

The lattice ghost propagator in Landau gauge up to three loops using Numerical Stochastic Perturbation Theory

F. Di Renzo

Università di Parma & INFN, Viale Usberti 7/A, I-43100 Parma, Italy

E-mail: francesco.direnzo@fis.unipr.it

E.-M. Ilgenfritz

*Institut für Theoretische Physik, Ruprecht-Karls-Universität Heidelberg, Philosophenweg 19,
D-69120 Heidelberg, Germany*

E-mail: ilgenfri@physik.hu-berlin.de

H. Perlt

Institut für Theoretische Physik, Universität Leipzig, PF 100 920, D-04009 Leipzig, Germany

E-mail: Holger.Perlt@itp.uni-leipzig.de

A. Schiller*

Institut für Theoretische Physik, Universität Leipzig, PF 100 920, D-04009 Leipzig, Germany

E-mail: Arwed.Schiller@itp.uni-leipzig.de

C. Torrero

*Institut für Theoretische Physik, Universität Regensburg, Universitätsstr. 31,
D-93053 Regensburg, Germany*

E-mail: christian.torrero@physik.uni-regensburg.de

We complete our high-accuracy studies of the lattice ghost propagator in Landau gauge in Numerical Stochastic Perturbation Theory up to three loops. We present a systematic strategy which allows to extract with sufficient precision the non-logarithmic parts of logarithmically divergent quantities as a function of the propagator momentum squared in the infinite-volume and $a \rightarrow 0$ limits. We find accurate coincidence with the one-loop result for the ghost self-energy known from standard Lattice Perturbation Theory and improve our previous estimate for the two-loop constant contribution to the ghost self-energy in Landau gauge. Our results for the perturbative ghost propagator are compared with Monte Carlo measurements of the ghost propagator performed by the Berlin Humboldt university group which has used the exponential relation between potentials and gauge links.

The XXVII International Symposium on Lattice Field Theory

July 26-31, 2009

Peking University, Beijing, China

*Speaker.

1. NSPT and Langevin equation

It is known that standard diagrammatic Lattice Perturbation Theory (LPT) becomes very complicated when studying higher orders of typical physical quantities as renormalization factors.

As an alternative, Numerical Stochastic Perturbation Theory (NSPT) (see e.g. [1]) is a powerful tool to study higher-loop contributions in LPT: thanks to it, higher-loop results are in fact obtained without computing vast numbers of Feynman diagrams. Several applications of NSPT have been reported over the last years, for some latest developments see the additional contributions of F. Di Renzo, M. Brambilla, C. Torrero and H. Perlt to this conference.

Here we extend our results reported earlier [2, 3] and study the three-loop ghost propagator in Landau gauge to make predictions for standard diagrammatic LPT and compare with non-perturbative calculations.

We use the lattice Langevin equation with stochastic time t

$$\frac{\partial}{\partial t} U_{x,\mu}(t; \eta) = i (\nabla_{x,\mu} S_G[U] + \eta_{x,\mu}(t)) U_{x,\mu}(t; \eta), \quad (1.1)$$

where η is Gaussian random noise, S_G the gauge action and $\nabla_{x,\mu}$ the left Lie derivative within the gauge group. Discretizing the time $t = n\tau$, the equation is integrated numerically in the Euler scheme by iteration:

$$U_{x,\mu}(n+1; \eta) = \exp(-F_{x,\mu}[U, \eta]) U_{x,\mu}(n; \eta) \quad (1.2)$$

with the force

$$F_{x,\mu}[U, \eta] = i(\tau \nabla_{x,\mu} S_G[U] + \sqrt{\tau} \eta_{x,\mu}). \quad (1.3)$$

