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Two-Loop Fermionic Integrals in Perturbation Theory on a Lattice

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A comprehensive number of one-loop integrals in a theory with Wilson fermions at $r = 1$ is computed using the Burgio–Caracciolo–Pelissetto algorithm. With the use of these results, the fermionic propagator in the coordinate representation is evaluated, thus making it possible to extend the Lüscher-Weisz procedure for two-loop integrals to the fermionic case. Computations are performed with FORM and REDUCE packages.

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1. Introduction

Perturbative calculations in lattice gauge theories (for a review, see [1]) are of interest from several points of view.

Firstly, they are needed to determine the Λ_{LAT} parameter of QCD in the lattice regularization and its relation to the respective value Λ_{QCD} in the continuum theory.

Secondly, every lattice action defines a specific regularization scheme, and thus one needs a complete set of renormalization computations in order for the results obtained in Monte Carlo simulations be understood properly. Perturbation theory is required to establish the connection of the matrix elements computed on a lattice with their values in the continuum theory [2], [3]. In this connection, it should be emphasized that the use of one-loop perturbative renormalization constants gives rise to large systematic uncertainties in lattice calculations of the momenta of hadronic structure functions [3] and respective two-loop computations are needed.

Thirdly, perturbative calculations provide the only possibility for an analytical control over the continuum limit in QCD. One can also mention anomalies, proof of renormalizability, Symanzik improvement program and other fields of application of lattice perturbation theory.

Here we consider one- and two-loop diagrams with Wilson ($r = 1$) fermions at zero external momenta [4]. We outline the Burgio-Caracciolo-Pelissetto (BCP) method [5] of calculations of one-loop integrals and describe the respective computer algorithm [6]. This algorithm allows to compute the fermionic propagator in the coordinate representation and, therefore, to extend the Lüscher-Weisz (LW) method [7] to the fermionic case; such extension is presented in Section 4.

1.1 Notation

We use the following designations: $\tilde{n}$ stands for the set n_1, n_2, n_3, n_4; $x = (x_1, x_2, x_3, x_4)$, where x_μ are integer-valued coordinates of an infinite four-dimensional lattice $\Lambda = \{x : x_\mu \in \mathbb{Z}\}$; we also need the lattice $\Lambda' = \Lambda \backslash \{0\}$ with removed site x=(0,0,0,0);

$$|x| = |x_1| + |x_2| + |x_3| + |x_4|, \qquad [x^n] = x_1^n + x_2^n + x_3^n + x_4^n. \tag{1.1}$$

Then we give the expressions for the denominators of bosonic and fermionic propagators,

$$\begin{aligned} \Delta_B(k) &= 4 + \mu_R^2 - \cos(k_1) - \cos(k_2) - \cos(k_3) - \cos(k_4); \\ \Delta_F(k) &= 10 - 4\sum_{\mu=1}^{4}\cos(k_\mu) + \sum_{1\le\mu<\nu\le 4}\cos(k_\mu)\cos(k_\nu) + \mu_R^2 \end{aligned} \tag{1.2}$$

where μ_R is the fictitious mass for infrared regularization. We also use $D_F = 2\,\Delta_F$ and $D_B = 2\,\Delta_B$ normalized in the standard way ($D_{B(F)(k)} \simeq 1/k^2$ as $k \to \infty$). These propagators in the coordinate representation are defined as follows:

$$G_{B(F)}(x) = \int_{BZ} \frac{dk}{(2\pi^4)} \frac{e^{-ikx}}{D_{B(F)}(k)}, \tag{1.3}$$

where BZ is the Brillouin zone, $BZ = \left\{ p : -\dfrac{\pi}{a} \le p_\mu \le \dfrac{\pi}{a} \right\}$.

2. The Burgio–Caracciolo–Pelissetto method

2.1 Bosonic Intefgrals

The integrals under study are defined as follows: $F(q;\tilde{n}) = \lim_{\delta\to 0} F_\delta(q;\tilde{n})$, where

$$F_\delta(q;\tilde{n}) = \int_{BZ} dk \, \frac{\cos(k_1)^{n_1}\cos(k_2)^{n_2}\cos(k_3)^{n_3}\cos(k_4)^{n_4}}{\Delta_B^{q+\delta}}. \tag{2.1}$$

