

## New ideas for $g - 2$ on the lattice

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Using iso-spin symmetry the quark-connected and disconnected contributions to the hadronic vacuum polarisation in a theory with  $N_f = 2$  flavours can be described as independent correlation functions, respectively. We show how this allows to use twisted boundary conditions for the connected contribution in order to improve the  $q^2$ -resolution in lattice QCD. Furthermore we derive an exact relation between the connected and the disconnected contributions at NLO in chiral perturbation theory. We discuss extensions to theories with more than 2 dynamical flavours.

*The XXVII International Symposium on Lattice Field Theory - LAT2009*

*July 26-31 2009*

*Peking University, Beijing, China*

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$e^+e^-$ -annihilation	$690.3(5.3) \times 10^{-10}$	[1]
LQCD	$713(15) \times 10^{-10}$	[2]
	$748(21) \times 10^{-10}$	
	$446(23) \times 10^{-10}$	[3]
	in progress	[4]

**Table 1:** Some recent results for the leading hadronic contribution to the muon anomalous moment,  $a_\mu^{\text{had}}$ .

## 1. Introduction

As recently summarized in [1] the current experimental value for the full muon anomalous magnetic moment is  $a_\mu = 11659208(6.3) \times 10^{-10}$  which has to be compared to the Standard Model prediction of  $a_\mu = 11659179(6.5) \times 10^{-10}$ . Although there are discussions about how to correctly estimate the size of the systematic errors there persists a tension between experiment and theory which amounts to around three standard deviations ( $3.2\sigma$  in the comparison quoted above). The muon anomalous moment is a remarkable observable in that all three sectors of the SM contribute considerably. While perturbation theory is employed in order to predict QED and weak contributions, the leading and next-to-leading hadronic contributions are non-perturbative effects. Lacking reliable and precise theory computations from first principles the current SM predictions for the leading hadronic contribution  $a_\mu^{\text{had}}$  are derived from experimental measurements of  $e^+e^-$ -annihilation into hadrons leading to a world average [1] for the hadronic contribution of  $a_\mu^{\text{had}} = 690.3(5.3) \times 10^{-10}$ .

Table 1 summarizes previous attempts to compute  $a_\mu^{\text{had}}$  in lattice QCD in comparison with the determination from  $e^+e^-$  annihilation. Without going into the technical details of the various attempts to compute  $a_\mu^{\text{had}}$  on the lattice it is obvious that currently the purely theoretical predictions can't match the level of precision which can be reached via the experimental determination. But yet, an independent confirmation of the result from  $e^+e^-$ -annihilation matching it in precision and by means of a SM calculation is clearly desirable.

The leading hadronic contribution to the muon anomalous moment is the convolution integral

$$a_\mu^{\text{had}} = 4\pi^2 \left( \frac{\alpha_{EM}}{\pi} \right)^2 \int_0^\infty dK^2 f(K^2) (\Pi(K^2) - \Pi(0)), \quad (1.1)$$

where  $K$  is the Euclidean momentum and the function  $f(K^2)$  in (1.1) diverges for  $K^2 \rightarrow 0$  (see e.g. the discussion in [5]).  $\Pi(K^2)$  is the vacuum polarization which for a theory with  $N_f$  quark flavours is defined through

$$\Pi_{\mu\nu}^{(N_f)}(q) = (q_\mu q_\nu - g_{\mu\nu} q^2) \Pi(q^2) = i \int d^4x e^{iqx} \langle j_\mu^{(N_f, EM)}(x) j_\nu^{(N_f, EM)}(0) \rangle, \quad (1.2)$$

where  $j_\mu^{(N_f, EM)}$  is the corresponding electromagnetic current.