Rescaling $\varepsilon = \beta \tau$ and expanding the gauge links ($g \propto \beta^{-1/2}$)

$$U_{x,\mu} \rightarrow 1 + \sum_{l>0} \beta^{-l/2} U_{x,\mu}^{(l)}, \quad (1.4)$$

the Langevin equation at finite time step ε turns into a system of updates for each perturbative order $U_{x,\mu}^{(l)}$. The algebra-valued gauge potentials $A_{x,\mu}$ are related to the gauge lattice link fields $U_{x,\mu}$ by

$$A_{x,\mu} = \log U_{x,\mu}. \quad (1.5)$$

Their expansion is given in the form

$$A_{x,\mu} \rightarrow \sum_{l>0} \beta^{-l/2} A_{x,\mu}^{(l)} \quad (1.6)$$

and is used to enforce unitarity to all orders in $1/\sqrt{\beta}$.

Each simultaneous Langevin update is augmented by a stochastic gauge-fixing step and by subtracting zero modes from $A^{(l)}$. From the resulting fields the Green functions of interest can be numerically constructed order by order.

To measure gauge-dependent quantities, exact gauge fixing is needed. We use the Landau gauge which is reached by iterative Fourier-accelerated gauge transformations [4].

2. The ghost propagator in perturbation theory

2.1 The ghost propagator in NSPT

It is known that the ghost propagator is defined from the inverse of the Faddeev-Popov (FP) operator M . In Landau gauge this operator is constructed by using the lattice covariant $D(U)$ and left partial derivatives, $M = -\nabla \cdot D(U)$. Following [5] we use here a definition of M which is most suitable for NSPT.

We introduce the physical lattice momenta $\hat{p}_\mu(k_\mu) = \frac{2}{a} \sin\left(\frac{\pi k_\mu}{L_\mu}\right) = \frac{2}{a} \sin\left(\frac{ap_\mu}{2}\right)$ and define the color diagonal propagator in momentum space as the color trace in the adjoint representation ($N_c = 3$)

$$G(p(k)) = \frac{1}{N_c^2 - 1} \langle \text{Tr}_{\text{adj}} M^{-1}(k) \rangle_U. \quad (2.1)$$

In (2.1) $M^{-1}(k)$ is the Fourier transform of the inverse FP operator in lattice coordinate space.

A perturbative expansion is based on the mapping $A_{x,\mu}^{(l)} \rightarrow M^{(l)} \rightarrow [M^{-1}]^{(l)}$ which allows to calculate the inverse FP operator in NSPT recursively (i.e., to any finite order inverting the matrix M results in a closed form)

$$[M^{-1}]^{(0)} = [M^{(0)}]^{-1}, \quad [M^{-1}]^{(l)} = -[M^{(0)}]^{-1} \sum_{j=0}^{l-1} M^{(l-j)} [M^{-1}]^{(j)}. \quad (2.2)$$

The momentum space ghost propagator is obtained by sandwiching $[M^{-1}]^{(l)}$ between plane-wave vectors. The propagator has to be computed from scratch for each chosen momentum tuple (k_1, k_2, k_3, k_4) and different colors of the plane wave. Since we are interested in finding the momentum behavior as good as possible, these measurements become relatively expensive.

Multiplying the measured lattice momentum ghost propagator either with $(ap)^2$ or \hat{p}^2 , two forms of the so-called ghost dressing function are defined:

$$J^{(l)} = (ap)^2 G^{(l)}, \quad \hat{J}^{(l)} = \hat{p}^2 G^{(l)}. \quad (2.3)$$

The perturbative construction of M in terms of A differs from the definition adopted in most Monte Carlo calculations where a linear relation between the gauge links and gauge potentials is used.

2.2 The ghost propagator in standard LPT

In the RI'-MOM scheme, the renormalized ghost dressing function $J^{\text{RI}'}$ is defined as

$$J^{\text{RI}'}(p, \mu, \alpha_{\text{RI}'}) = \frac{J(a, p, \alpha_{\text{RI}'})}{Z_{\text{gh}}(a, \mu, \alpha_{\text{RI}'})} \quad (2.4)$$

with the renormalization condition $J^{\text{RI}'}(p, \mu, \alpha_{\text{RI}'})|_{p^2=\mu^2} = 1$. Therefore, the ghost dressing function $J(a, p, \alpha_{\text{RI}'})$ is just the ghost wave function renormalization constant $Z(a, \mu, \alpha_{\text{RI}'})$ at $\mu^2 = q^2$.

