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New extended Crewther-type relation and the consequences of multiloop perturbative results

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We discuss the current status of the investigations of the conformal symmetry breaking contributions to three forms for the QCD generalizations of the Crewther relation. The new third form of the extension of this relation is considered in more detail. Particular attention is paid to the discussions of the applications of the β-expansion formalism proposed previously by one of us (SVM). Several relations between 5-loop contributions to the Adler D-function D_A^{NS} and to the polarized Bjorken sum rule $S_{\rm Bjp}$ are presented. One of them gives the additional confirmation of the correctness of the advanced analytical computer calculations of order α_s^4 contributions to D_A^{NS} and $S_{\rm Bjp}$ in the general $SU(N_c)$ gauge group.

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1. Introduction

There are several complementing each other QCD extensions of the original Crewther relation, which was derived in [1] from the axial-vector-vector (AVV) triangle amplitude in the quark-parton model. This fundamental relation has the following form:

$$D \cdot C^{Bjp} = 1 \ . \tag{1.1}$$

The first entry in the l.h.s of Eq.(1.1) is defined as the quark-parton model expression for the non-singlet part of the e^+e^- Adler function, D_A^{NS},

$$D_A^{NS}(a_s) = \left(N_c \sum_f Q_f^2\right) \cdot D(a_s) \tag{1.2}$$

while the second term in the l.h.s. is the quark-parton model limit of the non-singlet coefficient function C^{Bjp} of the Bjorken sum rule of the polarized lepton-hadron deep-inelastic scattering (DIS)

$$S_{Bjp}(a_s) = \left(\frac{1}{6}\frac{g_A}{g_V}\right) \cdot C^{Bjp}(a_s). \tag{1.3}$$

C^{Bjp} also enters into the non-singlet part of the Ellis-Jaffe sum rule of the polarized lepton-hadron DIS.

The derivation of Eq.(1.1) is essentially based on the concept of the conformal symmetry (CS). It is known that CS , which is valid in the quark-parton model limit, is the symmetry under the following transformations of coordinates (see e.g. the reviews [2], [3]) :

1. scale transformation or dilatation $x^{'\mu} = \rho x^\mu$ with 1 parameter ρ>0,
2. special conformal transformations $x^{'\mu} = \dfrac{x^\mu + \beta^\mu x^2}{1 + 2\beta x + \beta^2 x^2}$ with 4 parameters β^μ and
3. translations $x^{'\mu} = x^\mu + \alpha^\mu$ with 4 parameters α^μ,
4. homogeneous Lorentz transformations $x^{'\mu} = \Lambda^\mu_\nu x^\nu$ that also contain 4 parameters.

Note that in the case of perturbative quenched QED, namely, in the approximation when QED diagrams containing internal photon vacuum polarization contributions are neglected, the original Crewther relation of Eq.(1.1) is also valid [4]. Other important quantum field theory studies based on the concept of the CS are described, e.g., in [3], [5], [6].

However, it is known that conformal symmetry is broken by the normalization of the coupling constant in the renormalized massless quantum field models (for details see, e.g., [7]). One of the basis results of this procedure is nonzero normalization group β–function that can be defined within perturbation theory. The factor $\beta(a_s)/a_s$ appears as the result of renormalization of the trace of the energy-momentum tensor [8, 1, 9, 10, 11, 12] and generates the conformal anomaly. Here $a_s = \alpha_s/\pi$, and α_s is the QCD coupling.

The original Crewther relation of Eq.(1.1) can be generalized to the QCD case in the following way:

$$D(a_s(Q^2)) \cdot C^{Bjp}(a_s(Q^2)) = 1 + \Delta_{\rm csb}(a_s(Q^2)), \tag{1.4}$$

where Q^2 is the Euclidean transfered momentum and $\Delta_{\rm csb}(a_s(Q^2))$ is the conformal-symmetry breaking (CSB) term.

