

The repulsive core of the NN potential and the operator product expansion

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We investigate the short distance behavior of the nucleon–nucleon (NN) potential defined through the Bethe-Salpeter wave function, by perturbatively calculating anomalous dimensions of 6–quark operators in QCD. Thanks to the asymptotic freedom of QCD, the 1-loop estimations give exact results for the potential in the zero distance limit. We show that the chiral symmetry of the gauge interaction implies the existence of an operator whose anomalous dimension is zero for a given quantum number. Furthermore we find that non-zero anomalous dimensions of other operators are all negative. These results predict the functional form of the NN potential at short distance, which is a little weaker than r^{-2} . On the other hand, the computation of the anomalous dimension spectrum alone can not determine whether the potential is repulsive or attractive at short distance. An additional analytic non-perturbative analysis suggests that the force at short distance is indeed repulsive at low energy as found numerically. Some extensions of the method are briefly discussed.

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1. Introduction

In a recent paper [1], the nucleon-nucleon(NN) potentials have been calculated in lattice QCD from the Bethe-Salpeter (BS) wave function through the Schrödinger equation. The results qualitatively resemble phenomenological NN potentials which are widely used in nuclear physics. The force at medium to long distance ($r \geq 2$ fm) is shown to be attractive. This feature has long well been understood in terms of pion and other heavier meson exchanges. At short distance, a characteristic repulsive core is reproduced by the lattice QCD simulation [1]. No simple theoretical explanation, however, exists so far for the origin of the repulsive core.

A hint to understand the repulsive core theoretically has appeared in Ref.[2], where properties of the BS wave function of the Ising field theory in 2-dimensions,

$$\varphi(r, \theta) = i \langle 0 | \sigma(x, 0) \sigma(0, 0) | \theta, -\theta \rangle^{\text{in}}, \quad r = |x| \quad (1.1)$$

are considered analytically. Here θ is the rapidity of the one particle state. From the operator product expansion (OPE)

$$\sigma(x, 0) \sigma(0, 0) \underset{r \rightarrow 0}{\sim} G(r) \mathbf{1} + c r^{3/4} \mathcal{E}(0) + \dots, \quad (1.2)$$

the BS wave function at short distance becomes

$$\varphi(r, \theta) \sim C r^{3/4} \sinh(\theta) + O(r^{7/4}), \quad (1.3)$$

which predicts the short distance behavior of the potential as

$$V_{\theta}(r) = \frac{\varphi''(r, \theta) + \sinh^2 \theta \varphi(r, \theta)}{\varphi(r, \theta)} \sim -\frac{3}{16} \frac{1}{r^2}. \quad (1.4)$$

The OPE in this case predicts not only the r^{-2} behavior of the potential at short distance but also its coefficient $-3/16$. The potential at short distance does not depend on the energy (rapidity) of the state; it is universal.

In this report, the OPE analysis is applied to QCD, with the aim to theoretically better understand the repulsive core of the NN potential.

2. Operator Product Expansion and potentials at short distance

2.1 General argument

Let us generalize the argument in the introduction, which relates the operator product expansion(OPE) to the short distance behavior of the potential. We write the equal time Bethe-Salpeter (BS) wave function as

$$\varphi_{AB}^E(\vec{r}) = \langle 0 | O_A(\vec{r}, 0) O_B(\vec{0}, 0) | E \rangle \quad (2.1)$$

where $|E\rangle$ is the eigen-state of the system with the energy E , and O_A, O_B are some operators of the system. Here we suppress further quantum numbers of the state $|E\rangle$ other than E for simplicity. The OPE of O_A and O_B is written as

$$O_A(\vec{r}, 0) O_B(\vec{0}, 0) \simeq \sum_C D_{AB}^C(\vec{r}) O_C(\vec{0}, 0), \quad r = |\vec{r}| \rightarrow 0. \quad (2.2)$$

We here assume that the coefficient function D_{AB}^C behaves as

$$D_{AB}^C(\vec{r}) \simeq (-\log r)^{\beta_{AB}^C} \quad (2.3)$$

if O_C has the same mass dimension of $O_A O_B$. Therefore the BS wave function becomes

$$\varphi_{AB}^E(\vec{x}) \simeq \sum_C (-\log r)^{\beta_{AB}^C} D_C(E), \quad (2.4)$$

where $D_C(E) = \langle 0 | O_C(\vec{0}, 0) | E \rangle$.

