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Quantum Interactions Between Instantons, from Lattice Simulations

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We present a new method to compute non-perturbatively the instanton-antiinstanton interaction, using Lattice simulations. In this contribution, we test such method on the simple toy model defined by a quantum particle in a 1-dimensional double well potential. We evaluate the effective quantum interactions between instantons and antiinstantons both for a large barrier (where semi-classical arguments are supposed to hold) and for a low-barrier, where a fully non-perturbative and non-semiclassical approach is required.

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1. Introduction

The microscopic understanding of the dynamical mechanisms responsible for the hadron structure implies the identification of the vacuum gauge field fluctuations, which dominate the path integral and the correlation functions. In the last few decades, a large effort has been put towards understanding the role played by instantons [1],in connection with chiral symmetry breaking (see e.g. [2]) and hadron structure (see e.g. [3]). A limitation of existing instanton vacuum models is that they are affected from uncontrolled model dependence generated by the unknown structure of the interaction between the pseudo-particles. By definition, instanton-instanton interactions are quantum effects. Hence, their discussion requires a genuine, non-semiclassical, non-perturbative approach. A detailed knowledge of such interaction is expected to explain why large sized instantons are not present in the QCD vacuum and to determine the average packing fraction of the vacuum —the fraction of space time filled by the instanton fields— which is the main phenomenological input of instanton models.

In this work, we present the first step in the development of a rigorous non-perturbative method, based on lattice simulations, which allows to determine the effective interaction for multi-instanton field configurations, without relying on any semi-classical argument. We expect that this method should help evaluate microscopically all the phenomenological parameters in the instanton models. Hence, one could determine the role of the dynamics generated by such a set of low-energy vacuum field configurations in different correlation functions. In this talk, we briefly illustrate the formalism, in a simple toy model consisting of a one-dimensional quantum mechanical particle. The extension to QCD is presently under development.

2. Effective Interaction for the Vacuum Field Configurations

We start by considering a family of vacuum configurations $\tilde{x}(t;\gamma)$ parametrized by a set of collective coordinates γ. Specifically, in the case of the instanton model for QCD vacuum, $\tilde{x}(t;\gamma)$ represents a multi-instanton configuration and γ represents the set of all instanton positions, color orientation and sizes. In order to introduce the integral over the collective coordinates into the path integral

$$Z = \int \mathscr{D}x\, e^{-S[x]}. \tag{2.1}$$

we decompose every path in two orthogonal components

$$x(t) \,=\, \tilde{x}(t;\gamma) + y(t), \tag{2.2}$$

the first component —$\tilde{x}(t;\gamma)$— belongs to a functional manifold $\mathscr{M}$, parametrized by the curvilinear coordinates γ; the second component $y(t)$ — which we shall refer to as to the "fluctuation field"— is defined as the field orthogonal to the space tangent to the manifold $\mathscr{M}$, on the point $\bar{\gamma}$:

$$(y \cdot g_{\gamma_i}(\bar{\gamma})) = 0; \quad g_{\gamma_i}(t;\gamma) = \partial_{\gamma_i}\tilde{x}(t;\gamma), \quad \forall i = 1,\ldots,k \tag{2.3}$$

where $\{g_{\gamma_i}(t;\bar{\gamma})\}$ is an orthogonal basis of the tangent space. Such a decomposition is illustrated in Fig.(1). The path integral can then be written as

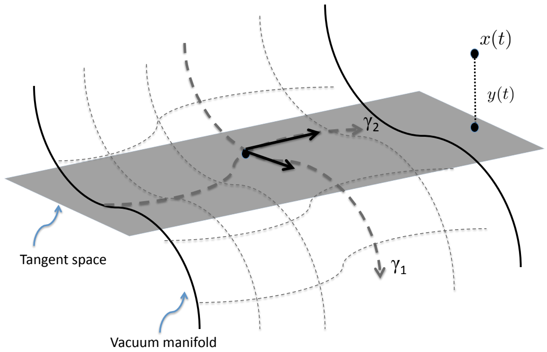

Figure 1: Pictorical representation of the projection of the field $x(t)$ onto a tangent space of the vacuum field manifold $\mathscr{M}$. $x(t)$ is represented by a point in this picture. The constraints (2.3) imply that the fluctuation field $y(t)$ is perpendicular to the plane tangent to the manifold in the point of the curvilinear abscissas $\gamma = \bar{\gamma}$.