Three forms for the QCD generalizations of the Crewther relations are known at present. Within the first generalization the additional correction to the "Crewther unity" in the $\overline{MS}$-scheme reads

$$\Delta_{\rm csb}(a_s) = \left(\frac{\beta(a_s)}{a_s}\right)P(a_s) = \left(\frac{\beta(a_s)}{a_s}\right)\sum_{m\geq 1} K_m a_s^m . \qquad (1.5)$$

This expression was discovered in [13], where the analytical expressions of two terms K_i with $i \leq 2$ was fixed. The work [13] was based on careful inspection of the $SU(N_c)$ group structure of the available (to the moment) perturbative approximations for the functions $D(a_s)$ and $C^{Bjp}(a_s)$ and of the 2-loop expression for the QCD β-function. The NLO corrections to the D-function were known from the calculations performed both analytically [14] and numerically [15] (the analytical results were confirmed in [16]), while the analytical N^{2}LO perturbative corrections were evaluated in [17], [18] and confirmed later on in [19] with the help of a different theoretical technique. Analogous NLO corrections to C^{Bjp} calculated in [20] were confirmed later on in [21]. The corresponding N^{2}LO corrections were evaluated in [22].

The validity of the expression (1.5) in all orders of perturbation theory was explored in the momentum space [23] (for some details see [24]) and proved rigorously in the coordinate space [25] (for additional discussions see [3]). The explicit expression for the N^{3}LO term K_3 in Eq.(1.5) was obtained recently [26]. It was found as the result of multiplication of $SU(N_c)$ group expression for the $\mathcal{O}(a_s^4)$ corrections to the $D(a_s)$[1] and $C^{Bjp}(a_s)$ functions the analytically calculated in [26] with taking into account the 3-loop analytical approximation of the QCD β-function, originally evaluated in [28] and confirmed later on in [29].

In the second generalization of the Crewther relation formulated in [30] at N^{2}LO, the CSB-term $\Delta_{\rm csb}(a_s(Q^2))$ in Eq.(1.4) is absorbed in the scale Q_D^{*2} of the effective charge of the D-function $\hat{a}_{Bjp}(Q^2) = \alpha_s^{eff}(Q^2)/\pi$ by using the BLM scale-fixing procedure [31] extended firstly to the N^{2}LO in [32]. As the result, the second generalized form of the Crewther relation takes the following N^{2}LO form:

$$\left(1 + \hat{a}_D(Q_D^{*2})\right)\left(1 - \hat{a}_{Bjp}(Q^2)\right) = 1 \qquad (1.6)$$

where $\hat{a}_D(Q^2)$ and $\hat{a}_{Bjp}(Q^2)$ are the effective charges of the $D(a_s(Q^2))$ and $C^{Bjp}(Q^2)$ functions. The concept of the effective charges was introduced in the process of creation of the effective charges approach developed in [33], [34], [35].

However, the all-order proof of the second generalization of the Crewther relation of Eq.(1.6) is absent. In view of this the analysis of [26], which obviously demonstrated, that Eq.(1.6) is also satisfied at N^{3}LO, can be considered as an important step of the verification of the second form of the generalization of the Crewther relation in high orders.

It is also possible to use the N^{2}LO variant of the BLM prescription [32] to absorb the N^{2}LO approximation of the CSB-term $\Delta_{\rm csb}(a_s(Q^2))$ in Eq.(1.4) into the BLM scale of the effective charge

[1] In the case of N_c=3 numbers of colours, this correction is known from analytical calculations of [27].

$\hat{a}_{Bjp}(Q^2)$ [36] and to obtain N^{2}LO variant of the results of Ref.[30] in the following form:

$$\Big(1+\hat{a}_D(Q^2)\Big)\Big(1-\hat{a}_{Bjp}(Q^{*2}_{Bjp})\Big)=1\ . \tag{1.7}$$

The studies, analogous to those in [26], should demonstrate the N^{3}LO validity of this expression for the second generalization of the Crewther relation.

Within the third variant of the generalized Crewther relation, proposed at N^{2}LO in [37] and considered in detail at the N^{3}LO in [38], the CSB-term in Eq.(1.4) is expressed through the following double expansion:

$$\Delta_{\rm csb}(a_s)=\sum_{n\geq 1}\left(\frac{\beta(a_s)}{a_s}\right)^n\mathscr{P}_n(a_s) = \sum_{n\geq 1}\sum_{r\geq 1}\left(\frac{\beta(a_s)}{a_s}\right)^n P_n^{(r)}a_s^r \tag{1.8}$$

$$= \sum_{n\geq 1}\sum_{r\geq 1}\left(\frac{\beta(a_s)}{a_s}\right)^n P_n^{(r)}[{\rm k},{\rm m}]{\rm C_F^k C_A^m}\ a_s^r, \tag{1.9}$$

