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Scaling study of the relativistic corrections to the static potential

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The relativistic corrections to the static potential, i.e. the $O(1/m)$ correction, the $O(1/m^2)$ spin-dependent and momentum-dependent corrections are investigated in SU(3) lattice gauge theory. These corrections are relevant ingredients of an effective field theory for heavy quarkonium called potential nonrelativistic QCD. Utilizing the multilevel algorithm for the field strength correlator on the quark-antiquark source, these corrections are determined at the distances ranged from 0.25 to 1.2 fm. A reasonable scaling behavior and long-range nonperturbative contributions are observed.

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1. Introduction

A promising way of studying heavy quarkonium systematically in QCD is to employ an effective field theory called potential nonrelativistic QCD (pNRQCD) [1, 2]. pNRQCD is derived from QCD by integrating out the scale above the heavy quark mass $m \gg \Lambda_{\rm QCD}$ and the scale of momentum transfer mv, where v is quark velocity.

The effective hamiltonian of pNRQCD consists of the nonrelativistic kinetic terms of a heavy quark and a heavy antiquark with the inter-quark potentials classified in powers of $1/m$. Up to $O(1/m^2)$ the effective hamiltonian has the form [2]

$$\begin{aligned} H \;=\; & \frac{\vec{p}_1^{\,2}}{2m_1} + \frac{\vec{p}_2^{\,2}}{2m_2} + V^{(0)}(r) + \frac{1}{m_1}V^{(1,0)}(r) + \frac{1}{m_2}V^{(0,1)}(r) \\ & + \frac{1}{m_1^2}V^{(2,0)}(r) + \frac{1}{m_2^2}V^{(0,2)}(r) + \frac{1}{m_1 m_2}V^{(1,1)}(r) + O(1/m^3)\,, \end{aligned} \tag{1.1}$$

where m_1 and m_2 denote the masses of quark and antiquark, placed at $\vec{r}_1$ and $\vec{r}_2$, respectively. $V^{(0)}(r \equiv |\vec{r}|)$, where $\vec{r} \equiv \vec{r}_1 - \vec{r}_2$, is the static potential. Note that m_1 and m_2 can be different. $V^{(1,0)}(r) = V^{(0,1)}(r)$ $(\equiv V^{(1)}(r))$ are the corrections at $O(1/m)$. $V^{(2,0)}(r)$, $V^{(0,2)}(r)$, and $V^{(1,1)}(r)$ are the corrections at $O(1/m^2)$, which contain the leading order spin-dependent corrections [5, 6] and momentum-dependent corrections [3, 4].

The spin-dependent (SD) part of the $O(1/m^2)$ correction is conventionally written as

$$\begin{aligned} V_{\rm SD}(r) \;=\; & \left(\frac{\vec{s}_1\cdot\vec{l}_1}{2m_1^2} - \frac{\vec{s}_2\cdot\vec{l}_2}{2m_2^2}\right)\left(\frac{V^{(0)\prime}(r)}{r} + 2\frac{V_1'(r)}{r}\right) + \left(\frac{\vec{s}_2\cdot\vec{l}_1}{2m_1 m_2} - \frac{\vec{s}_1\cdot\vec{l}_2}{2m_1 m_2}\right)\frac{V_2'(r)}{r} \\ & + \frac{1}{m_1 m_2}\left(\frac{(\vec{s}_1\cdot\vec{r})(\vec{s}_2\cdot\vec{r})}{r^2} - \frac{\vec{s}_1\cdot\vec{s}_2}{3}\right)V_3(r) + \frac{\vec{s}_1\cdot\vec{s}_2}{3m_1 m_2}V_4(r)\,, \end{aligned} \tag{1.2}$$

where $\vec{s}_1$ and $\vec{s}_2$ denote the spins, and $\vec{l}_1 = -\vec{l}_2 = \vec{l}$ the orbital angular momenta. Although the original expression of the SD correction in pNRQCD contains the matching coefficient c_F [2], it is assumed to be one here for simplicity. For the actual application one has to compute the matching coefficient in a perturbative or a nonperturbative manner depending on the matching scale between pNRQCD and QCD. $V_1'(r), V_2'(r)$ are responsible for the spin-orbit corrections (fine splitting), while $V_3(r), V_4(r)$ are for the spin-spin corrections (hyper-fine splitting).

