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Determination of α_s in 2+1-flavor QCD through vacuum polarization function

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We show a lattice determination of the strong coupling constant $\alpha_s(M_Z)$ from the vacuum polarization function (VPF) calculated on $N_f = 2+1$ QCD gauge configurations generated by the JLQCD and TWQCD collaborations with dynamical overlap fermions. Fitting lattice data of VPF to a perturbative formula supplemented by the operator product expansion (OPE), we determine the parameter $\Lambda^{(3)}_{\overline{MS}}$. Evolving the running coupling constant to the Z boson mass scale, we obtain $\alpha_s(M_Z) = 0.1181(8)(^{+4}_{-2})(^{+5}_{-6})$ as our preliminary result, where the first error is statistical one, and the second and third errors are systematic errors due to discretization effect and other remaining uncertainties in this calculation.

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1. Introduction

Lattice QCD provides a natural framework to calculate the vacuum polarization functions (VPFs). In fact, the VPFs in the vector (V) and axial-vector (A) channels can be extracted from two-point correlation function with space-like momentum insertions, which cover the the energy region from zero to the lattice cutoff. Our recent study [1] in $N_f = 2$ QCD proposed to use VPFs to determine the strong coupling constant by matching them with the perturbative calculations. This method gives a consistent result with in other lattice calculations at $N_f = 2$ [2, 3]. We extend this study to the realistic $2+1$-flavor QCD with dynamical light and strange quarks and determine the strong coupling constant through OPE formula including terms up to a mass dimension $n = 4$. One of the advantages of our approach is that we can use the gauge configurations already produced for spectrum studies. Another advantage is that the finite size correction is under control, since we consider physical observables at short distances. A similar strategy was also proposed in [4], which considers the quarkonium two-point function instead.

Simulations are carried out on the $N_f = 2+1$ dynamical overlap fermion configurations generated by the JLQCD and TWQCD collaborations. Sea quark masses in lattice units are am_{ud}= 0.015, 0.025, 0.035, and 0.050 for the up and down quarks, and am_s= 0.08, 0.10 for the strange quark [5]. The valence light quark masses are set to be equal to the sea quark masses. The set of up and down quark masses covers the range of $0.2m_s$–$0.8m_s$, while the set of the strange quark mass is close to its physical value. The inverse lattice spacing is $a^{-1} = 1.83$ GeV, which is obtained from the static quark potential with $r_0 = 0.49$ fm, and lattice volume is $16^3 \times 48$. The physical volume is thus about $(1.8\text{ fm})^3$. Topological charge of our configurations is fixed to zero, but this only gives small finite size effects to long distance physics [6]. We expect that this gives negligible effects to the short distance physics as considered in this work.

2. Lattice calculation of vacuum polarization functions

In the continuum theory the transverse ($\Pi_J^{(1)}$) and longitudinal ($\Pi_J^{(0)}$) parts of VPF are defined through the two point correlation function of vector ($J = V$) and axial-vector ($J = A$) currents,

$$\langle J_\mu J_\nu \rangle(q) = (\delta_{\mu\nu} q^2 - q_\mu q_\nu)\Pi_J^{(1)}(q^2) - q_\mu q_\nu \Pi_J^{(0)}(q^2) \tag{2.1}$$

with four-dimensional momentum q after a Fourier transformation. The longitudinal part in the vector channel vanishes $\Pi_V^{(0)}(q^2) = 0$, while in the axial-vector channel it is proportional to the quark mass. The above parameterization respects the Ward-Takahashi (WT) identity.

Defining $\Pi_J(q^2) \equiv \Pi_J^{(0)}(q^2) + \Pi_J^{(1)}(q^2)$, the perturbative expansion of VPF for $\Pi_{V+A}(q^2) \equiv \Pi_V(q^2) + \Pi_A(q^2)$ in the $\overline{MS}$ scheme is generally given by

$$\begin{aligned}\Pi_{V+A}|_{\text{pert}}(Q^2, \alpha_s) \;=\; & c + C_0^{V+A}(Q^2, \mu^2, \alpha_s) + C_m^{V+A}(Q^2, \mu^2, \alpha_s)\frac{m_r^2(Q^2)}{Q^2} \\ & + \sum_{q=u,d,s} C_{\bar{q}q}^{V+A}(Q^2, \alpha_s)\frac{\langle m\bar{q}q\rangle}{Q^4} + C_{GG}^{V+A}(Q^2, \alpha_s)\frac{\langle(\alpha_s/\pi)GG\rangle}{Q^4} + \mathcal{O}(Q^{-6})\end{aligned} \tag{2.2}$$

