several angular momenta from the relativistic quantum field theory.

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1. Introduction

Calculation of the scattering phase shift represents an important step for expanding our understanding of the strong interaction based on lattice QCD to dynamical aspects of hadrons. Since Lüscher derived a finite size formula for the two-meson system on 1986 [1], which give us a relation between the phase shift and the energy eigenvalue on the finite volume, many lattice calculations of the scatting length and the phase shift of the two-meson systems have been carried with his formula. Recently his formula was extended to that for the $N\pi$ system by Bernard *et al.* by using the non-relativistic effective theory [2]. QCDSF collaboration calculated the phase shift of this system with this extended formula and study $\Delta(1232)$ resonance [3].

The extension of formula is necessary to extend our study to many systems. In the present work I consider a derivation of the formula for the elastic NN scattering system, where the formula only for spin singlet state in the non-relativistic limit, which is same as that for the two-meson system given by Lüscher, has been known. My derivation is based only on the relativistic quantum field theory and any effective theories for the two-nucleon interaction are not assumed. Further the extension to the $N\pi$ system can be easily done as discussed latter.

2. Wave function in infinite volume

First we consider the wave function in the infinite volume defined by

$$\phi^{\infty}_{\alpha\beta}(\mathbf{x};\mathbf{k}) = \langle 0|\; n_\alpha(\mathbf{x}/2)\; p_\beta(-\mathbf{x}/2)\; |\mathbf{k},\lambda_n,\lambda_p\rangle \;, \tag{2.1}$$

where $n_\alpha(\mathbf{x})$ and $p_\beta(\mathbf{x})$ are interpolating operators of the nucleons and $|\mathbf{k},\lambda_n,\lambda_p\rangle$ is the asymptotic NN state with momentum $\mathbf{k}$, $-\mathbf{k}$ and the helicities λ_n, λ_p . Using LSZ reduction formula, the wave function can be written by

$$\phi^{\infty}_{\alpha\beta}(\mathbf{x};\mathbf{k}) = U_{\alpha\beta}(\mathbf{k},\lambda_n,\lambda_p)\mathrm{e}^{i\mathbf{x}\cdot\mathbf{k}} + \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \sum_{\xi_n\xi_p} U_{\alpha\beta}(\mathbf{p},\xi_n,\xi_p)\mathrm{e}^{i\mathbf{x}\cdot\mathbf{p}}\, \frac{T(\mathbf{p},\xi_n,\xi_p;\mathbf{k},\lambda_n,\lambda_p)}{p^2-k^2-i\varepsilon} \;, \tag{2.2}$$

where $U_{\alpha\beta}(\mathbf{k},\lambda_n,\lambda_p)$ is a spinor for two free nucleons given by $U_{\alpha\beta}(\mathbf{k},\lambda_n,\lambda_p) = u_\alpha(\mathbf{k},\lambda_n)u_\beta(-\mathbf{k},\lambda_p)$ with the one nucleon spinor $u(\mathbf{k},\lambda)$. $T(\mathbf{p},\xi_n,\xi_p;\mathbf{k},\lambda_n,\lambda_p)$ is the off-shell scattering amplitude for a process $n(\mathbf{k},\lambda_n)p(-\mathbf{k},\lambda_p) \rightarrow\; n(\mathbf{p},\xi_n)p(-\mathbf{p},\xi_p)$.

We can estimate (2.2) in the region $|\mathbf{x}| > R$ for the two-nucleon interaction range R, by using a integral formula

$$\int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{j_l(px)}{p^2-k^2-i\varepsilon} f(p) = \frac{k}{4\pi}\left(i\cdot j_l(kx) + n_l(kx)\right) f(k) \qquad \text{for } F(x)\text{=}0 \tag{2.3}$$

where $F(x)$ is the inverse Fourier transformation of $f(p)$. $j_l(x)$ is the spherical Bessel and $n_l(x)$ is the Neumann function, whose conventions agree with those in [5] as adopted in [1]. This formula is a extension of (A.11) in Appendix A in Ref. [4] to that for arbitrary value of l and can be derived by similar calculations of that paper.