The spin-independent (SI) part of the $O(1/m^2)$ correction is written as

$$\begin{aligned} V_{\rm SI}(r) \;=\; & \frac{1}{m_1^2}\left(\frac{1}{2}\{\vec{p}_1^{\,2}, V_{p^2}^{(2,0)}(r)\} + \frac{1}{r^2}V_{l^2}^{(2,0)}(r)\vec{l}_1^{\,2} + V_r^{(2,0)}(r)\right) \\ & + \frac{1}{m_2^2}\left(\frac{1}{2}\{\vec{p}_2^{\,2}, V_{p^2}^{(0,2)}(r)\} + \frac{1}{r^2}V_{l^2}^{(0,2)}(r)\vec{l}_1^{\,2} + V_r^{(0,2)}(r)\right) \\ & + \frac{1}{m_1 m_2}\left(-\frac{1}{2}\{\vec{p}_1\cdot\vec{p}_2, V_{p^2}^{(1,1)}(r)\} - \frac{1}{2r^2}V_{l^2}^{(1,1)}(r)(\vec{l}_1\cdot\vec{l}_2 + \vec{l}_2\cdot\vec{l}_1) + V_r^{(1,1)}(r)\right)\,. \end{aligned} \tag{1.3}$$

The radial functions specified by the subscripts p^2 and l^2, are related to the momentum-dependent potentials, $V_b(r), V_c(r), V_d(r)$ and $V_e(r)$ defined in Refs. [3, 4], by

$$\begin{aligned} & V_{p^2}^{(2,0)}(r) = V_{p^2}^{(0,2)}(r) = V_d(r) - \frac{2}{3}V_e(r)\,, \quad V_{l^2}^{(2,0)}(r) = V_{l^2}^{(0,2)}(r) = V_e(r)\,, \\ & V_{p^2}^{(1,1)}(r) = -V_b(r) + \frac{2}{3}V_c(r)\,, \quad V_{l^2}^{(1,1)}(r) = -V_c(r)\,. \end{aligned} \tag{1.4}$$

Once the functional forms of these corrections are determined in QCD, various properties of heavy quarkonium can be investigated systematically by solving the Schrödinger equation.

Recently, we investigated the $O(1/m)$ correction [7, 8], and the $O(1/m^2)$ spin-dependent [9, 10] and momentum-dependent corrections [8] in SU(3) lattice gauge theory utilizing a new method, and obtained remarkably clean signals up to distances of around 0.9 fm. We observed a certain deviation from the perturbative results at these distances.

In this report we present our new results of the relativistic corrections at longer distances up to 1.2 fm [11]. We then examine the scaling behavior of the corrections with respect to the change of lattice spacing.

2. Formulation and Numerical Procedures

According to pNRQCD, the $O(1/m)$ and $O(1/m^2)$ corrections can generally be expressed by the *matrix elements* and the *energy gaps* appeared in the spectral representation of the color-electric and color-magnetic field strength correlators (FSCs) on the quark-antiquark source [1, 2].

Let us explain how to determine the $O(1/m)$ correction as an example.

We write the eigenstate of the pNRQCD hamiltonian at $O(m^0)$ in the $\mathbf{3}\otimes\mathbf{3}^*$ representation of SU(3) color, which corresponds to the *static* quark-antiquark state, as $|n\rangle \equiv |n;\vec{r}_1,\vec{r}_2\rangle$. Then, the color-electric FSC, i.e. the correlator of two color-electric field strength operators $E_i = F_{4i}$ $(i = 1,2,3)$, attached to a quark at $\vec{r}_1$ and separated $t = t_1 - t_2$ in the time direction, takes the form

$$C(r,t) = \sum_{n=1}^{\infty} \langle 0|gE_i(\vec{r}_1)|n\rangle\langle n|gE_j(\vec{r}_1)|0\rangle e^{-(\Delta E_{n0}(r))t} , \tag{2.1}$$