Here δ is an infinitesimal parameter for an intermediate regularization [5]. This parameter makes it possible to derive[1] the recursion relations of the form

$$\begin{aligned} F(q;\ldots,n_\mu,\ldots) &= F(q;\ldots,n_\mu-2,\ldots) \; - \\ &- \frac{(n_\mu-1)F(q-1;\ldots,n_\mu-1,\ldots)}{q-1+\delta} + \frac{(n_\mu-2)F(q-1;\ldots,n_\mu-3,\ldots)}{q-1+\delta} \qquad (n_\mu \geq 2). \end{aligned} \tag{2.2}$$

With these relations and similar relations for $n_\mu \leq 1$ one can express the integrals (2.1) in terms of the quantities

$$G_\delta(q,\mu_R^2) = \int_{BZ} \frac{dk}{(2\pi)^4}\frac{1}{(\Delta_B)^{q+\delta}}. \tag{2.3}$$

Up to terms of the order $\mathscr{O}(\mu_R^2)$ and $\mathscr{O}(\delta)$, this expression has the form

$$F_\delta(q,\tilde{n}) = \sum_{r=q-n_1-n_2-n_3-n_4}^{0} A_{qr}^{(-)}(\delta,\tilde{n}) G_\delta(r,0) + \sum_{r=1}^{q} A_{qr}^{(+)}(\mu_R^2,\tilde{n}) G_\delta(r,\mu_R^2), \tag{2.4}$$

where $A_{qr}^{(-)}(\delta,\tilde{n})$ have a pole singularity in δ, and $A_{qr}^{(+)}(\mu_R^2,\tilde{n})$ are polynomials in μ_R^2. As for the function $G_\delta(r,\mu_R^2)$, the domains $r>0$ and $r\leq 0$ should be considered separately. At $r>0$, δ can be safely set to zero and the function $G_\delta(r,\mu_R^2)$ should be expanded in powers of μ_R^{-2}:

$$G_\delta(r,\mu_R^2) = \frac{1}{(2\pi)^2\Gamma(r)}\left[-b_{r-2} l_C + \sum_{k=1}^{r-2} \frac{b_{r-k-2}\Gamma(k)}{(\mu_R^2)^k} \right] + J(r) + \mathscr{O}(\mu_R^2) + \mathscr{O}(\delta), \tag{2.5}$$

where b_n are the coefficients of the asymptotic expansion at $z\to\infty$ of the function[2]

$$\exp(-4z) I_0^4(z) \simeq \frac{1}{(2\pi z)^2}\left(1 + \frac{b_1}{z} + \frac{b_2}{z^2} + \ldots\right), \tag{2.6}$$

$l_C = \ln(\mu_R^2) + C$, and C is the Euler-Mascheroni constant. At $r<0$, μ_R can be safely set to zero and the function $G_\delta(r,0)$ should be expanded in δ as follows: $G_\delta(r,0) = B(r) + J(r)\delta + \mathscr{O}(\delta^2)$.

The functions $J(q)$, in their turn, obey recursion relations of the type

$$c_0(q)J(q) + c_1(q)J(q+1) + c_2(q)J(q+2) + c_3(q)J(q+3) + c_4(q)J(q+4) = 0 \tag{2.7}$$

derived in [5]; the explicit expressions for the coefficients $c_i(q)$ can be found in [6]. Thus we express $J(q)$ at $q\geq 4$ and at $q\leq 0$ in terms of $J(0), J(1), J(2)$ and $J(3)$. It should be noted that $J(0)$ does not appear in ultimate expressions for the integrals (2.1). Then one can introduce the values

$$Z_0 = \frac{J(1)}{2}, \qquad F_0 = 4\pi^2 J(2), \qquad Z_1 = 32J(3) - 8J(2) + \frac{13}{6\pi^2} + \frac{1}{4}, \tag{2.8}$$

which are equal to [1] $Z_0 \approx 0.15493339023, Z_1 \approx 0.10778131354, F_0 \approx 4.369225233874758$.

[1] Using integration by parts

[2] $I_0(z)$ is the Infeld function.