In the approach of Ref.[1, 3, 4], the potential is defined by the BS wave function through the Schrödinger equation as

$$V(\vec{r}) = E + \frac{1}{2\mu} \frac{\nabla^2 \varphi_{AB}^E(\vec{r})}{\varphi_{AB}^E(\vec{r})}, \quad (2.5)$$

where μ is the reduced mass. Since

$$\nabla^2 = \frac{1}{r^2} \frac{d}{dr} r^2 \frac{d}{dr} - \frac{L(L+1)}{r^2} \quad (2.6)$$

for the state with angular momentum L , the dominant contribution of the potential for non-zero L at short distance is trivially given by

$$V_L(r) \simeq -\frac{L(L+1)}{r^2} + \dots \quad (2.7)$$

As a non-trivial case let us consider the $L = 0$ case. There are two cases, depending on the maximum value of β_{AB}^C defined as $\beta_X = \max_C \beta_{AB}^C$.

(1) $\beta_X \neq 0$: In this case, the potential at short distance is universally given by

$$V(r) \simeq -\frac{\beta_X}{r^2 (-\log r)}. \quad (2.8)$$

If $\beta_X > 0$ the interaction is attractive at short distance, while it is repulsive if $\beta_X < 0$.

(2) $\beta_X = 0$: We denote $\beta_Y < 0$ is the second largest among β_{AB}^C 's. The potential at short distance becomes

$$V(r) \simeq \frac{D_Y(E)}{D_X(E)} \frac{-\beta_Y}{r^2 (-\log r)^{1-\beta_Y}}, \quad (2.9)$$

which is attractive for $D_Y(E)/D_X(E) < 0$ while repulsive for $D_Y(E)/D_X(E) > 0$.

2.2 OPE in QCD

Since QCD is an asymptotically free theory, the 1-loop calculation for anomalous dimensions becomes exact at short distance. The OPE in QCD is written as

$$O_A(\vec{r}, 0) O_B(\vec{0}, 0) = \sum_C D_{AB}^C(r, g, m, \mu) O_C(\vec{0}, 0) \quad (2.10)$$

where $g(m)$ is the renormalized coupling constant (quark mass) at scale μ . In the limit that $r = e^{-t}R \rightarrow 0$ ($t \rightarrow \infty$ with fixed R), the renormalization group analysis leads to

$$D_{AB}^C(r, g, m, \mu) \underset{r \rightarrow 0}{\sim} (-2\beta^{(1)} g^2 \log r) \gamma_{AB}^{C,(1)/(2\beta^{(1)})} D_{AB}^C(R, 0, 0, \mu), \quad (2.11)$$

where

$$\beta^{(1)} = \frac{1}{16\pi^2} \left(11 - \frac{2N_f}{3} \right) \quad (2.12)$$

is the QCD 1-loop beta-function coefficient, and

$$\gamma_{AB}^{C,(1)} = \gamma_A^{(1)} + \gamma_B^{(1)} - \gamma_C^{(1)}. \quad (2.13)$$

Here $\gamma_X^{(1)}$ is the 1-loop anomalous dimension of the operator O_X . An appearance of $D_{AB}^C(R, 0, 0, \mu)$ on the right-hand side tells us that it is enough to know the OPE only at tree level. From the above expression, β_{AB}^C in the previous subsection is given by

$$\beta_{AB}^C = \frac{\gamma_{AB}^{C,(1)}}{2\beta^{(1)}}. \quad (2.14)$$

Therefore our task is to calculate $\gamma_X^{(1)}$ for 3 and 6-quark operators.