with analytic functions C_X^{V+A} (X= 0, $\bar{q}q$, and GG) and vacuum expectation values of operators up to dimension 4. The first term c in (2.2) is a scheme-dependent constant and does not contribute

to the Adler function $Q^2 d\Pi/dQ^2$, which is a physical observable. The second and third terms are, on the other hand, scheme independent and perturbatively calculated at three loop order in the $\overline{MS}$ scheme [9, 10, 11]. The third term contains the running mass $m_r(Q^2)$ [12, 13]. The fourth and fifth terms correspond to higher order effects in OPE. The Wilson coefficients are calculated at three-loop order [12]. An important fact is that once the divergence of the current is renormalized, the perturbatively calculated VPFs must be equivalent among different renormalization scheme up to neglected higher order terms except for the scheme dependent constant term c. This also applies to the VPF obtained from lattice calculation when the chiral symmetry is exactly preserved. At finite lattice spacings, there are discretization effects of $O(a^2Q^2)$ which can be extrapolated away towards the continuum limit. It should be noted that the exact chiral symmetry of the overlap fermion partly eliminate the unphysical terms of $O(a^2Q^2)$ to satisfy the WT identities. We therefore use (2.2) to fit VPF. This is important to reduce the systematic errors and obtain a precise value of α_s. In our previous study in $N_f = 2$ QCD [1], we had to use more complicated method to extract the physical VPFs, because we used non-conserving (aixal-)vector currents. (Also, see below.)

With the overlap fermion formulation, the conserved vector current has a complicated form. By generally writing in a bilinear form $V_\mu(x) = \sum_{w,z} \bar{q}(w) K_\mu(w,z|x) q(z)$ with a non-local kernel $K_\mu(w,z|x)$, where K_μ is determined such that it forms a Noether current of the overlap fermion action $S = \sum_{x,y} \bar{q}(x) D_{ov}(x,y) q(y)$ under a local vector transformation [7]. The flavor non-singlet conserved vector and axial-vector currents are thus written as

$$V_\mu^{a\,\mathrm{cv}}(x) = \sum_{w,z} \bar{q}(w) T^a K_\mu(w,z|x) q(z), \quad A_\mu^{a\,\mathrm{cv}}(x) = \sum_{w,z} \bar{q}(w) T^a K_\mu(w,z|x) [\hat{\gamma}_5 q](z), \tag{2.3}$$

where T^a denotes the generator of $SU(N_f)$ and $\hat{\gamma}_5(x,y) \equiv \gamma_5(\delta_{x,y} - D_{ov}(x,y)/m_0)$ with $m_0 = 1.6$.

In this study we consider the two point correlation function of conserved and local currents, $\langle T\{J_\mu^{\mathrm{cv}}(x) J_\nu^{\mathrm{loc}}(0)\}\rangle$, where J_μ^{loc} is either $V_\mu^{\mathrm{loc}}(x) = Z\bar{q}(x)\gamma_\mu q(x)$ or $A_\mu^{\mathrm{loc}}(x) = Z\bar{q}(x)\gamma_\mu\gamma_5 q(x)$. The renormalization constant Z is determined through the RI/MOM scheme as $Z = 1.39$ [8]. The WT identities in the momentum space are then written as

$$\sum_\mu \hat{Q}_\mu \langle V_\mu^{\mathrm{cv}} V_\nu^{\mathrm{loc}}\rangle(Q) = 0, \quad \sum_\mu \hat{Q}_\mu \langle A_\mu^{\mathrm{cv}} A_\nu^{\mathrm{loc}}\rangle(Q) - 2m_q \langle P A_\nu^{\mathrm{loc}}\rangle(Q) + \langle \delta_A A_\nu^{\mathrm{loc}}\rangle = 0. \tag{2.4}$$