Apart from the usual sources of systematic errors in simulations of lattice QCD (unphysically heavy quark mass, finite volume, cut-off effects) we identify three sources of systematic errors that

have to be brought under control in order to be able to make precise predictions for  $a_\mu^{\text{had}}$  from first principles:

- 1) since  $f(K^2)$  in (1.1) diverges for  $K^2 \rightarrow 0$  it is crucial to have a good momentum resolution for  $\Pi$  for small values of  $K$ . Given that the smallest Fourier mode in eqn. (1.2) on the lattice for typical simulation parameters is  $\frac{2\pi}{2L} \approx 250\text{MeV}$  ( $L$  being the spatial extent of the lattice and usually the time-extent is chosen as  $T = 2L$ ) one currently extrapolates results for  $\Pi(q^2)$  from the region where lattice data is available using some phenomenological ansatz
- 2) carrying out the Wick contraction in (1.2) reveals contributions from quark-disconnected diagrams which are notoriously hard to compute in lattice QCD
- 3) Aubin and Blum [2] found strong signs for vector dominance in their lattice data which has to be modelled in order to allow for a reliable description of the data within an effective theory frame work.

Here we suggest a computational strategy that should allow to systematically reduce the uncertainties due to 1) and 2). Concerning 3) we will follow [2] as long as the simulated quark masses are unphysically large. They used a model where a vector particle is added to the chiral Lagrangian as an additional degree of freedom in order to better describe their lattice data for  $\Pi(q^2)$ . This work is part of a major effort by the Mainz group aiming at a precision computation of the leading hadronic contribution to the muon anomalous moment (see also Hartmut Wittig's talk [6]).

## 2. New strategy for $N_f = 2$

Partially twisted boundary conditions [7, 8, 9, 10, 11, 12, 13, 14, 15] have now been used successfully to improve computations of observables of processes that depend on the hadron momentum. However, as already stated in [9] the net effect of the twist in flavour neutral hadrons, like e.g. the  $\pi^0$  is zero since the twist of the quark and the anti-quark cancel. The situation is similar here - the electromagnetic current is flavour diagonal, hence a naive application of (partially) twisted boundary conditions is ruled out.

The correlator entering eqn. (1.2) for  $N_f = 2$  is

$$\langle j_\mu^{(2,EM)} j_\nu^{(2,EM)} \rangle = \frac{4}{9} \langle j_\mu^{uu} j_\nu^{uu} \rangle - \frac{2}{9} \langle j_\mu^{uu} j_\nu^{dd} \rangle - \frac{2}{9} \langle j_\mu^{dd} j_\nu^{uu} \rangle + \frac{1}{9} \langle j_\mu^{dd} j_\nu^{dd} \rangle, \quad (2.1)$$

where we make the flavor content of the quark bilinear currents explicit and where we have factorized out the electromagnetic charges of the quarks in units of  $e$  on the r.h.s.. The first and the last correlator on the r.h.s. receive contributions from both a quark-connected and a disconnected diagram and the second and third correlator are disconnected, only. After carrying out the Wick contractions and using iso-spin symmetry ( $m_q \equiv m_u = m_d$ ), the two point function one has to compute in a lattice simulation is

$$C_{\mu\nu}^{(2,EM)}(q) = \sum_x e^{iqx} \left\{ \left\langle \frac{5}{9} \text{Tr} \{ \bar{q}(x) \gamma_\nu q(x) \bar{q}(0) \gamma_\mu q(0) \} + \frac{1}{9} \text{Tr} \{ \gamma_\nu q(x) \bar{q}(x) \} \text{Tr} \{ \gamma_\mu q(0) \bar{q}(0) \} \right\rangle \right\}, \quad (2.2)$$

which again using iso-spin can be written in the equivalent form

$$C_{\mu\nu}^{(2,EM)}(q) = \sum_x e^{iqx} \left\{ \frac{5}{9} \underbrace{\left\langle \text{Tr} \{ \bar{u}(x) \gamma_\nu d(x) \bar{d}(0) \gamma_\mu u(0) \} \right\rangle}_{\equiv C_{\mu\nu}^{(2,\text{con})}(q)} + \frac{1}{9} \underbrace{\left\langle \text{Tr} \{ \gamma_\nu d(x) \bar{d}(x) \} \text{Tr} \{ \gamma_\mu u(0) \bar{u}(0) \} \right\rangle}_{\equiv C_{\mu\nu}^{(2,\text{disc})}(q)} \right\}. \quad (2.3)$$

By using iso-spin we are able to express the vacuum polarization in terms of two correlation functions  $C_{\mu\nu}^{(2,\text{con})}(q)$  and  $C_{\mu\nu}^{(2,\text{disc})}(q)$  with their individual continuum and infinite volume limits. The crucial point to note is that in this way the coupling of the photon to the quark in the connected piece has been replaced by a coupling to a flavour non-diagonal current and partially twisted boundary conditions can be applied. We note that a similar trick has been used for computations of the electromagnetic pion form factor in [16, 17].