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2.2 Fermion Integrals

In the fermionic case, we consider the quantities $F(p,q;\tilde{n}) = \lim_{\delta\to 0} F_\delta(p,q;\tilde{n})$, where

$$F_\delta(p,q;\tilde{n}) = \lim_{\delta\to 0} \int \frac{d^4k}{(2\pi)^4} \frac{\cos^{n_1}(k_1)\cos^{n_2}(k_2)\cos^{n_3}(k_3)\cos^{n_4}(k_4)}{\Delta_B^q \Delta_F^{p+\delta}}. \tag{2.9}$$

With the recursion relations similar to (2.2) these integrals are expressed in terms of the functions $G_\delta(p,q) = \int \frac{d^4k}{(2\pi)^4}\frac{1}{\Delta_B^q\Delta_F^{p+\delta}}$, which can be represented in the form

$$\begin{aligned} G_\delta(p,q) &= D(p,q;\mu_R^2) + B(p,q) + \delta\,(L(p,q;\mu_R^2) + J(p,q)) + O(\delta^2), \qquad p \le 0; \\ G_\delta(p,q) &= D(p,q;\mu_R^2) + J(p,q) + O(\delta), \qquad p > 0. \end{aligned} \tag{2.10}$$

The divergent parts $D(p,q;\mu_R^2)$ and $L(p,q;\mu_R^2)$ in the domain of interest can be determined by a straightforward procedure [5], whereas the functions $B(p,q)$ and $J(p,q)$ obey recursion relations of several types. These relations and the procedure of their derivation were described in [5]; their explicit form (very cumbersome) is given in [6]. With the use of these relations, the functions $F(p,q;\tilde{n})$ can be represented (see [1], [5]) as linear combinations of the constants F_0, Z_0, Z_1 and

$$\begin{aligned} Y_0 &= \frac{J(2,0)}{4} - \frac{F_0}{16\pi^2}, \qquad Y_1 = \frac{1}{48} - \frac{1}{4}Z_0 - \frac{1}{24}J(-1,2) + \frac{1}{12}J(0,1) + \frac{1}{12}J(1,0), \\ Y_2 &= \frac{1}{6} - \frac{1}{\pi^2} - Z_0 - \frac{1}{6}J(-1,2) + \frac{1}{3}J(0,1) - \frac{1}{24}J(1,-2) - \frac{1}{12}J(1,-1) - \\ &\quad -\frac{17}{8}J(1,0) + 4\,J(1,1) - \frac{1}{48}J(2,-2) + \frac{25}{6}J(2,-1) - 4\,J(2,0), \\ Y_3 &= -\frac{1}{384\pi^2} - F_0\frac{1}{128\pi^2} + \frac{1}{96}Z_0 - \frac{1}{48}J(-1,3) + \frac{1}{192}J(0,1) + \frac{1}{48}J(0,2) + \frac{1}{48}J(1,1); \end{aligned} \tag{2.11}$$

$$\begin{aligned} &Y_4 = \frac{J(1,0)}{2}, \qquad Y_5 = J(1,-1), \qquad Y_6 = 2J(1,-2), \qquad Y_7 = \frac{J(2,-1)}{2}, \\ &Y_8 = J(2,-2), \qquad Y_9 = \frac{J(3,-2)}{2}, \qquad Y_{10} = J(3,-3), \qquad Y_{11} = 2J(3,-4). \end{aligned} \tag{2.12}$$

The respective codes can be found on the web page of the ITEP Lattice group `http://www.lattice.itep.ru/∼pbaivid/lattpt/`. The results stored there are as follows: (i) the program for a computation of $F(p,q;\tilde{n})$ at $0 \le p,q \le 9$ and $n_1+n_2+n_3+n_4 \le 25$; (ii) the values of the functions $J(p,q)$ and $B(p,q)$ at $-26 \le p \le 0$, $-56-2p \le q \le 34$ and the values of $J(p,q)$ at $1 \le p \le 9$, $-28 \le q \le 33-p$; and (iii) The explicit expressions for $F(p,q;\tilde{n})$ at some particular values of p and q and all $n_1 \le 6$.

3. The Lüscher–Weisz method

To outline the LW method [7] of computation of two-loop diagrams in the coordinate representation, we consider the diagram in Fig.1, given by the expression

$$A_B(p) = \sum_{x\in\Lambda} e^{-ipx} G_B^3(x). \tag{3.1}$$

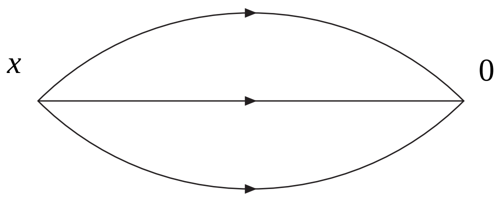

Figure 1

In the bosonic case, Lüscher and Weisz calculated $A_B(0)$ and its asymptotic expansion when $p \to 0$; they used the following representation:

