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CP invariance of chiral gauge theories and Majorana-Yukawa couplings on the lattice

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The construction of CP-invariant lattice chiral gauge theories and the construction of lattice Majorana fermions with chiral Yukawa couplings is subject to topological obstructions. In the present work we suggest lattice extensions of charge and parity transformation for Weyl fermions. This enables us to construct lattice chiral gauge theories that are CP invariant. For the construction of Majorana-Yukawa couplings, we discuss two models with symplectic Majorana fermions: a model with two symplectic doublets, and one with an auxiliary doublet.

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1. Introduction

Despite the considerable success in the formulation of chiral symmetry on the lattice [1, 2, 3, 4] based on the Ginsparg-Wilson relation, there remain several unsolved problems. One of them concerns the construction of CP invariant chiral gauge theories on the lattice [5, 6]. Another problem concerns the definition of Majorana fermions in the presence of Yukawa couplings [7, 8]. These problems are closely related to the requirements of locality and of avoiding species doublers, which are basic issues for chiral symmetry on the lattice. Loosely speaking, the above problems relate to the fact that chiral symmetry for a lattice Dirac action $\bar{\psi}D\psi$ with Ginsparg Wilson Dirac operator D requires an asymmetric treatment of ψ and $\bar{\psi}$. In turn, CP symmetry and Majorana-Yukawa couplings require a symmetric treatment of ψ and $\bar{\psi}$.

In this paper, we put forward possible solutions to these problems. We first discuss the obstructions in constructing lattice chiral gauge theories with CP invariance. Due to the Nielsen-Ninomiya no-go theorem [9, 10, 11, 12], consistent chiral projection operators necessarily depend on the Dirac operator, see e.g. [16]. It is natural to assume that the modified chiral symmetry on the lattice induces modifications of charge and parity transformations on the lattice. Here we define lattice extensions of charge and parity transformations for Weyl fermions [13] that explicitly depend on the chiral projection operators. This will be legitimate because CP is a discrete symmetry, and enables us to show CP symmetry in chiral gauge theories. We then construct Majorana-Yukawa actions by employing symplectic Majorana fermions. In addition to the model with two symplectic doublets discussed in [13, 14], we also construct a model with an auxiliary symplectic doublet following the idea given in [4, 15].

2. Obstructions in showing CP invariance of chiral gauge theory

Let us consider the lattice action of a chiral gauge theory,

$$S_{CGT} = \sum_{x,y\in\Lambda} \bar{\psi}(x)\Big(\frac{1-\gamma_5}{2}\Big)D(U)\ (x-y)\Big(\frac{1+\hat{\gamma}_5}{2}\Big)\psi(y)\,, \tag{2.1}$$

where the Dirac operator $D(U)$ with link variables U is used to define $\hat{\gamma}_5 = \gamma_5(1-aD(U))$. With these definitions the GW relation reads,

$$\gamma_5 D(U) + D(U)\hat{\gamma}_5 = 0\,. \tag{2.2}$$

The standard CP transformation is an operation

$$\psi \to -W^{-1}\,\bar{\psi}^T, \qquad \bar{\psi} \to \psi^T\,W\,, \tag{2.3}$$

where $W = CP$ is product of the parity $P = \gamma_4$ and the charge conjugation matrix C satisfies

$$C\gamma_\mu\,C^{-1} = -\gamma_\mu^T, \quad C\,C^\dagger = 1, \quad C = -C^T\,. \tag{2.4}$$