From (2.3) we know that all values in the numerator of the integrand in (2.2) can be replaced by the value at on-shell $p = k$. The off-shell scattering amplitude $T(\mathbf{p},\xi_n,\xi_p;\mathbf{k},\lambda_n,\lambda_p)$ is replaced by the on-shell amplitude, which can be expanded as [6]

$$T(k\mathbf{e}_p,\xi_n,\xi_p;\mathbf{k},\lambda_n,\lambda_p) = 16\pi^2\frac{\sqrt{s}}{k}\sum_{JM} T^{(J)}_{\xi_n\xi_p,\lambda_n\lambda_p}(k)\cdot {N_J}^2\, D^{(J)\,*}_{M\xi}(\Omega_p)\, D^{(J)}_{M\lambda}(\Omega_k)\ , \tag{2.4}$$

where $N_J=\sqrt{(2J+1)/(4\pi)}$, $\lambda=\lambda_n-\lambda_p$, $\xi=\xi_n-\xi_p$ and $\sqrt{s}=2\sqrt{m^2+k^2}$. In (2.4) the helicity amplitude in the subspace of the total energy $\sqrt{s}$ and the total angular momentum J is defined by $T^{(J)}_{\xi_n\xi_p,\lambda_n\lambda_p}(k)=\langle\xi_n\xi_p|\hat{T}^{(J)}(k)|\lambda_n\lambda_p\rangle$. The function $D^{(J)}_{MM'}(\Omega_p)=\langle JM|\exp(-i\alpha J_z)\exp(-i\beta J_y)$ $\exp(-i\gamma J_z)|JM'\rangle$ is the Wigner's D-function with the Euler angle $(\alpha,\beta,\gamma)=(\phi_p,\theta_p,-\phi_p)$ for momentum $\mathbf{p}=(p\sin\theta_p\cos\phi_p, p\sin\theta_p\cos\phi_p, p\cos\theta_p)$.

Using (2.3) and (2.4), we know that the wave function (2.2) in the region $|\mathbf{x}|>R$ is written by

$$\phi^\infty(\mathbf{x};\mathbf{k}) = \sum_{JM} N_J D_{M\lambda}(\Omega_k)\cdot\phi^\infty_{JM\lambda_n\lambda_p}(\mathbf{x};k) \qquad (\lambda=\lambda_n-\lambda_p)\ , \tag{2.5}$$

$$\phi^\infty_{JM\lambda_n\lambda_p}(\mathbf{x};k) = \sum_{\xi_n\xi_p}\left[J_{JM\xi_n\xi_p}(\mathbf{x};k)\cdot\alpha^{(J)}_{\xi_n\xi_p,\lambda_n\lambda_p}(k)+N_{JM\xi_n\xi_p}(\mathbf{x};k)\cdot\beta^{(J)}_{\xi_n\xi_p,\lambda_n\lambda_p}(k)\right]\ . \tag{2.6}$$

where $\alpha^{(J)}_{\xi_n\xi_p,\lambda_n\lambda_p}(k)=\langle\xi_n\xi_p|\hat{I}+i\hat{T}^{(J)}/2|\lambda_n\lambda_p\rangle$ and $\beta^{(J)}_{\xi_n\xi_p,\lambda_n\lambda_p}(k)=\langle\xi_n\xi_p|\hat{T}^{(J)}/2|\lambda_n\lambda_p\rangle$, which correspond to $\alpha^{(l)}=\cos\delta_l\cdot\exp(i\delta_l)$ and $\beta^{(l)}=\sin\delta_l\cdot\exp(i\delta_l)$ for the two-meson system with the scattering phase shift δ_l. In (2.6) the function $J_{JM\lambda_n\lambda_p}(\mathbf{x};k)$ is the wave function of two free nucleons with the total energy $\sqrt{s}$, the total angular momentum JM and the helicity $\lambda_n\lambda_p$. Its explicit form is given by

$$J_{JM\lambda_n\lambda_p}(\mathbf{x};k) = \hat{L}(\nabla)\ J^{\rm NR}_{JM\lambda_n\lambda_p}(\mathbf{x};k)\ \hat{R}(\overleftarrow{\nabla}) \equiv J^{\rm NR}_{JM\lambda_n\lambda_p}(\mathbf{x};k)\big|_{\rm R-EX}\ , \tag{2.7}$$

where differential operators $\hat{L}(\nabla)$ and $\hat{R}(\nabla)$ are defined by

$$\hat{L}(\nabla)=\begin{pmatrix} I \\ \dfrac{(\sigma\cdot\nabla/i)}{E+m}\end{pmatrix}\ ,\quad \hat{R}(\nabla)=\left(I\ ,\ \frac{-(\sigma^T\cdot\nabla/i)}{E+m}\right)\ , \tag{2.8}$$

with $E=\sqrt{k^2+m^2}$. In (2.7) the function $J^{\rm NR}_{JM\lambda_n\lambda_p}(\mathbf{x};k)$ is 2×2 non-relativistic spinor defined by