where $\Delta E_{n0}(r) \equiv E_n(r) - E_0(r)$ denotes the energy gap and $E_0(r) = V^{(0)}(r)$. The $O(1/m)$ correction is expressed with the matrix elements $\langle 0|gE_i(\vec{r}_1)|n\rangle$ and the energy gap $\Delta E_{n0}(r)$ as

$$V^{(1)}(r) = -\frac{1}{2}\delta^{ij}\sum_{n=1}^{\infty}\frac{\langle 0|gE_i(\vec{r}_1)|n\rangle\langle n|gE_j(\vec{r}_1)|0\rangle}{(\Delta E_{n0}(r))^2} . \tag{2.2}$$

Thus, once the matrix elements and the energy gaps are known from the behavior of FSCs, one can compute the correction.

We work in Euclidean space in four dimensions on a hypercubic lattice with lattice volume $V = L^3T$ and lattice spacing a, where we impose periodic boundary conditions in all directions. We use the Polyakov loop correlation function (PLCF, a pair of Polyakov loops P separated by a distance r) as the quark-antiquark source and evaluate the color-electric FSCs on the PLCF,

$$C(r,t) = \langle\langle gE_i(\vec{r}_1,t_1)gE_j(\vec{r}_1,t_2)\rangle\rangle_c = \langle\langle gE_i(\vec{r}_1,t_1)gE_j(\vec{r}_1,t_2)\rangle\rangle - \langle\langle gE_i(\vec{r}_1)\rangle\rangle\langle\langle gE_j(\vec{r}_1)\rangle\rangle , \tag{2.3}$$

using the multi-level algorithm [7, 10], where the double bracket represents the ratio of expectation values $\langle\langle\cdots\rangle\rangle = \langle\cdots\rangle_{PP^*}/\langle PP^*(r)\rangle$, while $\langle\cdots\rangle_{PP^*}$ means that the color-electric field is connected to the Polyakov loop in a gauge invariant way. The subtracted term on the r.h.s. of Eq. (2.3) can be nonzero as the color-electric field is even under CP transformations. The spectral representation of Eq. (2.3) derived with transfer matrix theory reads [10]

$$C(r,t) = 2\sum_{n=1}^{\infty}\langle 0|gE_i(\vec{r}_1)|n\rangle\langle n|gE_j(\vec{r}_1)|0\rangle e^{-(\Delta E_{n0})T/2}\cosh((\Delta E_{n0})(\frac{T}{2}-t)) + O(e^{-(\Delta E_{10})T}) , \tag{2.4}$$

Table 1: Simulation parameters used in this study. $N_{\rm tsl}$ is the number of time slices in a sublattice and $N_{\rm iupd}$ the number of internal update within a sublattice, both are parameters for the multilevel algorithm [10]. The lattice spacing a is set from the Sommer scale $r_0 = 0.5$ fm.

$\beta = 6/g^2$	a [fm]	$N_{\rm tsl}$	spin-dependent corrections			spin-independent corrections		
			$(L/a)^3(T/a)$	$N_{\rm iupd}$	$N_{\rm conf}$	$(L/a)^3(T/a)$	$N_{\rm iupd}$	$N_{\rm conf}$
5.85	0.123	3	24^4	50000	77	24^4	50000	133
6.00	0.093	4	$20^3 40$	7000	33	$24^3 32$	50000	100
6.20	0.068	5	$24^3 30$	10000	33	$30^3 40$	50000	33
6.30	0.059	6	24^4	6000	39	———		

where the last term represents terms with exponential factors equal to or smaller than $\exp(-(\Delta E_{10})T)$, which are negligible for a reasonably large T. We evaluate Eq. (2.3) via Monte Carlo simulations and determine the matrix element $\langle 0|gE_i(\vec{r}_1)|n\rangle\langle n|gE_j(\vec{r}_1)|0\rangle$ and the energy gap ΔE_{n0} in Eq. (2.4) by fitting, both of which are finally inserted into the definitions of the corrections, such as Eq. (2.2). Eq. (2.4) is reduced to the form like Eq. (2.1) in the infinite volume limit $T \to \infty$.