with a lattice momentum $\hat{Q}_\mu = 2\sin(aQ_\mu/2)e^{-iaQ_\mu/2}$, which corresponds to a backward derivative operator and $aQ_\mu = 2\pi n_\mu/L_\mu$ with $L_{\mu=1\sim4}$ the extent of the lattice in the μ-th direction. The second term for the axial-WT identity in (2.4) represents the correlation function of pseudo-scalar operator $P(x) = \bar{q}(x)\gamma_5(1 - D_{ov}/m_0)q(x)$ and the local axial-vector current. The third term is an extra contact term, which is a constant in the momentum space, derived from the axial transformation of A_ν^{loc} on the lattice . This term vanishes in the massless limit, and its magnitude is negligibly small. Vector and axial-vector VPFs are now given by

$$\langle J_\mu^{\mathrm{cv}} J_\nu^{\mathrm{loc}}\rangle(Q) \;=\; (\delta_{\mu\nu}|\hat{Q}|^2 - \hat{Q}_\mu^* \hat{Q}_\nu)\Pi_J^{(1)}(Q) - \hat{Q}_\mu^* \hat{Q}_\nu \Pi_J^{(0)}(Q) + \Delta_{\mu\nu}^J(Q). \tag{2.5}$$

Here, $\Pi_V^{(0)}(Q) = 0$ because of the conserved current, while $\Pi_A^{(0)}(Q)$ represents a remnant due to PCAC. The transverse part $\Pi_J^{(1)}(Q)$ is extracted as

$$\Pi_J^{(1)}(Q) = \langle J_\mu^{\mathrm{cv}} J_\mu^{\mathrm{loc}}\rangle(Q)/(|\hat{Q}|^2 - \hat{Q}_\mu^* \hat{Q}_\nu,). \tag{2.6}$$

The additional term $\Delta^J_{\mu\nu}(Q)$ comes from the violation of the current conservation for the local current $J^{\rm loc}_\nu$. $\Delta^J_{\mu\nu}(Q)$ can be expanded in terms of small $a\hat{Q}_\mu$ as

$$\Delta^J_{\mu\nu}(Q) = \sum_{m,n=1} \left(\delta_{\mu\nu}\sum_\rho |\hat{Q}_\rho|^{2m} - |\hat{Q}_\mu|^{2(m-1)}\hat{Q}^*_\mu\hat{Q}_\nu\right) a^{2n}Q_\nu^{2n}F_{mn}(\hat{Q}_\nu). \tag{2.7}$$

In this analysis we ignore this contribution to VPFs, because of its smallness in the range $(aQ)^2 < 1$, and will later estimate associated systematic errors. The VPF $\Pi_{V+A}(Q^2)$ thus extracted in our lattice calculations is plotted in Fig. 1. The statistical fluctuation is small enough to fit its Q^2 dependence and to extract α_s.

3. Fit with the perturbative formula

Here we discuss a fit of the lattice VPF data to the OPE formula. In this analysis the renormalization scale is set to $\mu = 2$ GeV, though the final results should not depend on μ up to higher order corrections. The gluon condensate $\langle(\alpha_s/\pi)GG\rangle$ is defined only through the perturbative expression like (2.2) due to the renormalon ambiguity [14], and we treat $\langle(\alpha_s/\pi)GG\rangle$ as a free parameter to describe the associated $1/Q^4$ correction. On the other hand, the quark condensate $\langle\bar{q}q\rangle$ is well-defined in the massless limit because there is no mixing with lower dimensional operators thanks to the exact chiral symmetry of the overlap fermion. Although it has a m/a^2 divergence at finite quark masses, this gives a tiny correction ($\sim$ 0.1–0.2%) compared to $\langle m\bar{q}q\rangle/Q^4$ at our value of lattice spacing. We therefore neglect this contribution. Then the quark mass dependence of $\Pi_{V+A}|_{\rm pert}(Q^2)$, which consists of the third and fourth terms in (2.2), is determined by α_s only, once the quark condensate is determined elsewhere. The third term is given by $m_r(Q^2) = Z_m(2\,\mathrm{GeV})m_q \times [\hat{m}(Q^2)/\hat{m}(4\,\mathrm{GeV}^2)]$ with $Z_m(2\,\mathrm{GeV}) \simeq 0.833$ [15]. In the fourth term, quark condensate is an input parameter, $\langle\bar{q}q\rangle = -[0.236(7)(+13)\ \mathrm{GeV}]^3$, which is taken from $N_f = 2$ hadron spectroscopy [15]. While the precise value of the quark condensate obtained in [16] has not been used here so far, this makes only a tiny difference to VPF since $C^{V+A}_{\bar{q}q}(Q^2,\alpha_s)\langle m\bar{q}q\rangle/Q^4$ is relatively small.