$$J^{\rm NR}_{JM\lambda_n\lambda_p}(\mathbf{x};k) = \sum_{ls} J^{\rm NR}_{JMls}(\mathbf{x};k)\cdot\langle JMls|JM\lambda_n\lambda_p\rangle\ , \tag{2.9}$$

$$J^{\rm NR}_{JMls}(\mathbf{x};k) = j_l(kx)Y^{ls}_{JM}(\Omega_x)/b_l(k)\ ,\quad Y^{ls}_{JM}(\Omega_x)=\sum_{m\mu}Y_{lm}(\Omega_x)\phi(s,\mu)\cdot C(lm;s\mu;JM)\ , \tag{2.10}$$

where the coefficient $\langle JMls|JM\lambda_n\lambda_p\rangle$ is the transformation coefficient from the helicity base to the orbit-spin base ($(JMls)$-base) with the angular momentum l and the spin s [6], and $C(lm;s\mu;JM)$ is the Clebsch-Gordan coefficient for angular momentum state $|lm\rangle\otimes|s\mu\rangle$ and $|JM\rangle$. Ω_x is the spherical coordinate for $\mathbf{x}$. $b_l(k)$ is the normalization constant of the state, which takes $1/b_l(k)=(4\pi)i^l\cdot(k^2+m^2)$ for the usual relativistic normalization ($u^\dagger u=2E$). $\phi(s,\mu)$ is 2×2 spin wave function for two spin $1/2$ particles with total spin $s\mu$. The function $N_{JM\lambda_n\lambda_p}(\mathbf{x};k)$ in (2.6) is given by replacing $j_l(kx)$ by $n_l(kx)$ in (2.7). We can regard (2.7) as a relativistic extension of the non-relativistic spinor $J^{\rm NR}_{JM\lambda_n\lambda_p}(\mathbf{x};k)$ to the relativistic one $J_{JM\lambda_n\lambda_p}(\mathbf{x};k)$, so that the spinor satisfies the Dirac equation. We use a notation $J^{\rm NR}|_{\rm R-EX}$ for this relativistic extension like as (2.7) in the follow.

Next we rewrite (2.5) and (2.6) by the $(JMls)$-base as

$$\phi^{\infty}(\mathbf{x};\mathbf{k}) = \sum_{JMls} C_{JMls}(\mathbf{k}) \cdot \phi^{\infty}_{JMls}(\mathbf{x};k) \ , \tag{2.11}$$

$$\phi^{\infty}_{JMls}(\mathbf{x};k) = \sum_{l's'} \Big[J_{JMl's'}(\mathbf{x};k) \cdot \alpha^{(J)}_{l's',ls}(k) + N_{JMl's'}(\mathbf{x};k) \cdot \beta^{(J)}_{l's',ls}(k) \Big] \ , \tag{2.12}$$

with some constant $C_{JMls}(\mathbf{k})$, where functions of the $(JMls)$-base are defined by

$$J_{JMls}(\mathbf{x};k) = \sum_{\lambda_n \lambda_p} J_{JM\lambda_n\lambda_p}(\mathbf{x};k) \cdot \langle JMls|JM\lambda_n\lambda_p\rangle \ , \tag{2.13}$$

$$\alpha^{(J)}_{l's',ls}(k) = \sum_{\xi_n\xi_p\lambda_n\lambda_p} \alpha^{(J)}_{\xi_n\xi_p,\lambda_n\lambda_p}(k) \cdot \langle JMl's'|JM\xi_n\xi_p\rangle \langle JMls|JM\lambda_n\lambda_p\rangle \ , \tag{2.14}$$

and $N_{JM\lambda_n\lambda_p}(\mathbf{x};k)$ and $\beta^{(J)}_{l's',ls}(k)$ are similarly defined. Here we should note that $J_{JMls}(\mathbf{x};k)$ and $N_{JMls}(\mathbf{x};k)$ are not eigenstates of the orbital angular momentum and the spin with l and s. These functions satisfy the Dirac equation, thus the upper and the lower components have different orbital angular momenta.