We represent the expansion of the bare $J(a, p, \alpha_{\text{RI}'})$ to n-loop order by

$$J^{\text{n-loop}}(a, p, \alpha_{\text{RI}'}) = 1 + \sum_{i=1}^n \alpha_{\text{RI}'}^i \sum_{k=0}^i z_{i,k}^{\text{RI}'} \left(\frac{1}{2} \mathcal{L}\right)^k, \quad \mathcal{L} = \log(ap)^2. \quad (2.5)$$

The leading log coefficients $z_{i,i}^{\text{RI}'}$ coincide with continuum perturbation theory (PT), the subleading log coefficients $z_{i,k}^{\text{RI}'|_{i>k>0}}$ have to be determined both from continuum PT and LPT in the used scheme, $z_{i,0}^{\text{RI}'}$ has to be found in LPT. Restricting ourselves in these proceedings to two-loop order in Landau gauge and to quenched approximation (compare e.g. [6]), we have

$$z_{1,1}^{\text{RI}'} = -\frac{9}{2}, \quad z_{2,2}^{\text{RI}'} = -\frac{315}{8}, \quad z_{2,1}^{\text{RI}'} = -\frac{2439}{24} + \frac{35}{2} z_{1,0}^{\text{RI}'}. \quad (2.6)$$

The coefficient $z_{1,0}^{\text{RI}'} = 13.8257$ has been calculated in [7], a first rough estimate of $z_{2,0}^{\text{RI}'}$ has been presented by us at Lattice08 [2].

From $\alpha_{\text{RI}'} = \alpha_0 + (-22 \log(a\mu) + 73.9355) \alpha_0^2 + \dots$ [8, 9], with the bare coupling $\alpha_0 = 3/(8\pi^2\beta)$, we get for the dressing function

$$J^{2\text{-loop}}(a, p, \beta) = 1 + \frac{1}{\beta} (J_{1,1}\mathcal{L} + J_{1,0}) + \frac{1}{\beta^2} (J_{2,2}\mathcal{L}^2 + J_{2,1}\mathcal{L} + J_{2,0}) \quad (2.7)$$

with

$$J_{1,1} = -0.0854897, \quad J_{1,0} = 0.525314, \quad J_{2,2} = 0.0215195, \quad J_{2,1} = -0.358423. \quad (2.8)$$

Among others we have to confirm $J_{1,0}$ from standard LPT and want to find a precise number for $J_{2,0}$ in Landau gauge quenched QCD using NSPT.

3. Results

3.1 Practice of measurements

Very precise measurements for different lattice sizes and different Langevin steps ε are needed. Typically we have of the order of 1000 measurements for each momentum tuple. Already at finite ε the non-integer $n = l/2$ (no-loop) contributions to the dressing function have to become negligible compared to the neighboring loop contributions. Examples of the dressing function \hat{J} for $n = 1, 2, 3$ and $n = 3/2$ vs. \hat{p}^2 at different volumes and $\varepsilon = 0.01$ are shown in Figure 1.