We are still left with the question of how to treat the disconnected part. To this end we resort to the description of the vacuum polarization in  $SU(2)$  chiral perturbation theory at NLO. In  $SU(2)$  the electromagnetic current receives a singlet contribution,

$$\begin{aligned} j_\mu^{uu} &= \frac{1}{2} \bar{\psi} (\sqrt{2} \sigma_0 + \sigma_3) \gamma_\mu \psi & j_\mu^{dd} &= \frac{1}{2} \bar{\psi} (\sqrt{2} \sigma_0 - \sigma_3) \gamma_\mu \psi \\ j_\mu^{ud} &= \frac{1}{2} \bar{\psi} (\sigma_1 + i \sigma_2) \gamma_\mu \psi & j_\mu^{du} &= \frac{1}{2} \bar{\psi} (\sigma_1 - i \sigma_2) \gamma_\mu \psi \end{aligned} \quad (2.4)$$

where  $\psi^T = (u, d)$ ,  $\sigma_i$  for  $i = 1, 2, 3$  are the Pauli matrices and  $\sigma_0 = \sqrt{1/2} \mathbf{1}_{2 \times 2}$ . The effective theory that describes the  $N_f = 2$  vacuum polarization is  $SU(2)$  chiral perturbation theory as formulated in [18] but including at  $O(p^4)$  terms with non-vanishing flavour-trace,

$$\mathcal{L}^{(4)} = \underbrace{2l_5 \langle U^\dagger \hat{L}_{\mu\nu} U \hat{R}^{\mu\nu} \rangle + 4h_2 \langle \hat{R}_{\mu\nu} \hat{R}^{\mu\nu} + \hat{L}_{\mu\nu} \hat{L}^{\mu\nu} \rangle}_{\text{flavour off-diagonal}} + \underbrace{4h_s \langle R_{\mu\nu} + L_{\mu\nu} \rangle \langle R^{\mu\nu} + L^{\mu\nu} \rangle}_{\text{flavour-diagonal}}, \quad (2.5)$$

where angular brackets indicate a flavour trace and the *hat* indicates that the latter has been subtracted (cf. [19] for further details on the notation). Note the additional low energy constant  $h_s$  multiplying the flavour diagonal term.

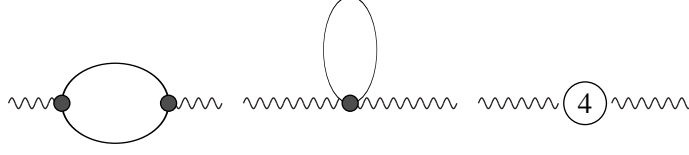
The diagrams contributing at NLO are illustrated in figure 1 and our preliminary result is

$$\begin{aligned} \Pi_{\mu\nu}^{(2)}(q) &= \Pi_{\mu\nu}^{(3,3)}(q) + \frac{1}{9} \Pi_{\mu\nu}^{(0,0)}(q) \\ &= -(q_\mu q_\nu - g_{\mu\nu} q^2) \left( i4\bar{B}_{21}(q^2, m_\pi^2) + 2l_5(\mu_0) + 4h_2(\mu_0) + \frac{4}{9}h_s + \frac{1}{48\pi^2} \ln \left( \frac{m_\pi^2}{\mu_0^2} \right) \right), \end{aligned} \quad (2.6)$$

where  $\mu_0$  is the renormalization scale and  $\bar{B}_{21}(q^2, m_\pi^2)$  is a loop integral (cf. [19]). The contributions  $\Pi_{\mu\nu}^{(a,a)}$  ( $a = 0, 1, 2, 3$ ) are the effective theory descriptions of 2pt-correlators constructed of the currents  $c \bar{\psi} \gamma_{\mu/\nu} \sigma_a \psi$ , where  $c \in \mathbb{C}$  is a normalization. We add that  $\Pi^{(0,0)}(q^2) = \Pi^{(0,0)}$  is momentum independent at NLO and only depends on the low energy constant  $h_s$ .