3. Wave function on the finite volume

Next we consider the wave function on the finite periodic box of volume L^3 defined by

$$\phi^{L}_{\alpha\beta}(\mathbf{x};k) = \langle 0|\ n_\alpha(\mathbf{x}/2)\ p_\beta(-\mathbf{x}/2)\ |k\rangle \ , \tag{3.1}$$

where $|k\rangle$ is the energy eigenstate with $\sqrt{s} = 2\sqrt{m^2+k^2}$ on the finite volume. Here we assume the condition $R < L/2$ for the two-nucleon interaction range R and the lattice size L, so that the boundary condition does not distort the shape of the two-nucleon interaction. In the region $R < |\mathbf{x}| < L$, the wave function satisfies following two equations and the boundary condition.

$$\big[\ i(\gamma\cdot\nabla) + \gamma^0 E - m\ \big]\, \phi^L(\mathbf{x};k) = 0 \ , \quad \phi^L(\mathbf{x};k) \Big[\ -i(\gamma\cdot\overleftarrow{\nabla}) + \gamma^0 E - m\ \Big]^{\mathrm{T}} = 0 \ , \tag{3.2}$$

$$\phi^L(\mathbf{x}+\mathbf{n}L;k) = \phi^L(\mathbf{x};k) \qquad (\ \mathbf{n}\in\mathbb{Z}^3\ ) \ , \tag{3.3}$$

where $E = \sqrt{m^2+k^2}$. The general solution of these equations can be written by the linear combination of the Green function defined by

$$G_{JMls}(\mathbf{x};k) = G^{\mathrm{NR}}_{JMls}(\mathbf{x};k)\big|_{\mathrm{R-EX}} \ , \tag{3.4}$$

$$G^{\mathrm{NR}}_{JMls}(\mathbf{x};k) = \mathscr{Y}^{ls}_{JM}(\nabla) \frac{1}{L^3} \sum_{\mathbf{p}\in\Gamma} \frac{1}{p^2-k^2} \mathrm{e}^{i\mathbf{p}\cdot\mathbf{x}} \ , \quad \mathscr{Y}^{ls}_{JM}(\mathbf{p}) = p^l \cdot Y^{ls}_{JM}(\Omega_p) \ , \tag{3.5}$$

where $\Gamma = \{\mathbf{p}|\mathbf{p} = (2\pi)/L\cdot\mathbf{n}\ ,\ \mathbf{n}\in\mathbb{Z}^3\}$ and $\mathbf{\Omega}_p$ is the spherical coordinate for $\mathbf{p}$. This Green function is related to that introduced in Ref. [1] $G_{lm}(\mathbf{x};k)$ by
$G^{\mathrm{NR}}_{JMls}(\mathbf{x};k) = \sum_{m\mu} G_{lm}(\mathbf{x};k)\cdot\phi(s,\mu)C(lm;s\mu;JM)$.

Using partial wave expansion of $G_{lm}(\mathbf{x};k)$ given in Ref. [1], we obtain

$$G_{JMls}(\mathbf{x};k) = a_l(k)b_l(k)\cdot N_{JMls}(\mathbf{x};k) + a_l(k) \sum_{J'M'l'} b_{l'}(k)\cdot J_{J'M'l's}(\mathbf{x};k)\cdot M^{(s)}_{J'M'l',JMl}(k) \ , \tag{3.6}$$

where $a_l(k) = (-1)^l k^{l+1}/(4\pi)$, $b_l(k)$ is the normalization constant appeared in (2.10) and

$$M^{(s)}_{J'M'l',JMl}(k) = \sum_{mm'\mu} M_{l'm',lm}(k) \cdot C(l'm';s\mu;J'M')C(lm;s\mu;JM) \ . \tag{3.7}$$

The function $M_{l'm',lm}(k)$ in (3.7) is defined by (3.34) in Ref. [1], which is given by

$$M_{l'm',lm}(k) = \sum_{l''m''} I_{l'm',l''m'',lm} W_{l''m''}(q) \ , \quad q = kL/(2\pi) \tag{3.8}$$

$$I_{l'm',l''m'',lm} = (-1)^l \cdot i^{l+l'} \cdot (2l''+1)\sqrt{\frac{2l+1}{2l'+1}} \cdot C(l0;l''0;l'0)C(lm;l''m'';l'm') \ , \tag{3.9}$$

$$W_{lm}(q) = \frac{1}{\pi^{3/2}q^{l+1}\sqrt{2l+1}} \sum_{\mathbf{n}\in\mathbb{Z}^3} \frac{1}{n^2-q^2} \mathscr{Y}_{lm}(\mathbf{n}) \ , \quad \mathscr{Y}_{lm}(\mathbf{n}) = n^l \cdot Y_{lm}(\Omega_n) \ . \tag{3.10}$$