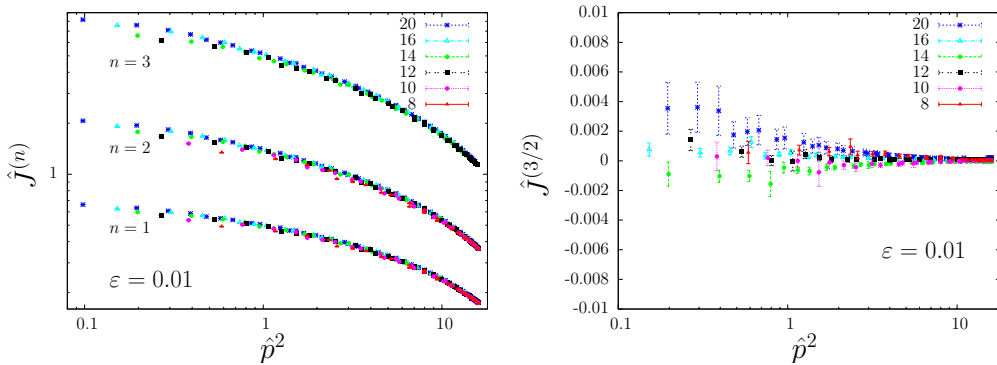


Figure 1: Measured ghost dressing function \hat{J} vs. \hat{p}^2 for all inequivalent lattice momentum 4-tuples near diagonal for $L = 8, 10, 12, 14, 16, 20$ and $\varepsilon = 0.01$. Left: The one-loop (β^{-1}), two-loop (β^{-2}) and three-loop (β^{-3}) contributions, right: the vanishing ($\propto \beta^{-3/2}$) contribution.

We have to take the zero Langevin step limit $\varepsilon \rightarrow 0$ for each 4-tuple (k_1, k_2, k_3, k_4) from the available finite ε measurements at fixed lattice size. This is shown for a particular momentum tuple at $L = 16$ in Figure 2. In order to make contact with standard LPT, the limits $L \rightarrow \infty$ and $ap \rightarrow 0$

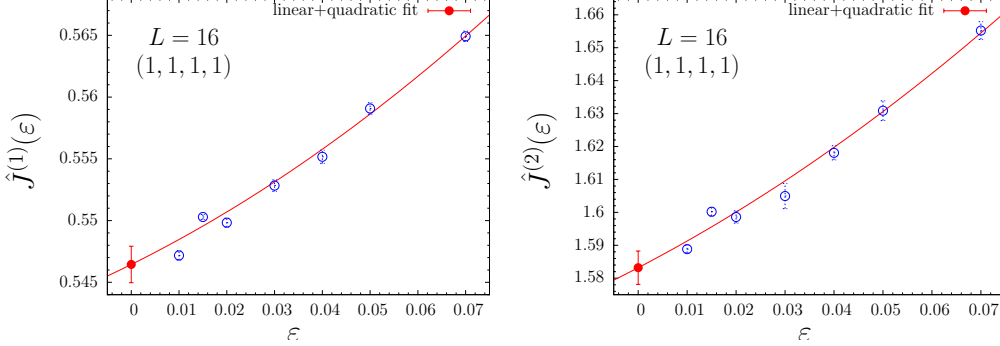


Figure 2: Extrapolation to $\varepsilon = 0$ of one- (left) and two-loop (right) ghost dressing function for lattice size 16^4 and momentum tuple $(1, 1, 1, 1)$: the fitting function contains both a linear and a quadratic term in ε .

have to be performed additionally.

3.2 Fitting logarithmic quantities on finite L

The dressing function J at zero Langevin step still suffers from finite $O(ap)$ and $O(pL)$ corrections. Our aim is to extract the finite constants in the power expansion of the lattice ghost dressing function $J_{i,0}$ with very high accuracy. In [10] it was pointed out that finite-size effects can be large when an anomalous dimension comes into play. Having at hand a variety of lattice sizes, we address a careful assessment of these effects (the main ideas entering the procedure can be found in [11]). Without entering into details, we summarize that strategy of fitting simultaneously $O(ap)$ and $O(pL)$ corrections together using several lattice sizes.

First we subtract all logarithmic pieces (supposed to be universal and known) from the dressing function for each momentum tuple and all lattice sizes. Next we select a range in $(ap)^2 = \sum (ap_\mu)^2$, $p_\mu = k_\mu(2\pi/L)$ with $p_{\min}^2 < p^2 < p_{\max}^2$. Within that range we identify a set S of momentum tuples (k_1, k_2, k_3, k_4) which is common to all chosen lattice sizes. The data in that set are assumed to have the same pL effects. Since finite-volume effects decrease with increasing momentum squared, we choose as reference fitting point – for an assumed behavior at $L = \infty$ – the data point at $p^2 \approx p_{\max}^2$ from the largest lattice size at our disposal.

