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Supersymmetric lattices - a brief introduction

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Recently, new theoretical ideas have allowed the construction of lattice actions which are explicitly invariant under one or more supersymmetries. These theories are local and free of doublers and in the case of Yang-Mills theories also possess exact gauge invariance. In this talk these ideas are reviewed with particular emphasis being placed on $\mathcal{N} = 4$ super Yang-Mills theory.

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1. Introduction

The problem of formulating supersymmetric theories on lattices has a long history going back to the earliest days of lattice gauge theory. However, after initial efforts failed to produce useful supersymmetric lattice actions the topic languished for many years. Indeed a folklore developed that supersymmetry and the lattice were mutually incompatible. However, recently, the problem has been re-examined using new tools and ideas such as topological twisting, orbifold projection and deconstruction and a class of lattice models have been constructed which maintain one or more supersymmetries exactly at non-zero lattice spacing.

While in low dimensions there are many continuum supersymmetric theories that can be discretized this way, in four dimensions there appears to a unique solution to the constraints – $\mathcal{N}=4$ super Yang-Mills. The availability of a supersymmetric lattice construction for this theory is clearly very exciting from the point of view of exploring the connection between gauge theories and string/gravitational theories. But even without this connection to string theory it is clearly of great importance to be able to give a non-perturbative formulation of a supersymmetric theory via a lattice path integral in the same way that one can formally define QCD as a limit of lattice QCD as the lattice spacing goes to zero and the box size to infinity. From a practical point of view one can also hope that some of the technology of lattice field theory such as strong coupling expansions and Monte Carlo simulation can be brought to bear on such supersymmetric theories.

In this talk I will outline some of the key ingredients that go into these constructions, the kinds of applications that have been considered so far and highlight the remaining difficulties.

First, let me explain why discretization of supersymmetric theories resisted solution for so long. The central problem is that naive discretizations of continuum supersymmetric theories break supersymmetry completely and radiative effects lead to a profusion of relevant supersymmetry breaking counterterms in the renormalized lattice action. The coefficients to these counterterms must then be carefully fine tuned as the lattice spacing is sent to zero in order to arrive at a supersymmetric theory in the continuum limit. In most cases this is both unnatural and practically impossible – particularly if the theory contains scalar fields. Of course, one might have expected problems – the supersymmetry algebra is an extension of the Poincaré algebra which is explicitly broken on the lattice. Specifically, there are no infinitesimal translation generators on a discrete spacetime so that the algebra $\{Q,\overline{Q}\}=\gamma_\mu p_\mu$ is already broken at the classical level. Equivalently it is a straightforward exercise to show that a naive supersymmetry variation of a naively discretized supersymmetric theory fails to yield zero as a consequence of the failure of the Leibniz rule when applied to lattice difference operators[1].

In the last five years or so this problem has been revisited using new theoretical tools and ideas and a set of lattice models have been constructed which retain exactly some of the continuum supersymmetry at non-zero lattice spacing. The basic idea is to maintain a particular subalgebra of the full supersymmetry algebra in the lattice theory. The hope is that this exact symmetry will constrain the effective lattice action and protect the theory from dangerous susy violating counterterms.

[1]Significant work has gone into generalizing the Leibniz rule to finite difference operators in the context of non-commutative models using the techniques of Hopf algebras see [1, 2, 3]. This approach will not be discussed in this talk

Two approaches have been pursued to produce such supersymmetric actions; one based on ideas drawn from the field of topological field theory [7, 8, 9, 10] and another pioneered by David B. Kaplan and collaborators using ideas of orbifolding and deconstruction [4, 5, 6]. Remarkably these two seemingly independent approaches lead to the same lattice theories – see [11, 12, 13] and the recent reviews [14, 15]. This convergence of two seemingly completely different approaches leads one to suspect that the final lattice theories may represent essentially unique solutions to the simultaneous requirements of locality, gauge invariance and at least one exact supersymmetry. We will only have time to discuss the approach via topological twisting in this talk.