4. Relation between ϕ^∞ and ϕ^L

In the following we restrict ourselves to the wave function for the irreducible representation of the rotational group on the finite volume (cubic group O), which is defined by

$$\phi^L_{\Gamma\alpha}(\mathbf{x};k) = \langle 0|\ n(\mathbf{x}/2)\ p(-\mathbf{x}/2)\ |k;\Gamma\alpha\rangle \ , \tag{4.1}$$

where $|k;\Gamma\alpha\rangle$ is the energy eigenstate with $\sqrt{s} = 2\sqrt{m^2+k^2}$ and belongs to the irreducible representation of O labeled by Γ and α ($\alpha = 1\ldots\dim\Gamma$, $\Gamma = \{A_1, A_2, E, T_1, T_2\}$). Projection of the irreducible representation of SU(2) ($|JM\rangle$) to that of O ($|\Gamma\alpha nJ\rangle$) is given by

$$|JM\rangle = \sum_{\Gamma\alpha n} |\Gamma\alpha nJ\rangle \cdot V(JM;\Gamma\alpha nJ)^* \ , \quad |\Gamma\alpha nJ\rangle = \sum_M |JM\rangle \cdot V(JM;\Gamma\alpha nJ) \ , \tag{4.2}$$

with known coefficient $V(JM;\Gamma\alpha nJ)$, where n is the multiplicity of the representation Γ.

In the previous section the wave functions were expanded in terms of functions of the $(JMls)$-base (J_{JMls} and N_{JMls}). But it is more convenient for the wave function (4.1) to expand in terms of functions of $(\Gamma\alpha nJls)$-base defined by

$$J_{\Gamma\alpha nJls}(\mathbf{x};k) = \sum_M J_{JMls}(\mathbf{x};k) \cdot V(JM;\Gamma\alpha nJ) \ , \tag{4.3}$$

with the coefficient $V(JM;\Gamma\alpha nJ)$.

In the region $|\mathbf{x}| > R$, the wave function (4.1) can be written by the linear combination of the Green function and also the wave function in the infinite volume as

$$\phi^L_{\Gamma\alpha}(\mathbf{x};k) = \sum_{nJls} E_{\Gamma\alpha nJls}(\mathbf{k}) \cdot G_{\Gamma\alpha nJls}(\mathbf{x};k) = \sum_{nJls} C_{\Gamma\alpha nJls}(\mathbf{k}) \cdot \phi^\infty_{\Gamma\alpha nJls}(\mathbf{x};k) \ , \tag{4.4}$$

with some coefficients $E_{\Gamma\alpha nJls}(\mathbf{k})$ and $C_{\Gamma\alpha nJls}(\mathbf{k})$, where $G_{\Gamma\alpha nJls}(\mathbf{x};k)$ and $\phi^\infty_{\Gamma\alpha nJls}(\mathbf{x};k)$ are functions of the $(\Gamma\alpha nJls)$-base obtained by the transformation (4.3) from $G_{JMls}(\mathbf{x};k)$ defined by (3.4) and $\phi^\infty_{JMls}(\mathbf{x};k)$ defind by (2.12). After some calculations we obtain

$$\phi^L_{\Gamma\alpha}(\mathbf{x};k) \ = \sum_{nJls} E_{\Gamma\alpha nJls}(\mathbf{k}) \left(b_l(k) \cdot N_{\Gamma\alpha nJls}(\mathbf{x};k) + \sum_{n'J'l'} b_{l'}(k) \cdot J_{\Gamma\alpha n'J'l's}(\mathbf{x};k) \cdot M^{(s)}_{n'J'l',nJl}(\Gamma;k) \right)$$

$$= \sum_{nJls} C_{\Gamma\alpha nJls}(\mathbf{k}) \sum_{l's'} \left(J_{\Gamma\alpha nJl's'}(\mathbf{x};k) \cdot \alpha^{(J)}_{l's',ls}(\Gamma;k) + N_{\Gamma\alpha nJl's'}(\mathbf{x};k) \cdot \beta^{(J)}_{l's',ls}(\Gamma;k) \right) , \quad (4.5)$$

where the constant $a_l(k)$ are removed by redefinition of the constant $E_{\Gamma\alpha nJls}(\mathbf{k})$, and

$$\delta_{\Gamma'\Gamma}\delta_{\alpha'\alpha} \cdot M^{(s)}_{n'J'l',nJl}(\Gamma;k) = \sum_{MM'} M^{(s)}_{J'M'l',JMl}(k) \cdot V(J'M';\Gamma'\alpha'n'J')V(JM;\Gamma\alpha nJ) , \quad (4.6)$$

$$\alpha^{(J)}_{l's',ls}(\Gamma;k) = \sum_{M} \alpha^{(J)}_{l's',ls} \cdot V(JM;\Gamma\alpha nJ)V(JM;\Gamma\alpha nJ) , \quad (4.7)$$

and $\beta^{(J)}_{l's',ls}(\Gamma;k)$ is similarly defined. The diagonal property of $M(\Gamma;k)$ in (4.6) for indices $(\Gamma\alpha)$ is result from the invariance of $M^{(s)}_{J'M'l',JMl}(k)$ under the rotation on the finite volume (see Ref. [1]).