Alternatively it is possible to express the quark-connected and disconnected contributions individually in the effective theory (the linear combination of which should of course again yield eqn. (2.6)). We computed the corresponding expressions as

$$\begin{aligned} \Pi^{(2,\text{con})}(q^2) &= \frac{10}{9} \Pi^{(3,3)}(q^2), \\ \Pi^{(2,\text{disc})}(q^2) &= \frac{1}{9} (\Pi^{(0,0)} - \Pi^{(3,3)}(q^2)). \end{aligned} \quad (2.7)$$



**Figure 1:** Diagrams in chiral perturbation theory that contribute to the vacuum polarization at NLO.

Since  $\Pi^{(0,0)}$  is independent of  $q^2$ , the quark-disconnected part turns out to have the same momentum dependence as the connected part up to a finite shift proportional to  $h_s$ ,

$$\Pi^{(2,\text{disc})}(q^2) = \frac{1}{9}\Pi^{(0,0)} - \frac{1}{10}\Pi^{(2,\text{con})}(q^2). \quad (2.8)$$

Assuming the validity of chiral perturbation theory at NLO the knowledge of  $h_s$  allows to fully predict the quark-disconnected contribution.

The leading hadronic contribution to the muon anomalous moment is computed from

$$\hat{\Pi}(q^2) \equiv \Pi(q^2) - \Pi(0). \quad (2.9)$$

One immediate observation is, that at NLO in chiral perturbation theory this expression is free of low energy constants since the loop integrals carry all the momentum dependence. A crucial consequence which to our knowledge hasn't been realized before is, that e.g. in the case of  $SU(2)$ ,

$$\frac{\hat{\Pi}(q^2)|_{\text{disc}}}{\hat{\Pi}(q^2)|_{\text{con}}} = -\frac{1}{10}. \quad (2.10)$$

This is a remarkable result which tells us that at this order of the effective theory the disconnected part shifts the central value of the connected part by minus ten per cent.

### 3. Conclusions and outlook

To summarize, we have shown how partially twisted boundary conditions can be used to improve the momentum resolution in lattice computations of the hadronic vacuum polarization of a photon. It was shown how to analytically predict at NLO in chiral perturbation theory the contribution of quark-disconnected diagrams to the leading hadronic contribution of the muon anomalous magnetic moment. It turns out that it's effect is to reduce the contribution of the connected part by 10%.

We are currently working on extending these arguments to the case of  $N_f = 2 + 1$  flavours. One complication there is that the iso-spin argument that allowed us to write the quark-connected and disconnected pieces as individual correlation functions in the case of  $N_f = 2$  does not work straight forwardly since the strange quark doesn't have an iso-spin partner. Naively the  $\bar{s}\gamma_\mu s$ -contribution to the EM current can therefore not be written in terms of a flavor off-diagonal current. Our strategy is to extend the flavour group from  $SU(3)$  to  $SU(3 + 1|1)$ , i.e. to a graded flavour group with an additional partially quenched quark (which we call  $r$ -quark) that behaves like a mass-degenerate *iso-partner* of the  $s$ -quark. Then the quark-connected and disconnected pieces can be written in terms of individual correlation functions.

Since quark-disconnected diagrams are extremely difficult to compute on the lattice, we will investigate if our method to predict their contribution in the frame work of chiral perturbation theory can be applied to other phenomenologically interesting observables in QCD.

Following [9] we are also computing the finite volume effects for the vacuum polarization including the effect of partial twisting.

Another task that remains to be finished is the inclusion of vector degrees of freedom in order to be able to extrapolate the lattice data to physical quark masses and to assess the vector particle's effect on the stability of eqn. (2.10).

As a joint effort the Mainz group has implemented correlation functions relevant for a lattice computation of  $a_\mu^{\text{had}}$  in a C-code. First computations on gauge configurations of  $N_f = 2$  non-perturbatively improved Wilson fermions which were generated as a collaborative effort within CLS (cf. the talks by Hartmut Wittig's [6] and Stefan Schäfer [20] at this conference) have been carried out on the Mainz Wilson-Cluster.

**Acknowledgements:** We warmly thank Dalibor Djukanovic, Jambul Gegelia, Chris Sachrajda, Stefan Scherer and Hartmut Wittig for valuable discussions.

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