2. Topological twisting

Perhaps the simplest way to understand how this subalgebra emerges is to reformulate the target theory in terms of "twisted fields". The basic idea of twisting goes back to Witten in his seminal paper on topological field theory [16] but actually had been anticipated in earlier work on staggered fermions [17]. In our context the idea is decompose the fields of the theory in terms of representations not of the original (Euclidean) rotational symmetry $SO_{\rm rot}(D)$ but a twisted rotational symmetry which is the diagonal subgroup of this symmetry and an $SO_{\rm R}(D)$ subgroup of the R-symmetry of the theory.

$$SO(D)' = {\rm diag}(SO_{\rm Lorentz}(D) \times SO_{\rm R}(D)) \tag{2.1}$$

To be explicit consider the case where the total number of supersymmetries is $Q = 2^D$. In this case I can treat the supercharges of the twisted theory as a $2^{D/2} \times 2^{D/2}$ matrix q. This matrix can be expanded on products of gamma matrices

$$q = \mathcal{Q}I + \mathcal{Q}_\mu \gamma_\mu + \mathcal{Q}_{\mu\nu}\gamma_\mu\gamma_n u + \ldots \tag{2.2}$$

The 2^D antisymmetric tensor components that arise in this basis are the twisted supercharges and satisfy a corresponding supersymmetry algebra following from the original algebra

$$\mathcal{Q}^2 = 0 \tag{2.3}$$
$$\{\mathcal{Q}, \mathcal{Q}_\mu\} = p_\mu \tag{2.4}$$
$$\cdots \tag{2.5}$$

The presence of the nilpotent scalar supercharge $\mathcal{Q}$ is most important; it is the algebra of this charge that we can hope to translate to the lattice. The second piece of the algebra expresses the fact that the momentum is the $\mathcal{Q}$-variation of something which makes plausible the statement that the energy-momentum tensor and hence the entire action can be written in $\mathcal{Q}$-exact form[2]. Notice that an action written in such a $\mathcal{Q}$-exact form is trivially invariant under the scalar supersymmetry provided the latter remains nilpotent under discretization.

The rewriting of the supercharges in terms of twisted variables can be repeated for the fermions of the theory and yields a set of antisymmetric tensors $(\eta, \psi_\mu, \chi_{\mu\nu}, \ldots)$ which for the case of $Q = 2^D$ matches the number of components of a real Kähler-Dirac field. This repackaging of the fermions

[2]Actually in the case of $\mathcal{N} = 4$ there is an additional $\mathcal{Q}$-closed term needed

of the theory into a Kähler-Dirac field is at the heart of how the discrete theory avoids fermion doubling as was shown by Becher, Joos and Rabin in the early days of lattice gauge theory [18, 19].

It is important to recognize that the transformation to twisted variables corresponds to a simple change of variables in flat space – one more suitable to discretization. A true topological field theory only results when the scalar charge is treated as a true BRST charge and attention is restricted to states annihilated by this charge. In the language of the supersymmetric parent theory such a restriction corresponds to a projection to the vacua of the theory. It is *not* employed in these lattice constructions.

3. An example: 2D super Yang-Mills

This theory satisfies our requirements for supersymmetric latticization; its R-symmetry possesses an $SO(2)$ subgroup corresponding to rotations of the its two degenerate Majorana fermions into each other. Explicitly the theory can be written in twisted form as

$$S = \frac{1}{g^2}\mathcal{Q}\int \mathrm{Tr}\left(\chi_{\mu\nu}\mathcal{F}_{\mu\nu} + \eta\,[\overline{\mathcal{D}}_\mu, \mathcal{D}_\mu] - \frac{1}{2}\eta\, d\right) \tag{3.1}$$

The degrees of freedom are just the twisted fermions $(\eta, \psi_\mu, \chi_{\mu\nu})$ previously described and a complex gauge field $\mathcal{A}_\mu$. The latter is built from the usual gauge field and the two scalars present in the untwisted theory $\mathcal{A}_\mu = A_\mu + iB_\mu$ with corresponding complexified field strength $\mathcal{F}_{\mu\nu}$.

Notice that the original scalar fields transform as vectors under the original R-symmetry and hence become vectors under the twisted rotation group while the gauge fields are singlets under the R-symmetry and so remain vectors under twisted rotations. This structure makes possible the appearance of a complex gauge field in the twisted theory. Notice though,that the theory is only invariant under the usual $U(N)$ gauge symmetry and not its complexified cousin.

The nilpotent transformations associated with $\mathcal{Q}$ are given explicitly by

$$\begin{aligned}
\mathcal{Q}\,\mathcal{A}_\mu &= \psi_\mu \\
\mathcal{Q}\,\psi_\mu &= 0 \\
\mathcal{Q}\,\overline{\mathcal{A}}_\mu &= 0 \\
\mathcal{Q}\,\chi_{\mu\nu} &= -\overline{\mathcal{F}}_{\mu\nu} \\
\mathcal{Q}\,\eta &= d \\
\mathcal{Q}\,d &= 0
\end{aligned}$$