From (4.5), we know that coefficients of functions $J_{\Gamma\alpha nJls}(\mathbf{x};k)$ and $N_{\Gamma\alpha nJls}(\mathbf{x};k)$ relate each other. After some calculations, we find that it is given by

$$\det[\, \mathbf{M}(\Gamma;k) \; - \; \mathbf{A}(\Gamma;k)/\mathbf{B}(\Gamma;k) \,] = 0 , \quad (4.8)$$

where we introduce a vector space spanned by indices $(nJls)$ at fixed $(\Gamma\alpha)$ and define linear operators on this vector space by

$$\big[\mathbf{M}(\Gamma;k)\big]_{n'J'l's',nJls} = \delta_{s's} \cdot M^{(s)}_{n'J'l',nJl}(\Gamma;k) , \quad \big[\mathbf{A}(\Gamma;k)\big]_{nJ'l's',nJls} = \delta_{n'n}\delta_{J'J} \cdot \alpha^{(J)}_{l's',ls}(\Gamma;k)/b_{l'}(k) , \quad (4.9)$$

and $\mathbf{B}(\Gamma;k)$ is similarly defined. Equation (4.8) is a finite size formula for the elastic NN scattering system, which gives us a relation between the energy eigenvalue on the finite volume and the quantity of the elastic scattering $\mathbf{A}/\mathbf{B}$.

5. Finite size formula for NN system

In this section we show the explicit matrix form of the finite size formula for the NN system (4.8). S-matrix at fixed J forms a 4×4 matrix. This matrix is reduced to sub-matrices by the eigenvalue of the global symmetry : the parity P and the particle exchange R $(= (-1)^I$ with the iso-spin I) as

$$\begin{aligned} S^{(J)} = \; & \big(\, 2\times 2 \text{ matrix} \;;\; P = (-1)^{J-1} \;,\; R = (-1)^{J-1} \,\big) \\ + & \big(\, 1\times 1 \text{ matrix} \;;\; P = (-1)^{J} \;,\; R = (-1)^{J} \,\big) \\ + & \big(\, 1\times 1 \text{ matrix} \;;\; P = (-1)^{J} \;,\; R = (-1)^{J-1} \,\big) . \end{aligned} \quad (5.1)$$

$\mathbf{A}(\Gamma;k)$ and $\mathbf{B}(\Gamma;k)$ in the finite size formula (4.8) also take same form.

We note that the basis of the partial wave expansion $J_{\Gamma\alpha nJls}(\mathbf{x};k)$ and $N_{\Gamma\alpha nJls}(\mathbf{x};k)$ in (4.5) are eigenstates of the parity and the particle exchange with $P = (-1)^l$ and $R = (-1)^l \cdot (-1)^{s-1}$. Thus the wave function for the state with $R = -P$, only functions with $s = 0$ appear in the partial wave expansion. For the state with $R = P$, only functions with $s = 1$ appear. The mixing between $s = 0$ and $s = 1$ is forbidden by the symmetry of the parity and the particle exchange (iso-spin). Therefore we can separately obtain the finite size formula for $P = -R$ $(s = 0)$ and $P = R$ $(s = 1)$.

In the case of $R = -P$ $(s = 0)$, the components of the matrix $\mathbf{M}(\Gamma;k)$ in the finite size formula (4.8) are given by

$$M^{(s)}_{n'J'l',nJl}(\Gamma;k) = \delta_{J'l'}\delta_{Jl} \cdot \sum_{MM'} M_{J'M',JM}(k) \cdot V(\Gamma\alpha nJ';J'M')V(\Gamma\alpha nJ;JM) . \quad (5.2)$$

This is the same matrix as that appeared in the finite size formula for the two-meson system. Further, $\alpha^{(J)}_{l's,ls}(k) = \delta_{Jl}\delta_{Jl'} \cdot \alpha_l(k)$ and $\beta^{(J)}_{l's,ls}(k) = \delta_{Jl}\delta_{Jl'} \cdot \beta_l(k)$ with the diagonal components $\alpha_l(k)$ and $\beta_l(k)$ also take same matrix form as that for the two-meson system. Thus the finite size formula for the NN system with $P = -R$ ($s = 0$) is same as that for the two-meson system in Ref. [1].