where the first expansion parameter is the function $(\beta(a_s)/a_s)$ and the second expansion parameter is the coupling a_s and $\rm C_F$ and $\rm C_A$ are the quadratic Casimir operators of $SU(N_c)$ group. The indices ${\rm k},{\rm m}$ and r in Eq.(1.9) are related as ${\rm k}+{\rm m}=r$ and the coefficients $P_n^{(r)}[{\rm k},{\rm m}]$ contain rational numbers and the odd ζ-functions. It should be stressed that the coefficients of $\mathscr{P}_n(a_s)$ in Eq.(1.8) do not depend on $\rm n_f$ which enter in the coefficients K_m of the first form of the generalized Crewther relation (1.5). However, contrary to the first generalization of the Crewther relation, the validity of this third generalization of the Crewther relation [38] is not yet proved to all orders of perturbation theory. Another interesting problem is related to the question whether special features of the third extension of the Crewther relation (1.8) and Eq.(1.9), namely, the $\rm n_f$ independence of its coefficients, can be effectively used in practise. To analyze this question in more detail, we supplement the discussions of [38] by extra considerations of the relations between the coefficients in the polynomials $\mathscr{P}_n(a_s)$ in Eq.(1.8) and the ones in $D(a_s)$, $C_{Bjp}(a_s)$ and $\beta(a_s)$ functions. The β-expansion formalism [39] will be applied to this task and the results will be studied in detail.

2. Applications of β-expansion formalism

Consider perturbative expansion of the normalized flavour non-singlet part of the Adler function D from Eq.(1.2) and the normalized C^{Bjp} function from Eq.(1.3), namely,

$$D(a_s)\ =\ 1+\sum_{n=1} d_n\ a_s^n\ ; \tag{2.1}$$

$$C^{Bjp}(a_s)\ =\ 1+\sum_{l=1} c_l\ a_s^l\ . \tag{2.2}$$

The QCD coupling constant a_s obey the renormalization group equation with the QCD β-function which we will define as

$$\mu^2\frac{d}{d\mu^2}a_s=\beta(a_s)=-a_s^2\left(\beta_0+\beta_1 a_s+\beta_2 a_s^2+\ldots\right). \tag{2.3}$$

Within the β-expansion formalism of Ref. [39], instead of commonly used representation of the coefficients of perturbative expansions for the renormalization-group invariant quantities in powers of $\mathrm{T_F n_f}$ and the colour group factors, one should consider their expansions in powers of the β_0, β_1, $\beta_2 \ldots$ of the β-function. For the quantities defined by the Eqs.(2.1) and (2.2) the coefficients of this expansion approach are $d_n[n_0,n_1,\ldots]$, $c_l[n_0,n_1,\ldots]$. Their first argument corresponds to the term with n_0 powers of β_0, $\beta_0^{n_0}$, the second one – n_1 powers of β_1, $\beta_1^{n_1}$, and so on. The elements $d_n[0,0,\ldots,0]$, $c_l[0,0,\ldots,0]$ represent "genuine" corrections with powers $n_i = 0$ of all coefficients β_i. The latter elements coincide with expressions for the coefficients d_n, c_l in the imaginary case of the nullified QCD β–function in all orders of perturbation theory is considered. This case corresponds to restoration of CS of some quantum field model and will be considered here as a technical trick. If all arguments n_i after index m of the elements $d_n[\ldots,m,0,\ldots,0]$ ($c_l[\ldots,m,0,\ldots,0]$) are equal to zero, then, for the sake of a simplified notation, we omit these arguments and write these elements as $d_n[\ldots,m]$. For the clarification of the β-expanded view of the coefficients in perturbation series for physical quantities we consider the corresponding representations of the several terms of Eq.(2.1), namely,

$$d_2 = \beta_0 d_2[1] + d_2[0], \tag{2.4}$$
$$d_3 = \beta_0^2 d_3[2] + \beta_1 d_3[0,1] + \beta_0 d_3[1] + d_3[0], \tag{2.5}$$
$$d_4 = \beta_0^3 d_4[3] + \beta_1 \beta_0 d_4[1,1] + \beta_2 d_4[0,0,1] + \beta_0^2 d_4[2] + \beta_1 d_4[0,1] + \beta_0 d_4[1] + d_4[0], \tag{2.6}$$
$$d_n = \beta_0^{n-1} d_n[n-1] + \ldots \tag{2.7}$$

The same ordering in the β-function coefficients can be applied to the coefficients c_l. The expressions like Eq.(2.4–2.7) are unique. The first of them, Eq.(2.4), is the basis of the standard BLM prescription [31].