We define the lattice color-electric field operator, $ga^2E_i(s)$, from the traceless part of $[U_{4i}(s) - U_{4i}^{\dagger}(s)]/(2i)$ with two-leaf modification (an average of $F_{4i}(s)$ and $F_{4i}(s-\hat{i})$), where $U_{\mu\nu}(s)$ is a plaquette variable defined on the site s. We multiply the Huntley-Michael factor [12] on the PLCF, Z_E [10], to the lattice color-electric field to cancel the self energies at least at $O(g^2)$.

Other relativistic corrections can be investigated similarly. See Refs. [8, 10] for definitions and further technical details.

3. Numerical results

We carry out simulations using the standard Wilson gauge action in SU(3) lattice gauge theory. Simulation parameters are summarized in Table 1.

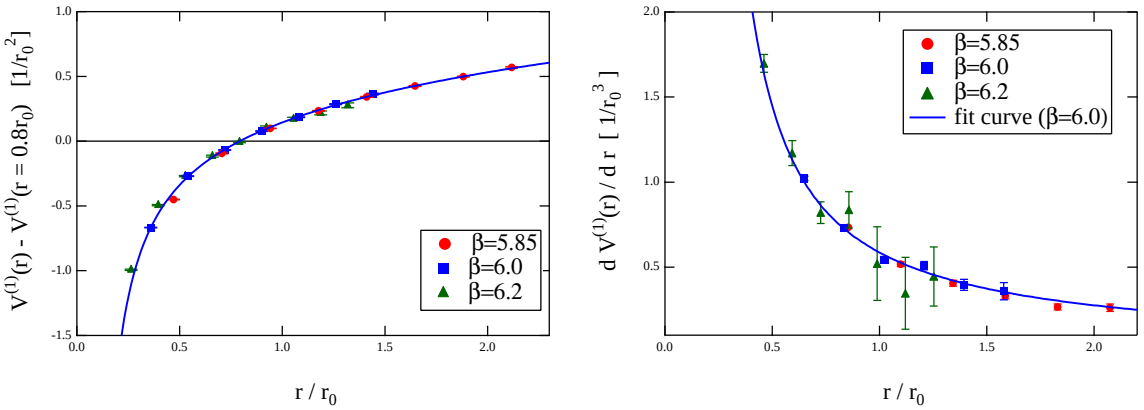

Figure 1: The $O(1/m)$ correction, $V^{(1)}(r)$, normalized at $r = 0.8\,r_0$ (left) and its derivative with respect to r (right) in units of r_0. The solid line is the fitting curve for the data at $\beta = 6.0$ with the functional form $V_{\rm ln}(r) = -A/r^2 + B\ln r + C$ and $dV_{\rm ln}(r)/dr$, where the data at $r/a = 2$ is not taken into account in the fit.