3. Anomalous dimensions for 6-quark operators

3.1 6 quark operators

The OPE of two baryon operators at tree level is given by

$$B_1(x)B_2(0) = B_1(0)B_2(0) + x^\mu (\partial_\mu B_1(0))B_2(0) + \frac{1}{2}x^\mu x^\nu (\partial_\mu \partial_\nu B_1(0))B_2(0) + \dots \quad (3.1)$$

where the first term corresponds to the $L = 0$ contribution, the second to the $L = 1$, and so on. In this report we consider the $L = 0$ case (the first term) only. We denote the general form of a gauge invariant 3-quark operator as

$$B_{\alpha\beta\gamma}^{fgh}(x) \equiv B_\Gamma^F(x) = \varepsilon^{abc} q_\alpha^{a,f}(x) q_\beta^{b,g}(x) q_\gamma^{c,h}(x) \quad (3.2)$$

where α, β, γ are spinor, f, g, h are flavor, a, b, c are color indices of quark field q . The 6-quark operator is constructed from two 3-quark operators as

$$B_{\Gamma_1, \Gamma_2}^{F_1, F_2}(x) = B_{\Gamma_1}^{F_1}(x) B_{\Gamma_2}^{F_2}(x) \quad (3.3)$$

where $\Gamma_i = \alpha_i \beta_i \gamma_i$ and $F_i = f_i g_i h_i$ ($i = 1, 2$). Since quarks are fermions, there are some linear dependencies among 6-quark operators. We have to determine a set of independent 6-quark operators. It is not so easy, however, to find them using a quark field basis. Instead we use a simpler method mentioned below.

As the choice of the gauge fixing in perturbation theory, we take the covariant gauge with gauge parameter λ . Since both the 3-quark operator B_Γ^F and 6-quark operator $B_{\Gamma_1, \Gamma_2}^{F_1, F_2}$ are gauge

invariant, $\gamma_{AB}^{C,(1)}$ must be independent of the gauge parameter λ . The λ dependent term in the calculation of $\gamma_{AB}^{C,(1)}$ at 1-loop becomes

$$\lambda \left(9B_{\Gamma_1, \Gamma_2}^{F_1, F_2} + 3 \sum_{i,j=1}^3 B_{(\Gamma_1, \Gamma_2)[ij]}^{(F_1, F_2)[ij]} \right), \quad (3.4)$$

where the i -th index of abc and the j -th index of def is interchanged in $(abc, def)[ij]$. For example, $(\Gamma_1, \Gamma_2)[11] = \alpha_2 \beta_1 \gamma_1, \alpha_1 \beta_2 \gamma_2$ or $(\Gamma_1, \Gamma_2)[21] = \alpha_1 \alpha_2 \gamma_1, \beta_1 \beta_2 \gamma_2$. Note that the interchange occurs simultaneously for both Γ_1, Γ_2 and F_1, F_2 in the above formula. The gauge invariance implies eq.(3.4) = 0, which yields constraints among the 6-quark operators. For example, let us consider the case that $\Gamma_1, \Gamma_2 = \alpha\alpha\beta, \alpha\beta\beta$ and $F_1, F_2 = ffg, ffg$, for which the constraint becomes

$$\begin{aligned} 3 \left(3B_{\alpha\alpha\beta, \alpha\beta\beta}^{ffg, ffg} + (3-2)B_{\alpha\alpha\beta, \alpha\beta\beta}^{ffg, ffg} + B_{\alpha\alpha\alpha, \beta\beta\beta}^{fff, fgg} + (2-1)B_{\alpha\beta\beta, \alpha\alpha\beta}^{fgg, fff} \right) &= 0 \\ \Rightarrow 4B_{\alpha\alpha\beta, \alpha\beta\beta}^{ffg, ffg} + B_{\alpha\alpha\alpha, \beta\beta\beta}^{fff, fgg} + B_{\alpha\beta\beta, \alpha\alpha\beta}^{fgg, fff} &= 0, \end{aligned} \quad (3.5)$$

where minus signs in the first line come from the property that $B_{\Gamma_2, \Gamma_1}^{F_2, F_1} = -B_{\Gamma_1, \Gamma_2}^{F_1, F_2}$. There are no further relations among 6-quark operators beyond (3.4).