$$
\begin{aligned}
Z &= \int \prod_{l=1}^{k} d\gamma_l \int \mathscr{D}y \left(\prod_i \delta^{(k)}(y \cdot g_{\gamma_i}(\bar{\gamma}) \right) \left| \det_{ij} \left(g_{\gamma_i}(\bar{\gamma}) \cdot g_{\gamma_j}(\gamma) \right) \right| e^{-S[\tilde{x}(t;\gamma)+y(t)]} \\
&\equiv \int d\gamma e^{-\frac{1}{\hbar}F(\gamma)},
\end{aligned} \tag{2.4}
$$

where $F(\gamma)$ represents the effective interaction for the instanton vacuum configurations $\tilde{x}(t;\gamma)$, and is defined as

$$
F(\gamma) = -\log \int \mathscr{D}y \left(\prod_i \delta^{(k)}(y \cdot g_{\gamma_i}(\bar{\gamma}) \right) \left| \det_{ij} \left(g_{\gamma_i}(\bar{\gamma}) \cdot g_{\gamma_j}(\gamma) \right) \right| e^{-S[\tilde{x}(t;\gamma)+y(t)]}. \tag{2.5}
$$

In order to evaluate the effective interaction $F(\gamma)$ we proceede in three steps:
1) We generate with lattice simulations an ensamble of N configurations $x_1(t),\ldots,x_N(t)$ and project them onto the tangent space of the vacuum manifold $\mathscr{M}$.

$$
\Psi_i(x_j) = (g_i(\bar{\gamma}) \cdot x_j); \quad \forall i = 1,\ldots,k; \; j = 1,\ldots,N. \tag{2.6}
$$

where $\Psi_i(x_j)$ are k functional of the fields $x_j(t)$.
2) Then we define $\Phi_i(\gamma)$, k function of the curvilinear coordinates γ, by projecting the vacuum manifold $\mathscr{M}$, on the tangent space

$$
\Phi_i(\gamma) = (g_i(\bar{\gamma}) \cdot \tilde{x}(\gamma)); \quad \forall i = 1,\ldots,k. \tag{2.7}
$$

3) Now, due to the orthogonality conditions in Eq.s(2.3), for every lattice configuration $x_j(t)$ we can construct a system of k equation for k unknowns γ_i, by equating the functionals $\Psi_i(x_j)$ in Eq.(2.6) with the functions $\Phi_i(\gamma)$ in Eq.(2.7)

$$
\begin{cases}
\Psi_1(x_j) = (g_1(\bar{\gamma}) \cdot x_j) = (g_1(\bar{\gamma}) \cdot \tilde{x}(\gamma)) = \Phi_1(\gamma) \\
\qquad\qquad\qquad\qquad \ldots \\
\Psi_k(x_j) = (g_k(\bar{\gamma}) \cdot x_j) = (g_k(\bar{\gamma}) \cdot \tilde{x}(\gamma)) = \Phi_k(\gamma) \quad ; \quad \forall j = 1,\ldots,N.
\end{cases} \tag{2.8}
$$

By solving this system of equations, one can determine a set of curvilinear coordinates $\{\gamma_i[x_j]\}$ for every configuration x_j. By counting the occurence of each set of $\{\gamma_i[x_j]\}$ one can then reconstruct the probability distribution $P(\gamma)$ of the collective coordinates and hence the effective interaction $F(\gamma)$, given by

$$F(\gamma) = -\log[P(\gamma)] \tag{2.9}$$

Now that we have presented our projection tecnique, we are going to test its validity, applying it to the simplest model presenting a non trivial vacuum structure: the 1-dimensional double well potential.

3. Double Well

In this first exploratory application, we choose to consider a simple toy model consisting by a quantum mechanical particle interacting with the double-well potential

$$U(x) = m\alpha\left(x^2 - \beta^2\right)^2, \tag{3.1}$$

where m is the mass of the particle. We consider the path integral with periodic boundary conditions

$$x_i = x_f = -\beta \qquad \text{(or equivalently: } x_i = x_f = +\beta \text{).} \tag{3.2}$$

In this specific system, the vacuum field manifold $\mathscr{M}$ is generated by the superposition of N instantons and N antiinstantons, with the curvilinear coordinates given by the pseudo-particle position one the Euclidean time axis, $\gamma_1 = t_1, \gamma_2 = \bar{t}_1, \ldots, \gamma_{2N-1} = t_N, \gamma_{2N} = \bar{t}_N$. Then, the path integral may be written as

$$Z[T;-\beta,-\beta] = \int dt_1, \int d\bar{t}_1 \ldots \int dt_N \int d\bar{t}_N \; e^{-\frac{1}{\hbar}F(t_1,\bar{t}_1,\ldots,t_N,\bar{t}_N)} \tag{3.3}$$