Performing the $\mathcal{Q}$-variation and integrating out the auxiliary field d yields

$$S = \frac{1}{g^2}\int \mathrm{Tr}\left(-\overline{\mathcal{F}}_{\mu\nu}\mathcal{F}_{\mu\nu} + \frac{1}{2}[\overline{\mathcal{D}}_\mu, \mathcal{D}_\mu]^2 - \chi_{\mu\nu}\mathcal{D}_{[\mu}\,\psi_{\nu]} - \eta\,\overline{\mathcal{D}}_\mu\,\psi_\mu\right) \tag{3.2}$$

To untwist the theory and verify that indeed in flat space it just corresponds to the usual theory one can do a further integration by parts to produce

$$S = \frac{1}{g^2}\int \mathrm{Tr}\,\left(-F_{\mu\nu}^2 + 2B_\mu D_\nu D_\nu B_\mu - [B_\mu, B_\nu]^2 + L_F\right) \tag{3.3}$$

where $F_{\mu\nu}$ is the usual Yang-Mills term. It is now clear that the imaginary parts of the gauge fields B_μ can now be given an interpretation as scalar fields in this parameterization. Similarly one can build spinors out of the twisted fermions and write the action in the manifestly Dirac form

$$L_F = \begin{pmatrix} \chi_{12} & \frac{\eta}{2} \end{pmatrix} \begin{pmatrix} -D_2 - iB_2 & D_1 + iB_1 \\ D_1 - iB_1 & D_2 - iB_2 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \tag{3.4}$$

4. Discretization

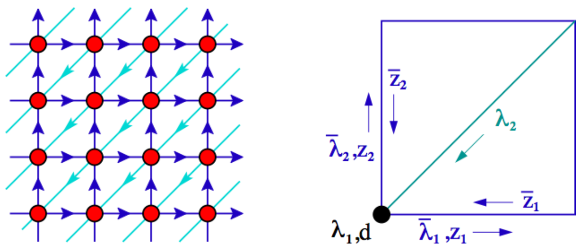
The prescription for discretization is somewhat natural. (Complex) gauge fields are represented as complexified Wilson gauge links $\mathcal{U}_\mu(x) = e^{\mathcal{A}}_\mu(x)$ living on links of a lattice which for the moment we can think of as hypercubic. These transform in the usual way under $U(N)$ lattice gauge transformations

$$\mathcal{U}_\mu(x) \to G(x)\mathcal{U}_\mu(x)G^\dagger(x) \tag{4.1}$$

Supersymmetric invariance then implies that $\psi_\mu(x)$ live on the same links and transform identically. The scalar fermion $\eta(x)$ is clearly most naturally associated with a site and transforms accordingly

$$\eta(x) \to G(x)\eta(x)G^\dagger(x) \tag{4.2}$$

The field $\chi_{\mu\nu}$ is slightly more difficult. Naturally as a 2-form it should be associated with a plaquette. In practice we introduce diagonal links running through the center of the plaquette and choose $\chi_{\mu\nu}$ to lie *with opposite orientation* along those diagonal links. This choice of orientation will be necessary to ensure gauge invariance.



To complete the discretization we need to describe how continuum derivatives are to be replaced by difference operators. A natural technology for accomplishing this in the case of adjoint fields was developed many years ago and yields expressions for the derivative operator applied to arbitrary lattice p-forms [20]. In the case discussed here we need just two derivatives given by the expressions

$$\mathcal{D}^{(+)}_\mu f_\nu = \mathcal{U}_\mu(x) f_\nu(x+\mu) - f_\nu(x)\mathcal{U}_\mu(x+\nu) \tag{4.3}$$

$$\overline{\mathcal{D}}^{(-)}_\mu f_\mu = f_\mu(x)\overline{\mathcal{U}}_\mu(x) - \overline{\mathcal{U}}_\mu(x-\mu) f_\mu(x-\mu) \tag{4.4}$$

The lattice field strength is then given by the gauged forward difference $\mathcal{F}_{\mu\nu} = D^{(+)}_\mu \mathcal{U}_\nu$ and is automatically antisymmetric in its indices. Furthermore it transforms like a lattice 2-form and yields a gauge invariant loop on the lattice when contracted with $\chi_{\mu\nu}$. Similarly the covariant backward difference appearing in $\overline{\mathcal{D}}_\mu \mathcal{U}_\mu$ transforms as a 0-form or site field and hence can be contracted with the site field η.

This use of forward and backward difference operators guarantees that the solutions of the theory map one-to-one with the solutions of the continuum theory and hence fermion doubling problems are evaded. Indeed, by introducing a lattice with half the lattice spacing one can map this Kähler-Dirac fermion action into the action for staggered fermions. Notice that, unlike the case of QCD, there is no rooting problem in this supersymmetric construction since the additional fermion degeneracy is already required by the continuum theory.