3.2 1-loop contributions

Only the divergent part of the gauge invariant contribution at 1-loop is necessary to calculate the anomalous dimension of 6-quark operators at 1-loop. The building block of 1-loop calculations is the gluon exchange between two quark lines. If both two quark lines belong to one operator, either $B_{\Gamma_1}^{F_1}$ or $B_{\Gamma_2}^{F_2}$, the contribution is canceled by the renormalization factor of the 3-quark operator $B_{\Gamma_i}^{F_i}$, so that the divergent term does not contribute to $\gamma_{AB}^{C,(1)} = \gamma_A^{(1)} + \gamma_B^{(1)} - \gamma_C^{(1)}$. If one quark line comes from $B_{\Gamma_1}^{F_1}$ and the other from $B_{\Gamma_2}^{F_2}$, the divergent term contributes to $\gamma_{AB}^{C,(1)}$. Suppose that one quark line has indices (α_A, f_A) at one end and (α_1, f_1) at the other end and the other quark line has (α_B, f_B) and (α_2, f_2) . The divergent contribution from the 1-gluon exchange can be expressed as

$$\frac{g^2}{96\pi^2} \frac{1}{\varepsilon} \left\{ \delta_{f_1 f_A} \delta_{f_2 f_B} [\delta_{\alpha_1 \alpha_A} \delta_{\alpha_2 \alpha_B} - 2\delta_{\alpha_2 \alpha_A} \delta_{\alpha_1 \alpha_B}] + 3\delta_{f_2 f_A} \delta_{f_1 f_B} [\delta_{\alpha_2 \alpha_A} \delta_{\alpha_1 \alpha_B} - 2\delta_{\alpha_1 \alpha_A} \delta_{\alpha_2 \alpha_B}] \right\} \quad (3.6)$$

for $(\alpha_1, \alpha_2) \in (R, R)$ or $(\alpha_1, \alpha_2) \in (L, L)$, where R and L means the right and the left handed component of the spinor indices. Other combinations, (R, L) or (L, R) , vanish. This property comes from the fact that the gluon coupling to quarks is chirally symmetric and the chirally non-symmetric quark mass term does not contribute to the divergence.

3.3 Chiral decomposition of 6-quark operators

The physical nucleon operators are constructed from general 3-quark operators as

$$B_{\alpha}^f = B_{\alpha\beta\gamma}^{fgh} (C\gamma_5)_{\beta\gamma} (i\tau_2)^{gh} \quad (3.7)$$

where C is the charge conjugation matrix, f, g, h are u or d , and τ_2 is the Pauli matrix in the flavor space. The spinor index α is restricted to the positive energy component such that $\alpha = 1, 2$ in the Dirac representation of the γ matrices.

In this report we consider $L = 0$ two nucleon states, which are 1S_0 and 3S_1 . Here we use the notation $^{2S+1}L_J$ where S is the total spin, L is the orbital angular momentum and J is the total angular momentum. The 6-quark operator for 1S_0 , which is the spin-singlet and isospin-triplet state, and for 3S_1 (the spin-triplet and isospin-singlet state) are given by

$$BB(^1S_0) = (i\sigma_2)_{\alpha\beta} B_\alpha^f B_\beta^f, \quad BB(^3S_1) = (i\tau_2)^{fg} B_\alpha^f B_\alpha^g, \quad (3.8)$$

where the summation is taken for the repeated index. Both 6-quark operators have the following chiral decomposition:

$$BB = B_{LL}B_{LL} + B_{LL}B_{LR} + B_{LL}B_{RL} + B_{LR}B_{LR} + B_{LL}B_{RR} + (L \leftrightarrow R) \quad (3.9)$$

where B_{XY} means $B_{\alpha, [\beta\gamma]}$ with $\alpha \in X$ and $[\beta, \gamma] \in Y$ for $X, Y = R$ or L .

3.4 Anomalous dimensions

We now give our main results in this report. We define

$$\gamma_{AB}^{C,(1)} = \gamma_A^{(1)} + \gamma_B^{(1)} - \gamma_C^{(1)} \equiv \frac{1}{32\pi^2} \gamma. \quad (3.10)$$

The eigen-operators of the anomalous dimension matrix γ are found to correspond to the chirally decomposed operators in the previous subsection. We give the eigenvalue of each operator in table 1, which shows that the operator with zero anomalous dimension always exists and other anomalous dimensions are all negative for both 1S_0 and 3S_1 states. This corresponds to the case (2) of the general discussion in the section 2.1:

- (2) ($\beta_X = 0, \beta_Y = -3/(33 - 2N_f)$) for 1S_0 and ($\beta_X = 0, \beta_Y = -1/(33 - 2N_f)$) for 3S_1 .