There are three unknown parameters, $\alpha_s(Q^2)$, c and $\langle(\alpha_s/\pi)GG\rangle$, in the fit of VPF using (2.2). The QCD scale $\Lambda^{(3)}_{\overline{MS}}$ controls the running coupling constant $\alpha_s(Q^2)$, which is evaluated using a four loop order formula [17]. Figure 1 shows a $(aQ)^2$ dependence of $\Pi_{V+A}(Q^2)$ in a window $0.4 \le (aQ)^2 \le 1.0$. Fit curves shown in this plot are (2.2) with the value for the unknown parameters extracted in the fit range $0.463 \le (aQ)^2 \le 0.994$.

The upper limit is chosen to avoid significant lattice artifact, which is estimated by a difference of the lattice momentum aQ_μ from another definition $2\sin(aQ_\mu/2)$. With a constraint $aQ_\mu \le \pi/4$, the results are unchanged within 1 σ level.

To determine the lower limit we investigate the stability of the fit results. Figure 2 plots the dependence of fit parameters on the lower limit. We observe that around $(aQ)^2_{\rm min}$ =0.4–0.5 all parameters are stable. It turned out that the contribution of the higher order terms in OPE, $1/Q^6$, is negligibly small around this region. In our study we set $(aQ)^2_{\rm min} \simeq 0.463$. In order to investigate where the quark and gluon condensate terms give significant contributions, we attempt a fit with (2.2) but without the fourth and fifth terms. $(aQ)^2_{\rm min}$ dependence for $\Lambda^{(3)}_{\overline{MS}}$ is also given in Fig. 2 by filled circles. There is a significant deviation from the full analysis below $(aQ)^2 = 0.48$. This

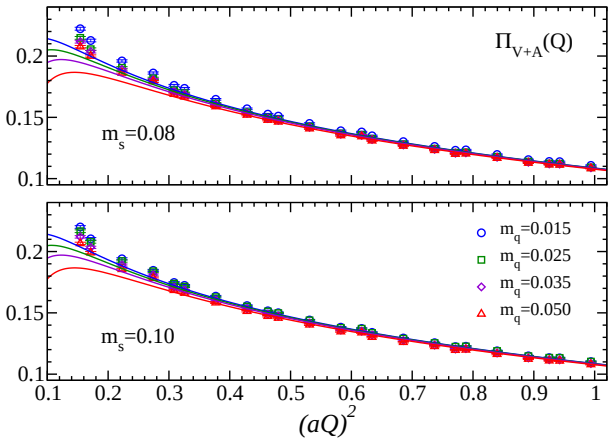

Figure 1: $(aQ)^2$ dependence of VPF, $\Pi_{V+A}(Q)$, at all valence quark masses: $m_q = 0.015$ (circle), 0.025 (square), 0.035 (diamond), 0.050 (triangle). Top panel is a result in $m_s = 0.08$; bottom is in $m_s = 0.1$. Solid lines show a fit function at each quark masses. Here we use a fit range determined as explained in a text.

implies that the contribution from the dimension-4 operators play an important role in $(aQ)^2 < 0.48$. The statistical error becomes small as the number of the fit parameters is reduced when we use the leading term only, however χ^2 become worse as $\chi^2/\mathrm{dof} \sim 3.0$ because the non-perturbative quark mass dependence is not described by the perturbative function.

After doing a simultaneous fit of the VPF data at all sea quark masses, the Λ parameter is obtained as $\Lambda^{(3)}_{\overline{MS}} = 0.247(11)$. At the Z boson mass scale the strong coupling constant is obtained as $\alpha_s(M_Z) = 0.1181(8)$.

4. Systematic errors

We estimate the size of systematic errors in our results.