$$\begin{aligned} A_B(0) &= G_B^3(0) + \sum_{x\in\Lambda'} G_{as}^3(x) \\ &+ \sum_{x\in\{\mathscr{F}_N\}} \Big(G_B^3(x) - G_{as}^3(x) \Big) \\ &+ \sum_{x\in\{\Lambda'\setminus\mathscr{F}_N\}} \Big(G_B^3(x) - G_{as}^3(x) \Big), \end{aligned} \tag{3.2}$$

where $\mathscr{F}_N = \{x : |x_1|+|x_2|+|x_3|+|x_4| \le N\}$, and $G_{as}(x)$ is an asymptotic approximation of $G_B(x)$ when $x \to \infty$,

$$G_{as}(x) = \frac{1}{[x^2]} + \left(\frac{2[x^4]-[x^2]^2}{[x^2]^4}\right) + \left(40\,\frac{[x^4]^2}{[x^2]^7} + 16\,\frac{[x^4]}{[x^2]^5} - 48\,\frac{[x^6]}{[x^2]^6} - 4\,\frac{1}{[x^2]^3}\right) + \ldots \tag{3.3}$$

In the domain $\mathscr{F}_N$, the propagator $G_B(x)$ can computed by the recursion formulas

$$G_B(x+\hat{\mu}) = G_B(x-\hat{\mu}) + \frac{2x_\mu}{\left(\sum_{\nu=1}^4 x_\nu\right)} \sum_{\lambda=1}^{4} (G_B(x) - G_B(x-\hat{\lambda})), \tag{3.4}$$

which allow to express it in terms of $G_B(0,0,0,0) = Z_0$ and $G_B(1,1,0,0) = -1/4 + Z_1 + Z_0$. The domain $\{\Lambda\setminus\mathscr{F}_N\}$ is chosen so that the propagator is fitted by its asymptotic expression (3.3) with a sufficient precision making it possible to neglect the third sum in the formula (3.2). Then the first sum can be calculated exactly using the summation formulas derived in [7] and the second sum can be expressed in terms of Z_1 and Z_2 by employing the relations (3.4).

4. Two-loop fermionic integrals.

In the fermionic case, calculations are performed by the same procedure, however, **we have no recursion relations similar to (3.4).** The fermionic propagator in x-representation

$$G_F(x_1,x_2,x_3,x_4) = \int \frac{d^4k}{(2\pi)^4} \frac{\cos(k_1x_1)\cos(k_2x_2)\cos(k_3x_3)\cos(k_4x_4)}{\Delta_F} \tag{4.1}$$

is expressed in terms of the quantities

$$F(p,q;n_1,n_2,n_3,n_4) = \int \frac{d^4k}{(2\pi)^4} \frac{\cos^{n_1}(k_1)\cos^{n_2}(k_2)\cos^{n_3}(k_3)\cos^{n_4}(k_4)}{\Delta_B^q \Delta_F^{p+\delta}} \tag{4.2}$$

by making use of the relations

$$\cos(nx) = 2^{n-1}\cos^n x + \frac{n}{2}\sum_{k=0}^{[n/2]-1} \frac{(-1)^{k+1}}{k+1} C_{n-k-2}^k (2\cos x)^{n-2k-2}. \tag{4.3}$$

To employ the LW method outlined above, we compile a table of values of $G_F(x)$ over the domain $x_1 \geq x_2 \geq x_3 \geq x_4 \geq 0,\ |x| \leq 48$ and derive an asymptotic approximation of $G_F(x)$ at $|x| \to \infty$ up to the terms of the order $1/[x^2]^4$. To treat integrals with nontrivial numerators, we should also compile the tables of the values

$$K_{B[F]} = \int \frac{dp}{(2\pi^4)} \frac{(e^{-ipx} \; - 1)}{D^2_{B[F]}(p)} \; , \quad L_{B[F]} = \int \frac{dp}{(2\pi^4)} \frac{\left(e^{-ipx} - 1 + \frac{x^2}{8} \, (4 \; - \sum_{\mu=1}^{4} \cos^2 k_\mu) \right)}{D^3_{B[F]}(p)} \; , \tag{4.4}$$

Each of these tables involves 14147 entries, each entry is a linear combination of the constants $F_0, Z_0, Z_1,\ Y_0, Y_1, ...Y_{11},\ \frac{1}{(2\pi)^2}$, and 1 with rational coefficients; from 5 to 20 MB per table in size. Fortunately, they can be conveniently treated with FORM [8].