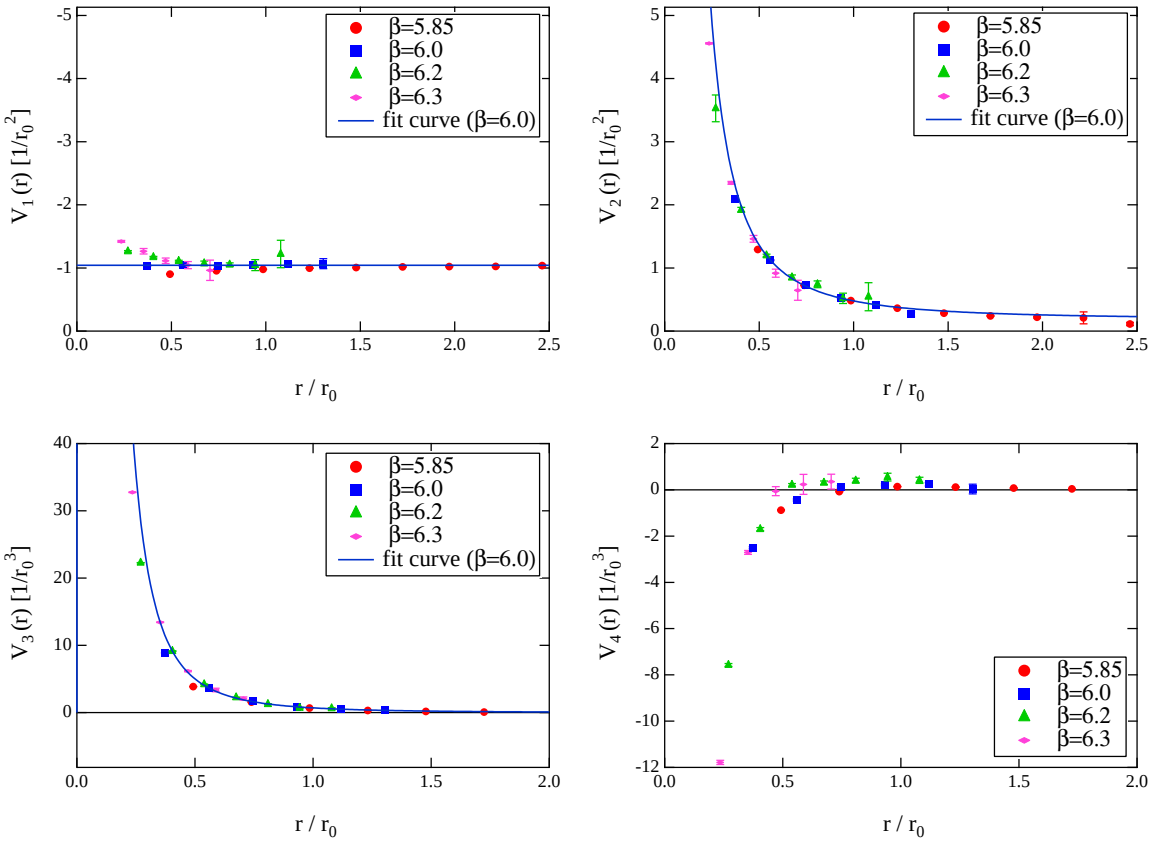

Figure 2: The spin-dependent corrections $V_1'(r)$, $V_2'(r)$, $V_3(r)$ and $V_4(r)$ in units of r_0.

3.1 The $O(1/m)$ correction

In Fig. 1, we show the $O(1/m)$ correction, $V^{(1)}(r)$, normalized at $r = 0.8\,r_0$ together with the fitting curve $V_{\ln}(r) = -A/r^2 + B\ln r + C$. The first term is motivated by perturbation theory at $O(\alpha_s^2)$, which provides $V_{\rm pert}(r) = -\frac{C_F C_A \alpha_s^2}{4r^2}$, where C_F and C_A are the Casimir charges of the fundamental and the adjoint representations, respectively, and $\alpha_s = g^2/(4\pi)$ the strong coupling [1]. We measure $V^{(1)}(r)$ up to $r = 2.35\,r_0 = 1.2$ fm. $V^{(1)}(r)$ for different β values show a reasonable scaling behavior, except for the data at $r/a = 2$, which are always suffered from a larger discretization error than the data at $r/a \geq 3$. We find that the $O(1/m)$ correction contains the radial behavior which cannot be explained by perturbation theory at $O(\alpha_s^2)$. The new data set at $r \gtrsim 1.5\,r_0$ suggests that the long distance behavior is described by a logarithmic function [11, 13].

3.2 The $O(1/m^2)$ spin-dependent corrections

In Fig. 2, we present the spin-dependent corrections, $V_1'(r)$, $V_2'(r)$, $V_3(r)$ and $V_4(r)$, where the data at $\beta = 6.0$ and 6.3 are already published in Ref. [10]. The $O(1/m^2)$ spin-orbit corrections, $V_1'(r)$ and $V_2'(r)$, are non-vanishing up to $r = 2.23\,r_0 = 1.12$ fm. The finite tail of $V_2'(r)$ is observed in Ref. [10], which are again observed at further longer distances.