Next we perform a non-linear fit using all data points of different L from that set S and the reference point correcting for finite size (C_m , no functional form) and assuming a functional behavior for $H(4)$ ($p^n = \sum_\mu p_\mu^n$):

$$J_{i,0}(k_1, k_2, k_3, k_4, p^2) = J_{i,0} + a^2 \left(\tilde{\alpha} p^2 + \tilde{\gamma} \frac{p^4}{p^2} \right) + a^4 \left(\tilde{\beta} (p^2)^2 + \tilde{\eta} p^4 + \tilde{\sigma} \frac{p^6}{p^2} \right) + \sum_{m \in S} C_m \delta[m, \{k_1, k_2, k_3, k_4\}], \quad i = 1, 2, \dots \quad (3.1)$$

Finally we vary the momentum squared window and find an optimal χ^2 region which allows us to find the 'best' $J_{i,0}$. In Figure 3 we show such a χ^2 -behavior for a non-linear fit to the one-loop dressing function.

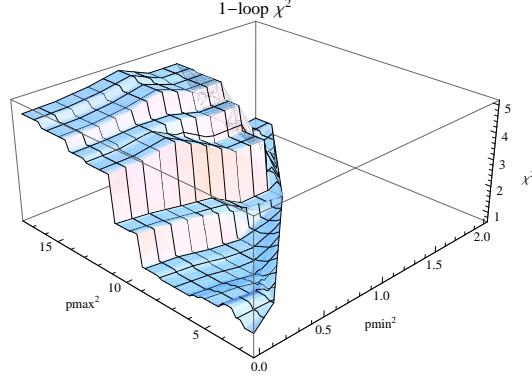


Figure 3: χ^2 -behavior of the non-linear fit for the one-loop dressing function in possible lattice momentum windows.

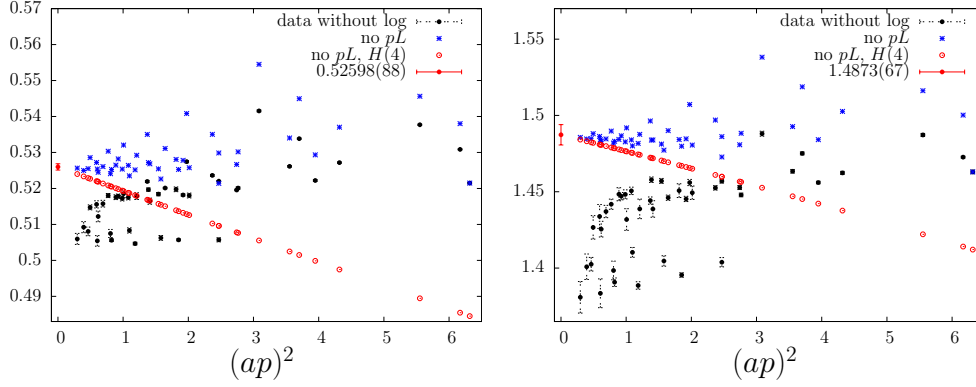


Figure 4: $L = 8, \dots, 20, S = 7$; black filled circles: log-subtracted data from allowed tuples; blue stars: pL effects removed; red open circles: $C_i = 0, H(4)$ effects removed. Left: one-loop order; right: two-loop order.