5. Twisted $\mathcal{N} = 4$ super Yang-Mills

In four dimensions the constraint that the target theory possess 16 supercharges singles out a single theory for which this construction can be undertaken – $\mathcal{N} = 4$ SYM.

The continuum twist of $\mathcal{N} = 4$ that is the starting point of the twisted lattice construction was first written down by Marcus in 1995 [21] although it now plays a important role in the Geometric-Langlands program and is hence sometimes called the GL-twist [22]. This four dimensional twisted theory is most compactly expressed as the dimensional reduction of a five dimensional theory in which the ten (one gauge field and six scalars) bosonic fields are realized as the components of a complexified five dimensional gauge field while the 16 twisted fermions naturally span one of the two Kähler-Dirac fields needed in five dimensions. Remarkably, the action of this theory contains a $\mathcal{Q}$-exact piece of precisely the same form as the two dimensional theory given in eqn. 3.1 provided one extends the field labels to run now from one to five. In addition the Marcus twist requires a new $\mathcal{Q}$-closed term which was not possible in the two dimensional theory.

$$S_{\rm closed} = -\frac{1}{8}\int {\rm Tr}\ \varepsilon_{mnpqr}\chi_{qr}\overline{\mathcal{D}}_p\chi_{mn} \tag{5.1}$$

The supersymmetric invariance of this term then relies on the Bianchi identity $\varepsilon mnpqr\mathcal{D}_p\mathcal{F}_{qr} = 0$.

The lattice that emerges from examining the moduli space of the lattice theory is called A_4^* and is constructed from the set of five basis vectors v_a pointing out from the center of a four dimensional equilateral simplex out to its vertices together with their inverses $-v_a$. It is the four dimensional analog of the 3D bcc lattice. Complexified Wilson gauge link variables $\mathcal{U}_a$ are placed on these links together with their $\mathcal{Q}$-superpartners ψ_a. Another 10 fermions are associated with the diagonal links $v_a + v_b$ with $a > b$. Finally, the exact scalar supersymmetry implies the existence of a single fermion for every lattice site. The lattice action corresponds to a discretization of the Marcus twist on this A_4^* lattice and can be represented as a set of traced closed bosonic and fermionic loops. It is invariant under the exact $\mathcal{Q}$ scalar susy, lattice gauge transformations and a global permutation symmetry S^5 and can be proven free of fermion doubling problems as discussed before.

While the supersymmetric invariance of the $\mathcal{Q}$-exact term is manifest in the lattice theory it is not clear how to discretize the continuum $\mathcal{Q}$-closed term. Remarkably, it is possible to discretize eqn. 5.1 in such a way that it is indeed exactly invariant under the twisted supersymmetry! The discrete gauge invariant form is given by

$$S_{\rm closed} = -\frac{1}{8}\sum_{\mathbf{x}} {\rm Tr}\ \varepsilon_{mnpqr}\chi_{qr}(\mathbf{x}+\mu_m+\mu_n+\mu_p)\overline{\mathcal{D}}^{(-)}_p\chi_{mn}(\mathbf{x}+\mu_p) \tag{5.2}$$

and can be seen to be supersymmetric since the lattice field strength satisfies an exact Bianchi identity of the form

$$\varepsilon_{mnpqr}\mathcal{D}^{(+)}_p\mathcal{F}_{qr} = 0 \tag{5.3}$$

6. Prospects

One of the key issues that still remains to be explored is the question of how much residual fine tuning will be required to achieve a continuum limit in which full supersymmetry is restored. This is controlled by the flows in all relevant operators which could be induced in the effective action as a result of quantum corrections. We have used the exact lattice symmetries together with power counting to enumerate the possible set of such lattice operators.

Only three terms appear and two of these correspond to renormalizations of kinetic terms already present in the bare lattice action. There is one additional term of the form

$$\eta\,\mathscr{U}_\mu\overline{\mathscr{U}}_\mu \tag{6.1}$$

which leads to supersymmetric mass terms for the fermions and scalars.

The question of the restoration of full supersymmetry then rests on whether the ratios of the coefficients to these operators flow away from their classical values as the lattice spacing is decreased. A one loop calculation is in progress which should shed light on this issue.