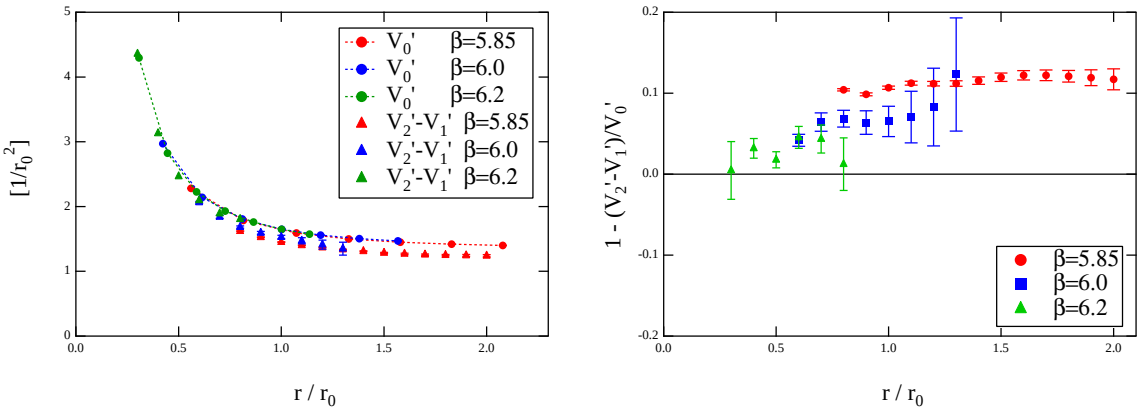

Figure 3: Test of the Gromes relation $V^{(0)\prime}(r) = V_2'(r) - V_1'(r)$. Comparison between the force $V^{(0)\prime}$ and the difference of the spin-orbit corrections $V_2' - V_1'$ (left), and the relative deviation from the Gromes relation $1 - (V_2' - V_1')/V^{(0)\prime}$ (right).

The Gromes relation [6], an important analytic relation derived from the Lorentz invariance,

$$V^{(0)\prime}(r) = V_2'(r) - V_1'(r) \,, \tag{3.1}$$

is approximately satisfied as shown in Fig. 3, where the comparison of the l.h.s and the r.h.s of Eq. (3.1) as well as the relative deviation from the relation, $1 - (V_2' - V_1')/V^{(0)\prime}$, are plotted. If the Gromes relation is satisfied, this quantity should be zero at all r. We find that the deviation is 10 to 12 % at $a = 0.123$ fm, while 4 to 10 % at $a = 0.093$ fm, and seems to be smaller at $a = 0.068$ fm. Namely, we find a tendency such that the deviation decreases as $a \to 0$.

For the spin-spin corrections, $V_3(r)$ and $V_4(r)$, we see that they have no long range contribution. However a detailed analysis shows that the functional form slightly deviates from the leading-order perturbative expressions at intermediate distances.

3.3 The $O(1/m^2)$ momentum-dependent corrections

In Fig. 4, we show the momentum-dependent corrections, $V_b(r)$, $V_c(r)$, $V_d(r)$ and $V_e(r)$, which are normalized at $r = 0.8\, r_0$. Motivated by the minimal area law model [3, 4], we fit the data to the functional form as the static potential $V(r) = -A/r + Br + C$, where the data at $r/a = 2$ is not taken into account. We find that the global structure of the data are well described with this function as seen in Fig. 4.

There are nonperturbative relations like the Gromes relation for the momentum-dependent corrections, called the BBMP relation [3, 4],

$$V_b(r) + 2V_d(r) = -\frac{1}{2}V^{(0)}(r) + \frac{r}{6}\frac{dV^{(0)}(r)}{dr} \,, \quad V_c(r) + 2V_e(r) = -\frac{r}{2}\frac{dV^{(0)}(r)}{dr} \,. \tag{3.2}$$

We examined these relations in Ref. [8] and found that they seem to be satisfied. Using the new data set, we are now investigating the BBMP relation carefully, which will be reported in a future publication. In any case, it is clear that the momentum-dependent corrections contain nonperturbative contribution as they are related to the static potential. The effect of these corrections to