The CP transformation (2.3) is not an invariance of the action (2.1), simply because

$$W\gamma_5\,W^{-1} = -\gamma_5^T \neq -\hat{\gamma}_5^T\,. \tag{2.5}$$

We observe, however, that the action (2.1) would be CP invariant with the standard parity transformation if the charge conjugation maps γ_5 to $\hat{\gamma}_5^T$. Therefore, one may construct a lattice extension of the charge conjugation matrix, $\hat{C}$, which satisfies

$$\hat{C}\gamma_5\,\hat{C}^{-1} = \hat{\gamma}_5^T\,. \tag{2.6}$$

A solution to this equation is given by $\hat{C} = C\,(1-aD/2)$, which vanishes for $D = 2/a$. Indeed, any attempt of constructing smooth mappings between two different types of chiral projection operators $P = (1+\gamma_5)/2$ and $\hat{P} = (1+\hat{\gamma}_5)/2$ fails. This can be understood as follows: For general chiral projection operators P and $\hat{P}$ satisfying

$$(1-P)\,D = D\,\hat{P}\,, \tag{2.7}$$

it has been shown in [16] that the projection operators P and $\hat{P}$ carry a winding number that is related to the total chirality χ of the system at hand,

$$\chi = n[\hat{P}] - n[1-P]\,, \qquad \text{with} \qquad n[P] \equiv \frac{1}{2!}\left(\frac{i}{2\pi}\right)^2 \int_{T^4} \mathrm{tr}\,P(\mathrm{d}P)^4\,. \in \mathbf{Z}\,. \tag{2.8}$$

Eq. (2.8) therefore entails that for odd total chirality, e.g. a single Weyl fermion, $\hat{P}\psi$ and $\bar{\psi}\,P$ live in topologically different spaces. Hence there are no smooth mappings connecting them. Note that this theorem applies to a wide class of Dirac operators including Ginsparg-Wilson Dirac operators as a special case.

3. Lattice extension of C and/or P transformation

The absence of smooth mappings between the two spaces specified by $\hat{P}\psi$ and $\bar{\psi}\,P$ may not be a problem, because CP is a discrete symmetry. We conclude that we simply have to include the chiral projection operators in the definition of C and/or P transformation. Consequently we define a lattice extension of charge conjugation for Weyl fermions [13]:

$$\begin{aligned}
\bar{\psi}(x)\Big(\frac{1\pm\gamma_5}{2}\Big) &\to \sum_{y\in\Lambda} \psi^T(y)C\Big(\frac{1\pm\tilde{\gamma}_5(U^C)}{2}\Big)(y,x) \\
\sum_{y\in\Lambda}\Big(\frac{1\pm\hat{\gamma}_5(U)}{2}\Big)(x,y)\,\psi(y) &\to -\Big(\frac{1\pm\gamma_5}{2}\Big)C^{-1}\bar{\psi}^T(x)\,,
\end{aligned}$$

where $\tilde{\gamma}_5 = (1-aD(U))\gamma_5$. For link variable U, we use the standard C and P transformations:

$$\begin{aligned}
U_\mu(x) \to U_\mu^C(x) &= \left(U_\mu^\dagger\right)^T(x) \\
U_\mu(x) \to U_\mu^P(x) &= \begin{cases} U_i^\dagger(x_P - a\hat{i}) \;\; for \;\; i = 1,2,3. \\ U_4(x_P) \end{cases}
\end{aligned} \tag{3.1}$$

We also use the standard parity transformation for spinors

$$\psi(x) \to \psi^P(x) = P^{-1}\psi(x_P), \qquad \bar{\psi}(x) \to \bar{\psi}^P(x) = \bar{\psi}(x_P)P\,. \tag{3.2}$$

Using (3.1) and (3.2), we obtain the relations

$$\begin{aligned} CD(U^C)C^{-1} &= \big(D(U)\big)^T \\ PD(U^P)P^{-1}(x,y) &= D(U)(x_P,y_P)\,. \end{aligned} \tag{3.3}$$

It is straightforward to show CP invariance of (2.1),

$$\begin{aligned} &\bar{\psi}(1-\gamma_5)D(U)(1+\hat{\gamma}_5)\psi \\ &\xrightarrow{C} -\psi^T C(1-\tilde{\gamma}_5(U^C))D(U^C)(1+\gamma_5)C^{-1}\bar{\psi}^T \\ &\quad = \bar{\psi}(1+\gamma_5)D(U)(1-\hat{\gamma}_5(U))\psi \\ &\xrightarrow{P} \bar{\psi}(1-\gamma_5)D(U)(1+\hat{\gamma}_5(U))\psi\,. \end{aligned} \tag{3.4}$$

Some remarks are in order:

(1) Performing the charge conjugation (3.1) twice, one finds

$$\bar{\psi}(1+\gamma_5)/2 \to \bar{\psi}(1+\gamma_5)/2, \quad (1+\hat{\gamma}_5)\psi/2 \to (1+\hat{\gamma}_5)\psi/2, \tag{3.5}$$

as it should be.