An example of a combined fit at low χ^2 is shown in Figure 4 for the first and the second loop as a function of $(ap)^2$. We observe that the numerical data at $\varepsilon = 0$ from the chosen set – with all logarithmic contributions subtracted – scatter significantly (black filled circles). Switching off the $O(pL)$ corrections ($C_m = 0$ in the fit form (3.1)), the blue stars line up in ‘rows’ according to the different hypercubic invariants at infinite lattice volume. The reference point (here the rightmost point) is of course unchanged. Finally, after removing also the non-rotational hypercubic effects (leaving only $\alpha \neq 0$ in (3.1)), we obtain a smooth (almost linear) curve formed by the red open circles which directly points towards the fitting constant $J_{i,0}$ in the zero lattice spacing limit.

To make a more realistic estimate of $J_{i,0}$, we have collected fit results from five different sets S with minimal χ^2 . From these sets we obtain the following (preliminary) constants in Landau gauge ($J_{1,0}^{\text{exact}} = 0.525314$)

$$J_{1,0} = 0.52520(46), \quad J_{2,0} = 1.489(4). \quad (3.2)$$

The two-loop constant can be transformed into the RI'-MOM scheme in that gauge. Our prediction for that small contribution is $z_{2,0}^{\text{RI}'} = 9.2(2.7)$.

Using the found $J_{2,0}$ as input for the non-leading log contribution to three loops, we are in the position to estimate $J_{3,0}$ as well. This analysis is in progress.

3.3 Comparison to Monte Carlo data

Using the $A = -\log U$ definition as in NSPT, the Berlin Humboldt university group has produced Monte Carlo results for the ghost and gluon propagator in Landau gauge and different gauge couplings [12]. Since it is assumed that non-perturbative contributions dominate mainly the infrared, it is of interest to compare directly the perturbative ghost dressing function obtained in NSPT with its Monte Carlo counterpart for each common momentum tuple.

We calculate the perturbative dressing function at a given lattice volume summed up to loop order n_{\max} for a given lattice coupling β as follows:

$$\hat{f} = \sum_{n=1}^{n_{\max}} \frac{1}{\beta^n} \hat{f}^{(n)}. \quad (3.3)$$

In Figure 5 we compare the perturbative ghost dressing function at lattice size $L = 16$ to

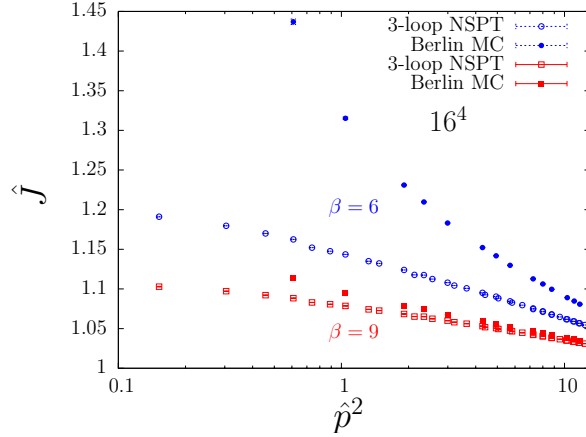


Figure 5: Three-loop NSPT of the ghost dressing function in comparison to Monte Carlo at two β values.

Monte Carlo data at two different β values. We observe that in not less than three-loop accuracy the perturbative ghost propagator at larger β is approximately able to describe the full two-point function in the large momentum squared region.

The situation becomes even worse in comparison with Monte Carlo when trying to define a perturbative running coupling from

$$\alpha_s^{3\text{-loop}}(\hat{p}, \beta) = \frac{6}{4\pi\beta} \hat{f}(\beta)^2 \hat{G}(\beta). \quad (3.4)$$

Here \hat{G} is the gluon dressing function (see [13, 3], used in NSPT in the same accuracy). This is demonstrated in Figure 6. So more loops would be necessary to find a satisfactory agreement with the non-perturbative data at largest lattice momenta.