Discretization effect can be estimated using perturbation theory, as the effect is more important at large momentum region. We calculate the one loop diagram of VPF, $\Pi^{\mathrm{PT}}_{V+A}(Q^2)$, with local and conserved currents in lattice perturbation theory. The result may be parameterized as $c - 1/(2\pi^2)\ln(aQ)^2 + 0.0062(aQ)^2$ for small a. The logarithmic term is the same as in the continuum perturbation theory and c is the scheme dependent constant as already noticed. The term $0.0062(aQ)^2$ comes from the discretization effect. By subtracting this term from the lattice data the final result for $\alpha_s(M_Z)$ changes by $+0.0002$ in $\alpha(M_z)$, which gives a conservative estimate of the leading a^2 effects. Other discretization error may come from the non-conserving J^{loc}_v as given in (2.7). The maximum magnitude of $F^{V+A}_{11}(Q)$ in (2.7) is less than 1% of $\Pi_{V+A}(Q)$ in the fit range $0.463 \le (aQ)^2 \le 0.994$. Thus a systematic error due to the violation of the WT identities has a similar order to the perturbatively estimated discretization effects. The difference due to the momentum definition on the lattice is about 0.5% of $\Pi_{V+A}(Q)$ that is negligible. The total discretization error is thus estimated to be from $+0.0004$ to -0.0002 in $\alpha_s(M_Z)$.

The uncertainty in the quark condensate leads to -0.0001 decrease of $\alpha_s(M_Z)$. The truncation effect of the perturbative expansion at $\mathcal{O}(\alpha_s^3)$ is less than 1%, whose value is estimated by a

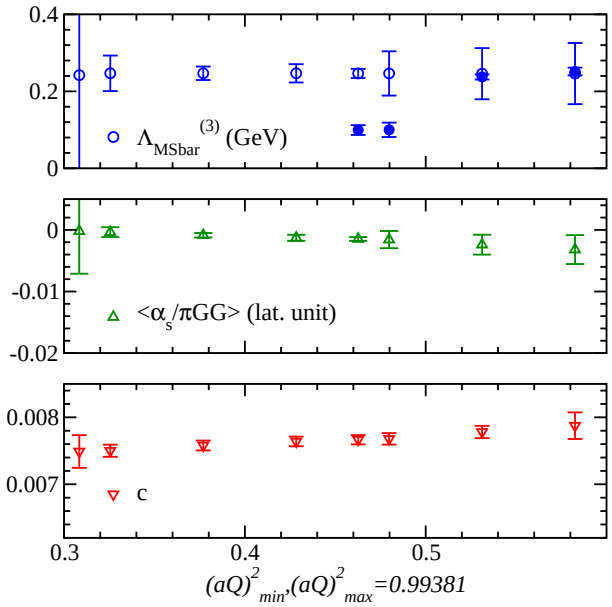

Figure 2: Dependence of the minimum fit range for three parameter. The maximum value is fixed in $(aQ)^2 \simeq 0.994$, which is determined as explained in text. Open and filled symbols show the results with and without the quark and gluon condensate terms.

comparison with lower loop order and four-loop order [18] for $C_0(Q^2, \mu^2.\alpha_s)$ in the VPF formula. Numerically it amounts to ± 0.0001 in $\alpha_s(M_Z)$. There is an uncertainty in the mass dependence of the renormalization constant, which is maximally 1% of $\Pi_{V+A}(Q^2)$. Total of those uncertainties, *i.e.* truncation effect and renormalization constant, is estimated as ± 0.0001 The uncertainties of quark mass of charm and bottom used in the perturbative matching procedure give ± 0.0003 error to $\alpha_s(M_Z)$.

Table 1 shows a summary of these systematic errors in our determination of $\alpha_s(M_Z)$. Our prelimimary result for the strong coupling constant at the Z boson mass scale is

$$\alpha_s(M_Z) = 0.1181(8)(^{+4}_{-2})(^{+5}_{-6}), \tag{4.1}$$

where the first and second errors are statistical error and systematic errors. This result is consistent with the other lattice QCD result $0.1170(12)$ [19] and the world average $0.1176(20)$ [20].

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Table 1: Summary table of the systematic errors in $\alpha_s(M_Z)$ and their values.

Sources	Systematic error in $\alpha_s(M_Z)$
Discretization effect in $\Pi_{V+A}(Q^2)$	$+0.0002$
$\Delta_{\mu\nu}^{V+A}$	± 0.0002
Momentum definition	negligible
Quark condensate	-0.0001
Truncation effects	± 0.0001
Renormalization constant	± 0.0001
Uncertainties of $m_{c,b}$	± 0.0003

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