(2) In the continuum limit, the charge conjugation (3.1) tends towards the standard one,

$$\psi \to -C^{-1}\,\bar{\psi}^T \quad \bar{\psi} \to \psi^T\,C\,. \tag{3.6}$$

(3) For the functional measure introduced by Lüscher,

$$\begin{aligned} &\mathscr{D}\psi\mathscr{D}\bar{\psi} = \prod_j (dc_j d\bar{c}_j) \\ &\psi(x) = \sum_j v_j(x)c_j, \quad \bar{\psi}(x) = \sum_j \bar{v}_j\bar{c}_j \\ &\left(\frac{1+\hat{\gamma}_5}{2}\right)v_j = v_j, \quad \bar{v}_j\left(\frac{1-\gamma_5}{2}\right) = \bar{v}_j\,, \end{aligned} \tag{3.7}$$

a CP transformation with (3.1) acts as $c_j \Leftrightarrow \bar{c}_j$, and therefore $\mathscr{D}\psi\mathscr{D}\bar{\psi}$ remains invariant.

In the construction put forward above we have modified the charge conjugation. Alternatively we could use the standard charge conjugation while modifying the parity transformations for Weyl fermions:

$$\begin{aligned} &\left(\frac{1\pm\gamma_5}{2}\right)\psi(x) \to P^{-1}\sum_{y\in\Lambda}\left(\frac{1\mp\hat{\gamma}_5(U)}{2}\right)(x_P,y_P)\psi(y_P) \\ &\sum_{y\in\Lambda}\bar{\psi}(y)\left(\frac{1\pm\tilde{\gamma}_5(U)}{2}\right)(y,x) \to \bar{\psi}(x)\left(\frac{1\mp\gamma_5}{2}\right)P\,. \end{aligned} \tag{3.8}$$

The modified CP-transformations with (3.8) are an invariance of the theory.

4. Majorana fermions and Yukawa couplings

Majorana spinors are defined by imposing a reality condition with $B = \gamma_5 C$,

$$\psi^* = B\psi \qquad \Rightarrow \qquad \psi^{**} = B^* B\psi\,. \tag{4.1}$$

For four dimensional Euclidean space, we have $B^*B = -1$ which leads to $\psi^{**} = \ -\psi$. Therefore, there are no Majorana spinors satisfying the reality constraint [17, 18]. This difficulty can be circumvented by doubling the fermions and implementing a symplectic Majorana condition

$$\begin{aligned}
&\psi_1^* = B\,\psi_2, \quad \psi_2^* = -B\,\psi_1 \\
&\Rightarrow \psi_a^{**} = \varepsilon_{ab}B^*\psi_b^* = \varepsilon_{ab}\varepsilon_{bc}B^*B\psi_c = \psi_a \quad (a,b = 1,2)\,.
\end{aligned} \tag{4.2}$$

On the lattice, a further doubling of degrees of freedom is needed for a chirally invariant theory with GW Dirac operator D [13, 14]. Introducing symplectic pairs of Majorana spinors, (Ψ_1,Ψ_2) and (ψ_1,ψ_2), we construct a chiral Yukawa theory:

$$\begin{aligned}
S &= S_0 + S_Y \\
S_0 &= \sum\left(\psi_1^T CD\Psi_1 + \psi_2^T CD\Psi_2\right) \\
S_Y &= \frac{g}{4}\sum\Big[\Big\{\psi_1^T C(1+\gamma_5)\varphi(1+\hat{\gamma}_5)\Psi_1 \\
&\qquad + \psi_1^T C(1-\gamma_5)\varphi^*(1-\hat{\gamma}_5)\Psi_1\Big\} + \Big\{1\to 2\Big\}\Big]\,.
\end{aligned} \tag{4.3}$$