4. Summary

We have presented a detailed perturbative calculation of the lattice ghost propagator in Landau gauge using NSPT. The one-loop constant $J_{1,0}$ perfectly agrees with known $V \rightarrow \infty$ result. The two-loop constant $J_{2,0}$ is determined with good accuracy for the first time. We have performed a very

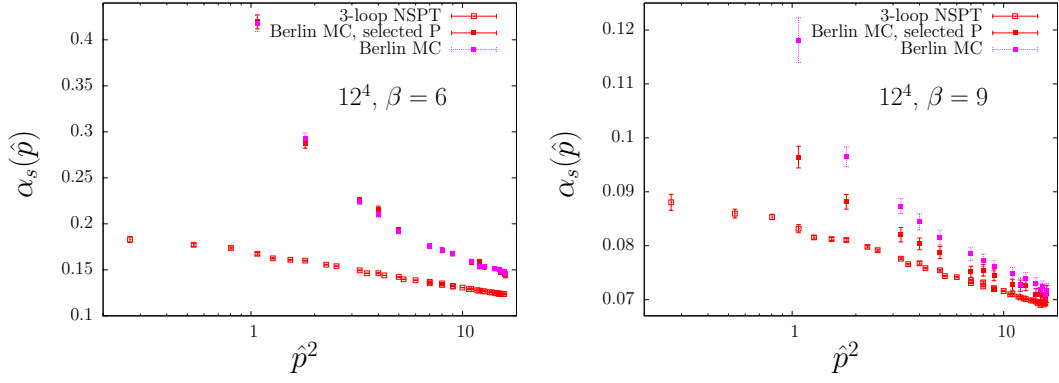


Figure 6: Three-loop NSPT of α_s in comparison to Monte Carlo (partly with specially chosen Polyakov sectors) for two β values.

careful analysis of all necessary limits. A technique to simultaneously deal with both $O(ap)$ and $O(pL)$ corrections is described in some detail. A comparison with Monte Carlo data of the ghost propagator and the running coupling shows that additional loops are needed to better describe the asymptotically prevailing perturbative tail at large lattice momenta.

References

- [1] F. Di Renzo and L. Scorzato, *JHEP* **10** (2004) 073 [arXiv:hep-lat/0410010].
- [2] F. Di Renzo, E.-M. Ilgenfritz, H. Perlt, A. Schiller and C. Torrero, PoS(LATTICE 2008)217 [arXiv:0809.4950[hep-lat]].
- [3] F. Di Renzo, E. M. Ilgenfritz, H. Perlt, A. Schiller and C. Torrero, PoS(Confinement8)050 [arXiv:0812.3307[hep-lat]].
- [4] C. T. H. Davies et al., *Phys. Rev.* **D 37** (1988) 1581.
- [5] H. J. Rothe, *World Sci. Lect. Notes Phys.* **59** (1997) 1.
- [6] J. A. Gracey, *Nucl. Phys. B* **662** (2003) 247 [arXiv:hep-ph/0304113].
- [7] H. Kawai, R. Nakayama and K. Seo, *Nucl. Phys. B* **189** (1981) 40.
- [8] A. Hasenfratz and P. Hasenfratz, *Phys. Lett. B* **93** (1980) 165.
- [9] M. Lüscher and P. Weisz, *Nucl. Phys. B* **452** (1995) 234 [arXiv:hep-lat/9505011].
- [10] F. Di Renzo, V. Miccio, L. Scorzato and C. Torrero, *Eur. Phys. J. C* **51** (2007) 645 [arXiv:hep-lat/0611013].
- [11] F. Di Renzo, L. Scorzato and C. Torrero, PoS(LATTICE 2007)240 [arXiv:0710.0552[hep-lat]]; F. Di Renzo, M. Laine, Y. Schroder and C. Torrero, *JHEP* **0809** (2008) 061 [arXiv:0808.0557[hep-lat]].
- [12] C. Menz, diploma thesis, Humboldt-Universität zu Berlin (2009); we acknowledge receiving those data prior to publication.
- [13] E.-M. Ilgenfritz, H. Perlt, and A. Schiller, PoS(LATTICE 2007)251 [arXiv:0710.0560[hep-lat]].