The appearance of zero eigenvalues in both 1S_0 and 3S_1 states can be understood as follows. As mentioned in section 3.2, 1-loop contributions to the $\gamma_{AB}^{C,(1)}$ exist only if spinor indices of two quark lines, one from Γ_1 the other from Γ_2 in $B_{\Gamma_1} B_{\Gamma_2}$, belong to the same chirality (left or right). Since $B_{LL}B_{RR} + B_{RR}B_{LL}$ has no such combination, $\gamma_{AB}^{C,(1)}$ is always zero for this type of operators. As pointed out before, this property is the consequence of the chiral symmetry in QCD interactions.

Table 1: The eigenvalue γ for anomalous dimension of each eigen operator in 1S_0 and 3S_1 states. In the table (XY, ZW) means $B_{XY}B_{ZW} + (R \leftrightarrow L)$.

	(LL, LL)	(LL, LR)	(LL, RL)	(LR, LR)	(LR, RL)	(LL, RR)
$\gamma(^1S_0)$	-12	-4	-8	-8	-6	0
$\gamma(^3S_1)$	-28/3	-4/3	-8	-16/3	-6	0

4. Conclusion

The OPE and renormalization group analysis in QCD predicts the universal functional form of the nucleon-nucleon potential at short distance:

$$V(r) \simeq \frac{D_Y(E)}{D_X(E)} \frac{-\beta_Y}{r^2(-\log r)^{1-\beta_Y}}, \quad r \rightarrow 0, \quad (4.1)$$

which is a little weaker than a $1/r^2$ singularity. We obtain

$$\beta_Y(^1S_0) = -\frac{3}{33-2N_f}, \quad \beta_Y(^3S_1) = -\frac{1}{33-2N_f}. \quad (4.2)$$

The anomalous dimension spectrum, however, cannot alone tell whether the potential at short distance is repulsive or attractive. If we evaluate $D_X(E)$ and $D_Y(E)$ by the chiral effective theory at the leading order (*i.e.* the tree level), we obtain

$$\frac{D_Y(E)}{D_X(E)}(^1S_0) = \frac{2E}{m_N}, \quad \frac{D_Y(E)}{D_X(E)}(^3S_1) = \frac{2m_N}{E}, \quad (4.3)$$

where $E = \sqrt{\vec{p}^2 + m_N^2}$ and \vec{p} is the relative momentum of two nucleons. We find a repulsive core for both states. In particular, in the low energy region such that $\vec{p}^2 \ll m_N^2$, the repulsive potential at short distance is almost energy independent, since

$$\frac{D_Y(E)}{D_X(E)}(^1S_0) \simeq \frac{D_Y(E)}{D_X(E)}(^3S_1) \simeq 2. \quad (4.4)$$

Furthermore no low energy constant of the effective theory is required at leading order to obtain the above result.

There are several interesting extensions of the analysis using the OPE. The extension of the OPE analysis to the 3-flavor case may reveal the nature of the repulsive core in the baryon-baryon potentials. Since quark mass can be neglected in this OPE analysis, the calculation can be done in the exact SU(3) symmetric limit. It is also interesting to investigate the existence or the absence of the repulsive core in the 3-body nucleon potential. In this case, we have to calculate anomalous dimensions of 9-quark operators at 2-loop level. More precise evaluations of the matrix element $D_X(E) = \langle 0|O_X|E \rangle$ would be preferable. The most straightforward extension is to analyze the tensor force and *LS* force by the OPE. Preliminary results indicate that

$$V_T(r) \simeq C_0 + C_1(-\log r)^{\beta_Y-1}, \quad V_{LS}(r) \simeq -\frac{12}{m_N r^2}, \quad (4.5)$$

where *LS* force has the strong attractive core at short distance, while no repulsive core exists for the tensor potential. Absence of the repulsive core predicted in the tensor force is consistent with recent numerical simulations[4, 5].

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