In the case of $R = P$ ($s = 1$), the matrix $\mathbf{M}(\Gamma;k)$ and $\mathbf{A}(\Gamma;k)/\mathbf{B}(\Gamma;k)$ have complicated structure. In the following we show explicit matrix form of the finite size formula for some channels as example. We neglect contributions of $J \geq 5$. In this case the multiplicity n is 1 for all irreducible representations Γ, thus we omit the index n in the formula (4.8) for simplicity, as

$$\det\left[\, \mathbf{M}(\Gamma;k) \;-\; \mathbf{A}(\Gamma;k)/\mathbf{B}(\Gamma;k)\,\right] = 0 \;, \tag{5.3}$$

$$\left[\mathbf{M}(\Gamma;k)\right]_{J'l',Jl} = M^{(s)}_{n'J'l',nJl}(\Gamma;k) \;, \quad \left[\mathbf{A}(\Gamma;k)\right]_{J'l',Jl} = \delta_{J'J} \cdot \alpha^{(J)}_{l's',ls}(\Gamma;k)/b_{l'}(k) \;, \tag{5.4}$$

where $n = n' = 1$, $s = s' = 1$ and the matrix $\mathbf{B}(\Gamma;k)$ is similarly defined.

The first example is the deuteron state. We have to consider the NN state with the total angular momentum $J = 1$ and the parity $P = +1$, which corresponds to ${}^3S_1$ and ${}^3D_1$ states in the non-relativistic limit. The $J = 1$ state belongs to the irreducible representation of the cubic group $\Gamma = T_1$, thus we consider the finite size formula for $\Gamma = T_1$ for the study of the deuteron. The other angular momentum states also belong to T_1 as $T_1 = 1+3+4$ up to $J \geq 5$ and the finite size formula includes contributions from all these states. For each values of J, possible values of l are given by

$$\begin{array}{ll} l = 0,\ 2 & \text{for}\ \ J = 1 \\ l = 2,\ 4 & \text{for}\ \ J = 3 \\ l = 4 & \text{for}\ \ J = 4 \end{array} \;, \tag{5.5}$$

from the parity conservation and the theory of addition of the angular momentum. Thus matrices $\mathbf{M}(\Gamma;k)$ and $\mathbf{A}(\Gamma;k)/\mathbf{B}(\Gamma;k)$ in the finite size formula (5.3) take :

$$\mathbf{M} = \begin{pmatrix} M_{10,10} & M_{10,12} & M_{10,32} & M_{10,34} & M_{10,44} \\ M_{12,10} & M_{12,12} & M_{12,32} & M_{12,34} & M_{12,44} \\ M_{32,10} & M_{32,12} & M_{32,32} & M_{32,34} & M_{32,44} \\ M_{34,10} & M_{34,12} & M_{34,32} & M_{34,34} & M_{34,44} \\ M_{44,10} & M_{44,12} & M_{44,32} & M_{44,34} & M_{44,44} \end{pmatrix} , \quad \mathbf{A}/\mathbf{B} = \begin{pmatrix} \multicolumn{2}{|c|}{\multirow{2}{*}{\textit{J=1}}} & 0 & 0 & 0 \\ & & 0 & 0 & 0 \\ 0 & 0 & \multicolumn{2}{|c|}{\multirow{2}{*}{\textit{J=3}}} & 0 \\ 0 & 0 & & & 0 \\ 0 & 0 & 0 & 0 & \boxed{\textit{J=4}} \end{pmatrix} , \tag{5.6}$$

where the boxes in the matrix $\mathbf{A}/\mathbf{B}$ which enclose the values of J refer to 2×2 or 1×1 matrices expanded by the possible values of l. In (5.6) components of the matrix $\mathbf{M}$ are denoted by $M_{J'l',Jl} \equiv [\mathbf{M}(\Gamma;k)]_{J'l',Jl}$ and are given by