For a very high barrier, $\alpha \to \infty$, the instanton and antiinstanton will behave as non-interacting quasi-particles: in such a regime, the Dilute Instanton Gas Approximation (DIGA) is supposed to work and the effective interaction $F(\gamma)$ can be obtained analytically. If the height of the barrier is sufficiently low, the DIGA fails, and the vacuum fields behave as an interacting liquid. In such a regime the effective interaction can be decomposed as a sum of pairwise terms

$$F(\bar{t}_1,\ldots,\bar{t}_N) \simeq \sum_{i=1}^{N} F_2^{IA}(\bar{t}_i - t_i) + F_2^{AI}(t_{i+1} - \bar{t}_i), \tag{3.4}$$

where F_2^{IA} (F_2^{AI}) express the two-body instanton-antiinstanton (antiinstanton- instanton) correlations [1]. It is convenient to integrate out all pseudo-particles except two, from the path integral. This way, one obtains an expression in terms of just a single instanton-antiinstanton pair:

$$Z[T;-\beta,-\beta] = \frac{1}{2}\left(\int dt' \int dt \; e^{-\frac{1}{\hbar}\,F_2^{IA}(t'-t)} + \int dt \int dt' \; e^{-\frac{1}{\hbar}\,F_2^{AI}(t'-t)}\right). \tag{3.5}$$

[1] Eq. (3.4) can be generalized to include higher-order (e.g. three-body, four-body, etc...) correlations.

In order to extract the instanton-antiinstanton effective interaction F_2^{IA} we then parametrize a generic configuration $x(t)$ using the sum ansatz for an instanton-antiiinstanton pair,

$$x(t) = \tilde{x}^{IA}(t;t_1,t_2) + y(t) \quad (3.6)$$
$$= -\beta\left\{1 - \tanh\left[\sqrt{2\alpha}\,\beta\,(t-t_1)\right] + \tanh\left[\sqrt{2\alpha}\,\beta\,(t-t_2)\right]\right\} + y(t) \quad (3.7)$$

where $y(t)$ is a configuration of boundary conditions $y(\pm T/2) = 0$, and t_1 and t_2 are the coordinates of the two pseudoparticles, in the Euclidean time axis. Notice that we can define a variable $\xi = t_2 - t_1$ which represents the "relative distance" between the instanton and antiinstanton and that Eq.(3.5) implies

$$F_2(t_1,t_2) = F_2(t_2 - t_1) \equiv F_2(\xi), \quad (3.8)$$

We recall that the multi-instanton field configuration and the fluctuation field have to fulfill the orthogonality conditions (2.3), which is enforced in a specific point $\gamma = \bar{\gamma}$ of the manifold $\mathscr{M}$. The basis vector of the tangent space of the manifold defined by the sum ansatz (3.6) are, for an arbitrary point $\gamma = (\bar{t}_1, \bar{t}_2)$

$$g_{t_1}(t;\bar{t}_1,\bar{t}_2) = \left.\partial_{t_1}\tilde{x}_{S_2}(t;t_1,t_2)\right|_{t_1=\bar{t}_1,t_2=\bar{t}_2} = -\sqrt{2\alpha}\beta^2\mathrm{sech}^2\left[\sqrt{2\alpha}\beta(t-\bar{t}_1)\right] \quad (3.9)$$
$$g_{t_2}(t;\bar{t}_1,\bar{t}_2) = \left.\partial_{t_2}\tilde{x}_{S_2}(t;t_1,t_2)\right|_{t_1=\bar{t}_1,t_2=\bar{t}_2} = \sqrt{2\alpha}\beta^2\mathrm{sech}^2\left[\sqrt{2\alpha}\beta(t-\bar{t}_2)\right] \quad (3.10)$$