The precision of 20 significant digits in determination of the constants $Y_4 \div Y_{11}$ [1], [5] is not sufficient for computation of $G_F(x)$ at $|x| > 6$. Using the procedure proposed in [7] for calculation of Z_0 and Z_1, we obtain

$$\begin{aligned}
Y_4 &= 0.08539036359532067913516702888533412058194147127443265(1) \\
Y_5 &= 0.46936331002699614475347539705751803482046295887523184(1) \\
Y_6 &= 3.39456907367713000586008689702374496453685272313733503(1) \\
Y_7 &= 0.05188019503901136636490228766471579940968012757291508(1) \\
Y_8 &= 0.23874773756341478520233613930386970445280194983477988(1) \\
Y_9 &= 0.03447644143803223145396188144243193600121277124715784(1) \\
Y_{10} &= 0.13202727122781293085314731098196596971197144795959477(1) \\
Y_{11} &= 0.75167199030295682253543148590778110991011277193144803(1)
\end{aligned}$$

At $|x| > 48$, $G_F(x)$ is approximated by the function

$$G_F^{(as)}(x) = \frac{1}{[x^2]} + \left(\frac{8[x^4] - 4[x^2]^2}{[x^2]^4} \right) + \left(640 \, \frac{[x^4]^2}{[x^2]^7} - 768 \, \frac{[x^6]}{[x^2]^6} + 208 \, \frac{[x^4]}{[x^2]^5} - \frac{40}{[x^2]^3} \right) + \ldots, \tag{4.5}$$

To provide an example, let us consider the following two-loop fermionic integrals:

$$Q_1^{BBB} = \int_{BZ} \frac{d^4k}{(2\pi)^4} \, \frac{d^4q}{(2\pi)^4} \sum_{\mu=1}^{4} \frac{\hat{k}_\mu^2 \hat{q}_\mu^2}{D_B(k) D_B(q) D_B(r)} \tag{4.6}$$

$$Q_1^{BBF} = \int_{BZ} \frac{d^4k}{(2\pi)^4} \, \frac{d^4q}{(2\pi)^4} \sum_{\mu=1}^{4} \frac{\hat{k}_\mu^2 \hat{q}_\mu^2}{D_B(k) D_B(q) D_F(r)}$$

and similar quantities with other combinations of bosonic and fermionic propagators. We can also consider

$$Q_2^{BBF} = \int_{BZ} \frac{d^4k}{(2\pi)^4} \, \frac{d^4q}{(2\pi)^4} \sum_{\mu=1}^{4} \frac{\hat{k}_\mu^2 \hat{q}_\mu^2 \hat{r}_\mu^2}{D_B(k) D_B(q) D_F(r)} \tag{4.7}$$

etc. The results of the computations are as follows:

$$
\begin{aligned}
Q_1^{BBB} &= 0.042306368(1) & Q_1^{FBB} &= 0.020079702(3) \\
Q_1^{BBF} &= 0.024555253(3) & Q_1^{FFB} &= 0.00969896(1) \\
Q_1^{BFF} &= 0.01173224(1) & Q_1^{FFF} &= 0.00576013(3) \\
Q_2^{BBB} &= 0.05462397818(1) & Q_2^{BBF} &= 0.02659175158(3) \\
Q_2^{BFF} &= 0.0130373237(1) & Q_2^{FFF} &= 0.0064945681(3)
\end{aligned}
\tag{4.8}
$$

5. Summary and Outlook

The BCP algorithm has been realized on a computer. The basic fermionic integrals $G(p,q)$ are found over a sufficiently large domain of values of p and q. This allows (i) to express one-loop intergals involving fermionic denominators in terms of the constants F_0, Z_0, Z_1 and $Y_0 \div Y_{11}$ and (ii) to compute $G_F(x)$ at $|x| \leq 96$.

The LW method is extended to the case of fermions; asymptotic behavior of the fermionic propagator at $|x| \to \infty$ is found. Therewith, $G_F(x)$ is expressed at $|x| \leq 48$ in terms of the constants $Y_4 \div Y_{11}$, the values of which are computed to a precision of 54 significant digits. This is really needed for calculation of two-loop integrals. A new feature of FORM - a possibility to work with database-like structures - proved to be useful for summation over the domain $|x| \leq 48$. As an illustration, several two-loop fermionic integrals are evaluated at zero external momentum.

Operations with a table of precise values of the functions $G_{B(F)}(x)$ $K_{B(F)}(x)$ and $L_{B(F)}(x)$ allow to compute one-loop and two-loop diagrams of the propagator type at nonvanishing external momentum. The work is in progress!

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