Beyond this issue it would be very interesting to use Monte Carlo simulation to test the aspects of the AdS/CFT conjecture. Parallel code has been developed to study $\mathscr{N} = 4$ super Yang-Mills. At finite temperature this theory and its dimensional reductions should be dual to a variety of black hole solution in supergravity. We hope to report on the results of these soon. One would hope that the results of such simulations could be useful in the quest to understand how aspects of the quantum geometry can be understood in terms of the dual gauge theory.

References

[1] A. D'Adda, I. Kanamori, N. Kawamoto, K. Nagata, Twisted superspace on a lattice, Nucl. Phys. B707 (2005) 100–144. http://arxiv.org/abs/hep-lat/0406029

[2] A. D'Adda, I. Kanamori, N. Kawamoto, K. Nagata, Exact extended supersymmetry on a lattice: Twisted n = 2 super yang-mills in two dimensions, Phys. Lett. B633 (2006) 645–652. http://arxiv.org/abs/hep-lat/0507029

[3] A. D'Adda, I. Kanamori, N. Kawamoto, K. Nagata, Exact Extended Supersymmetry on a Lattice: Twisted N=4 Super Yang-Mills in Three Dimensions, Nucl. Phys. B798 (2008) 168–183. http://arxiv.org/abs/0707.3533

[4] A. G. Cohen, D. B. Kaplan, E. Katz, M. Unsal, Supersymmetry on a euclidean spacetime lattice. i: A target theory with four supercharges, JHEP 08 (2003) 024. http://arxiv.org/abs/hep-lat/0302017

[5] A. G. Cohen, D. B. Kaplan, E. Katz, M. Unsal, Supersymmetry on a euclidean spacetime lattice. ii: Target theories with eight supercharges, JHEP 12 (2003) 031. http://arxiv.org/abs/hep-lat/0307012

[6] D. B. Kaplan, M. Unsal, A euclidean lattice construction of supersymmetric yang- mills theories with sixteen supercharges, JHEP 09 (2005) 042. http://arxiv.org/abs/hep-lat/0503039

[7] S. Catterall, Lattice supersymmetry and topological field theory, JHEP 05 (2003) 038. http://arxiv.org/abs/hep-lat/0301028

[8] F. Sugino, A lattice formulation of super yang-mills theories with exact supersymmetry, JHEP 01 (2004) 015. http://arxiv.org/abs/hep-lat/0311021

[9] S. Catterall, S. Ghadab, Lattice sigma models with exact supersymmetry, JHEP 05 (2004) 044. http://arxiv.org/abs/hep-lat/0311042

[10] S. Catterall, A geometrical approach to n = 2 super yang-mills theory on the two dimensional lattice, JHEP 11 (2004) 006. http://arxiv.org/abs/hep-lat/0410052

[11] P. H. Damgaard, S. Matsuura, Lattice supersymmetry: Equivalence between the link approach and orbifolding, JHEP 09 (2007) 097. http://arxiv.org/abs/0708.4129 [hep-lat]

[12] P. H. Damgaard, S. Matsuura, Classification of Supersymmetric Lattice Gauge Theories by Orbifolding http://arxiv.org/abs/0704.2696 [hep-lat]

[13] S. Catterall, From Twisted Supersymmetry to Orbifold Lattices, JHEP 01 (2008) 048. http://arxiv.org/abs/0712.2532,

[14] J. Giedt, Progress in four-dimensional lattice supersymmetry http://arxiv.org/abs/0903.2443

[15] Simon Catterall, David B. Kaplan and Mithat Ünsal, Exact lattice supersymmetry http://arxiv.org/abs/0903.4881

[16] E. Witten, Topological quantum field theory, Commun. Math. Phys. 117 (1988) 353.

[17] S. Elitzur, E. Rabinovici, A. Schwimmer, Supersymmetric models on the lattice, Phys. Lett. B119 (1982) 165.

[18] J. M. Rabin, Homology theory of lattice fermion doubling, Nucl. Phys. B201 (1982) 315.

[19] P. Becher, H. Joos, The dirac-kahler equation and fermions on the lattice, Zeit. Phys. C15 (1982) 343.

[20] H. Aratyn, M. Goto, A. H. Zimmerman, A lattice gauge theory for fields in the adjoint representation, Nuovo Cim. A84 (1984) 255.

[21] N. Marcus, The other topological twisting of n=4 yang-mills, Nucl. Phys. B452 (1995) 331–345. http://arxiv.org/abs/hep-th/9506002

[22] A. Kapustin, E. Witten, Electric-magnetic duality and the geometric langlands program http://arxiv.org/abs/hep-th/0604151

[23] S. Catterall, First results from simulations of supersymmetric lattices, JHEP 01 (2009) 040. http://arxiv.org/abs/0811.1203

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