We are now ready to determine the effective interaction $F_2^{IA}(\xi)$ using the projection tecnique presented in Section 2. In order to show that our method yields the correct result, let us first consider the case for which the effective interaction can be evaluated analytically: the two-body part of the effective interaction in the DIGA is

$$F_{2,DIGA}^{IA}(\xi) = (\kappa T - 1)\log(T-\xi) + \text{const}, \quad (3.11)$$

where k is a constant which can be interpreted as the instanton density. In Fig.(2) we show the results obtained projecting equilibrium configurations sampled by the probability distribution $e^{-F_{2,DIGA}^{IA}}$ —each point corrispond to $\simeq 1000$ configurations—, while the dashed line is the expected theoretical result. The DIGA approximation holds only for ξ much larger than the instanton size —which is 0.26, in these units—: as we can see, in this regime our projection tecnique reproduces the correct result. In fact, a linear fit of the data for $\xi > 1$ yields a slope of 0.32 ± 1, in excellent agreement with the exact theoretical result, which is 0.31. Let us now consider quantum fluctuations. Fig. (3) shows the results of such a non-perturbative calculation for a well with $\alpha = 1$ (low barrier) and $\alpha = 7$ (high barrier). As one can see, at small ξ —i.e. when instanton and antiinstanton overlap— quantum interactions induce an effective repulsion, resulting in a minimum of the effective interaction at finite postitive ξ: this effect is enhanced in the lower barrier. In such a regime, the vacuum behaves like a one-dimensional liquid, rather than as an ideal gas; we note that this is precisely the physical picture underlying the instanton liquid model of the QCD vacuum. For higher values of ξ the effective interaction starts to raise and eventually reaches the dilute gas limit.

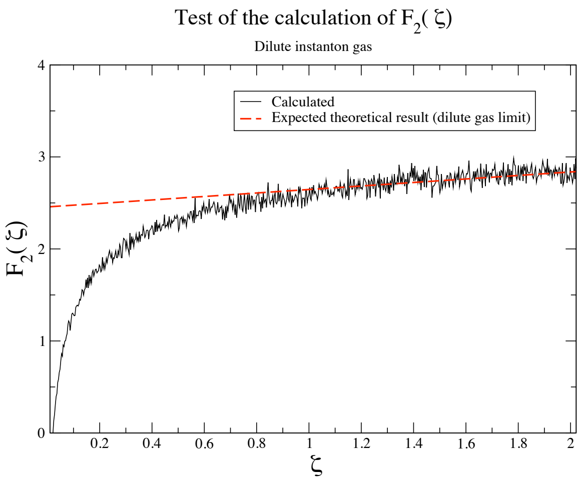

Figure 2: Non-perturbative calculation of the effective interaction $F_2(\xi)$ for a dilute instanton gas with $\alpha = 7$, $m = 1$, $\beta = 1$ in a volume $T = 200$.

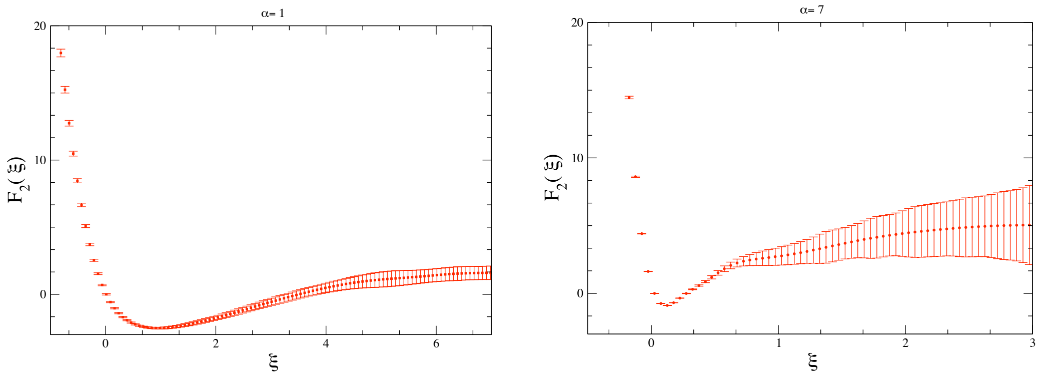

Figure 3: Non-perturbative calculation of the effective interaction $F(\xi)$ for $\alpha = 1$ (left panel) and $\alpha = 7$.

4. Conclusions

In this paper, we have presented an approach which allows to rigorously determine the instanton-antiinstanton interaction, directly from lattice simulations. The accuracy of our method has been assessed by correctly reconstructing the effective interaction used to generate an ensemble of sinthetic configurations. Since this framework does not rely on the semi-classical approximation, it can in principle be generalized to build more general effective theories for the vacuum, e.g. ones based on different types of vacuum field configurations. The details of the method briefly illustrated here will be presented in a forthcoming paper [4].The extension of the present formalism to QCD is presently in progress.

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