$$\begin{array}{lll} M_{10,10} = W_{00} & & \\ M_{12,10} = 0 & M_{12,12} = W_{00} & \\ M_{32,10} = 0 & M_{32,12} = -\frac{6}{7}\sqrt{6}W_{40} & M_{32,32} = W_{00} + \frac{6}{7}W_{40} \\ M_{34,10} = -2W_{40} & M_{34,12} = -\frac{5}{7}\sqrt{2}W_{40} & M_{34,32} = \frac{30}{77}\sqrt{3}W_{40} + \frac{50}{33}\sqrt{3}W_{60} \\ M_{44,10} = \frac{6}{7}\sqrt{7}W_{40} & M_{44,12} = -\frac{3}{7}\sqrt{14}W_{40} & M_{44,32} = \frac{18}{77}\sqrt{21}W_{40} + \frac{10}{11}\sqrt{21}W_{60} \end{array}$$

$$\begin{array}{ll} M_{34,34} = W_{00} + \frac{81}{77}W_{40} + \frac{25}{33}W_{60} & \\ M_{44,34} = -\frac{27}{77}\sqrt{7}W_{40} - \frac{15}{11}\sqrt{7}W_{60} & M_{44,44} = W_{00} + \frac{81}{143}W_{40} + \frac{1}{55}W_{60} + \frac{1792}{715}W_{80} \;, \end{array} \tag{5.7}$$

where $M_{J'l',Jl} = M_{Jl,J'l'}$ and the function W_{lm} is defined by (3.10).

Finally we consider the state with same J but opposite parity to the deuteron, *ie.* $J = 1, P = -1$. We also consider the representation $\Gamma = T_1$. Possible values of l for each J are different from those of the deuteron case as

$$\begin{array}{ll} l = 1 & \text{for } J = 1 \\ l = 3 & \text{for } J = 3 \\ l = 3,\ 5 & \text{for } J = 4 \end{array} \quad . \tag{5.8}$$

Thus matrices in the finite size formula take different forms :

$$\mathbf{M} = \begin{pmatrix} M_{11,11} & M_{11,33} & M_{11,43} & M_{11,45} \\ M_{33,11} & M_{33,33} & M_{33,43} & M_{33,45} \\ M_{43,11} & M_{43,33} & M_{43,43} & M_{43,45} \\ M_{45,11} & M_{45,33} & M_{45,43} & M_{45,45} \end{pmatrix}, \quad \mathbf{A}/\mathbf{B} = \begin{pmatrix} \boxed{J{=}1} & 0 & 0 & 0 \\ 0 & \boxed{J{=}3} & 0 & 0 \\ 0 & 0 & \multicolumn{2}{c}{\multirow{2}{*}{\boxed{J{=}4}}} \\ 0 & 0 & & \end{pmatrix}, \tag{5.9}$$

$$\begin{array}{ll} M_{11,11} = W_{00} & \\ M_{33,11} = \frac{3}{7}\sqrt{14}W_{40} & M_{33,33} = W_{00} + \frac{3}{11}W_{40} - \frac{25}{11}W_{60} \\ M_{43,11} = -\frac{1}{7}\sqrt{210}W_{40} & M_{43,33} = -\frac{3}{11}\sqrt{15}W_{40} - \frac{35}{33}\sqrt{15}W_{60} \\ M_{45,11} = \frac{2}{7}\sqrt{42}W_{40} & M_{45,33} = \frac{6}{11}\sqrt{3}W_{40} + \frac{70}{33}\sqrt{3}W_{60} \\ M_{43,43} = W_{00} + \frac{9}{11}W_{40} - \frac{5}{33}W_{60} & \\ M_{45,43} = \frac{18}{143}\sqrt{5}W_{40} - \frac{14}{165}\sqrt{5}W_{60} - \frac{896}{715}\sqrt{5}W_{80} & \\ M_{45,45} = W_{00} + \frac{126}{143}W_{40} - \frac{32}{165}W_{60} - \frac{448}{715}W_{80} \quad . & \end{array} \tag{5.10}$$

6. Summary

The finite size formula for the elastic NN scattering system is derived from the relativistic quantum field theory. The extension to other two-baryon system as the $N\Lambda$ system is trivial. Finally I give a comment for the $N\pi$ system. The formulation of this paper for the NN system is also valid for the system with the general value of the spin s. Thus the finite size formula for the $N\pi$ system can be easily obtained from that for the NN system (4.8) by set $s = 1/2$. In calculations of matrices $\mathbf{M}(\Gamma;k)$, $\mathbf{A}(\Gamma;k)$ and $\mathbf{B}(\Gamma;k)$ in (4.9), we change the coefficient $V(JM;\Gamma\alpha nJ)$ in (4.2) by that for the double covered cubic group (^{2}O) to deal with the half integer value of the total angular momentum J as discussed in Ref. [2]. I confirmed that my results are consistent with those obtained from the non-relativistic effective theory by Bernard *et al.* [2].

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