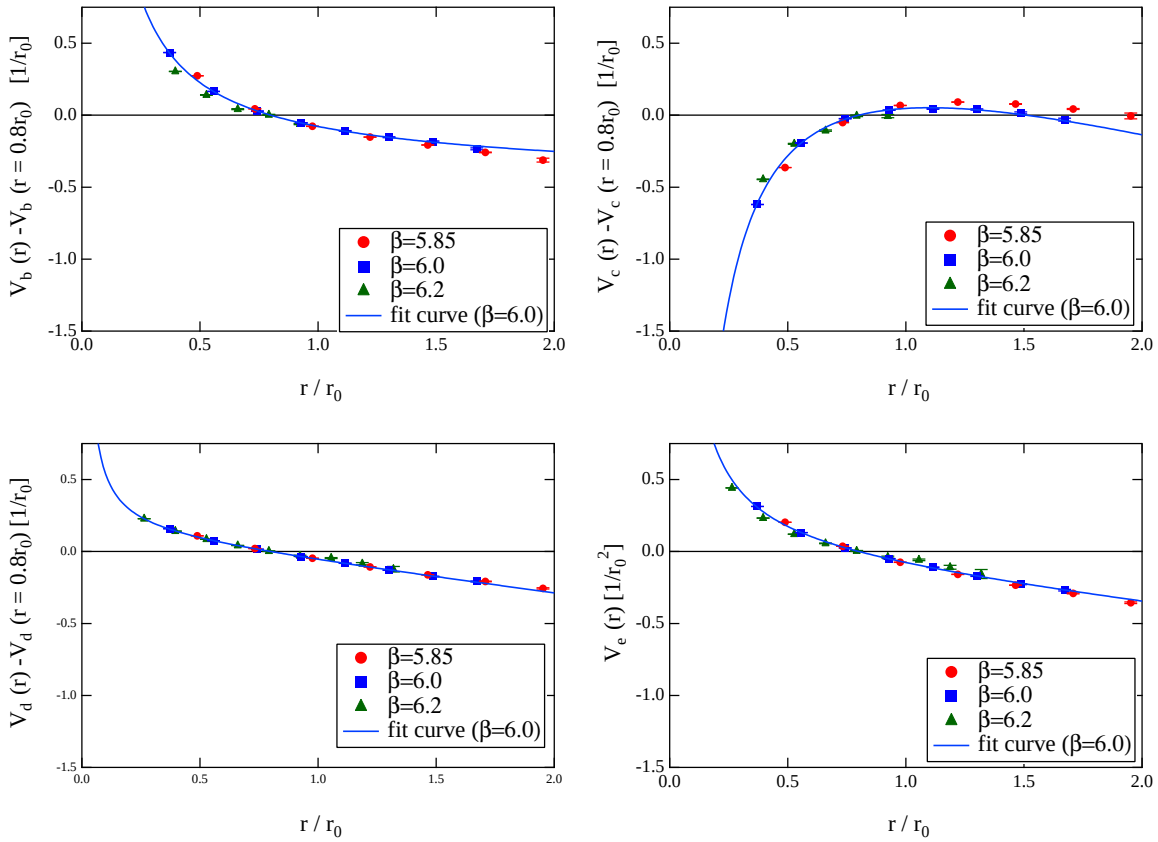

Figure 4: The momentum-dependent corrections $V_b(r)$, $V_c(r)$, $V_d(r)$ and $V_e(r)$ in units of r_0, normalized at $r = 0.8\,r_0$.

the spectroscopy is not yet well examined, so that it is quite interesting to solve the Schrödinger equation with these corrections.

4. Summary

We have investigated the relativistic corrections to the static potential, the $O(1/m)$ correction and the $O(1/m^2)$ spin-dependent and momentum-dependent corrections in SU(3) lattice gauge theory. These corrections are important ingredients of pNRQCD for heavy quarkonium spectroscopy. By evaluating the color-electric and color-magnetic FSCs on the PLCF with the multilevel algorithm, and exploiting the spectral representation of FSCs, we have obtained a very clean signal for these corrections in the region from 0.25 fm to 1.2 fm. We have observed long-range nonperturbative contribution to these corrections, which show a reasonable scaling behavior with respect to the change of lattice spacing.

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