The action is invariant under the chiral transformations

$$\begin{aligned}
\delta\psi_1 &= i\varepsilon\gamma_5\psi_1, \quad \delta\psi_2 = -i\varepsilon\gamma_5\psi_2 \\
\delta\Psi_1 &= i\varepsilon\hat{\gamma}_5\Psi_1, \quad \delta\Psi_2 = -i\varepsilon\hat{\gamma}_5\Psi_2 \\
\delta\varphi &= 2i\varepsilon\varphi\,.
\end{aligned} \tag{4.4}$$

In the above we have introduced two symplectic doublets. It is possible to make one of them an auxiliary doublet. Following [4, 15], we consider a free field action of two symplectic Majorana doublets,

$$S_0 = \sum\left[\left(\psi_1^T CD\psi_1 - \frac{2}{a}\Psi_1^T C\Psi_1\right) + \Big(1\to 2\Big)\right], \tag{4.5}$$

where (Ψ_1,Ψ_2) are auxiliary fields. The action (4.5) is invariant under the symmetric chiral transformations

$$\begin{aligned}
&\delta\psi_1 = i\varepsilon\gamma_5\left(1-\frac{a}{2}D\right)\psi_1 + i\varepsilon\gamma_5\Psi_1, \qquad \delta\Psi_1 = i\varepsilon\gamma_5\frac{a}{2}D\psi_1 \\
&\delta\psi_2 = -i\varepsilon\gamma_5\left(1-\frac{a}{2}D\right)\psi_2 - i\varepsilon\gamma_5\Psi_2, \quad \delta\Psi_2 = -i\varepsilon\gamma_5\frac{a}{2}D\psi_2 \\
&\delta\varphi = 2i\varepsilon\varphi\,.
\end{aligned} \tag{4.6}$$

Since

$$\delta(\psi_1+\Psi_1) = i\varepsilon\gamma_5(\psi_1+\Psi_1), \;\; \delta(\psi_2+\Psi_2) = -i\varepsilon\gamma_5(\psi_2+\Psi_2)\,, \tag{4.7}$$

$\psi_a + \Psi_a$ can be used to construct a chirally invariant Yukawa coupling,

$$S_Y = \frac{g}{2}\sum\Big[\Big\{(\psi_1+\Psi_1)^T C\varphi(1+\gamma_5)(\psi_1+\Psi_1) \\ +(\psi_2+\Psi_2)^T C\varphi^*(1-\gamma_5)(\psi_2+\Psi_2)\Big\} + \Big\{1\to 2\Big\}\Big]. \tag{4.8}$$

It is easy to see that the total action $S_0 + S_Y$ is invariant under (4.6).

5. Discussion and summary

We have discussed the construction of CP-invariant chiral gauge theories, as well as that of CP-invariant Majorana-Yukawa couplings on the lattice. Both problems are closely related to the fact that chiral projection operators on the lattice necessarily depend on the Dirac operator. This already suggests to include the chiral projection operators explicitly in the definition of charge conjugation and/or parity transformation for the Weyl fermions on the lattice. On the basis of these modified transformations we have constructed CP-invariant actions. We have also shown that the C and P transformations tend toward the standard C and P transformations in the continuum limit and leave the path integral measure invariant.

Let us also discuss some other approaches to the CP problem. In the interesting work [19, 20], chiral projection operators are constructed which are independent of gauge fields. Then CP invariance of chiral gauge theory is shown by using an 8-component notation. For this work, one has to examine whether the formulation gives the correct fermionic degrees of freedom in 4-component notation. In [21], a renormalization group approach is discussed to give a symmetric form of chiral projection operators. There, locality of those operators has to be examined.

For the construction of the Majorana-Yukawa couplings, we have built a model with an auxiliary doublet of symplectic Majorana fermions in addition to the two doublet model discussed in [13]. An extension of the formalism based on the symplectic Majorana condition to supersymmetric theory on the lattice will be interesting and challenging.

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