The coefficients $d_n[n-1]$ are identical to the terms generated by the renormalon chain insertions and can be obtained from the results of Ref.[13]. The clarification of the physical and mathematical origin of other elements is a separate and not straightforward task. The problem of getting diagrammatic representation for different contributions into the β-expanded coefficients was considered in [39]. Further on in this Section we specify how to obtain β-expanded results at the level of order a_s^3-corrections.

Together with Eq.(2.4–2.7), Eq.(1.8) gives the possibility to express the sum of the elements of n–loop β-expanded coefficients through the ones which result from $(n-1)$–loop calculations.

We will use the property, that Eq.(1.1) is satisfied in the CS limit of QCD, namely, in the case when the β- function has identically zero coefficients $\beta_i = 0$ for $i \geq 0$. In this model, the Crewther relation (1.1) can be rewritten as

$$D_0 \cdot C_0^{Bjp} = 1, \tag{2.8}$$

where the expansions for the functions D_0 and C_0^{Bjp}, analogous to the ones of Eq.(2.1), Eq.(2.2), will contain the coefficients of genuine content only, namely, d_n $(c_n) \equiv d_n[0]$ $(c_n[0])$.

Equation (2.8) provides evident relation between the genuine elements in any loops, namely,

$$c_n[0] + d_n[0] + \sum_{l=1}^{n-1} d_l[0] c_{n-l}[0] = 0. \tag{2.9}$$

In particular, they express the yet unknown genuine parts of the 5-loop terms d_4, c_4, through the 4-loop results already known from the analysis in [39]:

$$c_4[0] + d_4[0] = 2d_1 d_3[0] - 3d_1^2 d_2[0] + (d_2[0])^2 + d_1^4. \tag{2.10}$$

This equation contains contributions proportional to $\mathrm{C_F}$ and $\mathrm{C_A}$. Note that to check the perturbatively quenched QED approximation for d_4 [27], available from [40], it was suggested in [41] to use the relation that arises from the projection of the relation (2.10) onto the maximum power of $\mathrm{C_F}$, namely, $\mathrm{C_F}^4$.

The β–expanded form for the d_3-term was obtained in [39] by means of a careful consideration of the analytical $\mathscr{O}(a_s^3)$ expression for the Adler function $D(a_s, \mathrm{n_f}, \mathrm{n_{\tilde{g}}})$ with the $\mathrm{n_{\tilde{g}}}$ MSSM gluino multiplets, obtained in [19] [2]. The element $d_3[2]$, which is proportional to the maximum power β_0^2 in (2.5), can be fixed in a straightforward way. Then one should separate the contributions $\beta_1 d_3[0,1]$ and $\beta_0 d_3[1]$ to the d_3-term. They both are linear in the number of quark flavours $\mathrm{n_f}$. Their separation is possible if one uses additional degrees of freedom – the gluino contributions mentioned above and labelled here by their $\mathrm{n_{\tilde{g}}}$ multiplet number. In this way, one can get the explicit form for the functions $\mathrm{n_f} = \mathrm{n_f}(\beta_0, \beta_1)$ and $\mathrm{n_{\tilde{g}}} = \mathrm{n_{\tilde{g}}}(\beta_0, \beta_1)$. They can be obtained after taking into account the gluino contributions to the first two coefficients of the QCD β-functions known from the two-loop calculations performed in [44]. Finally, one arrives at the expressions for the coefficients in Eqs.(2.4–2.5) presented in [38],

$$d_1 = \frac{3}{4}\mathrm{C_F};\ d_2[1] = \left(\frac{33}{8} - 3\zeta_3\right)\mathrm{C_F};\ d_2[0] = -\frac{3}{32}\mathrm{C_F^2} + \frac{1}{16}\mathrm{C_F C_A}; \tag{2.11}$$

$$d_3[2] = \left(\frac{151}{6} - 19\zeta_3\right)\mathrm{C_F}$$

$$d_3[1] = \left(-\frac{27}{8} - \frac{39}{4}\zeta_3 + 15\zeta_5\right)\mathrm{C_F^2} - \left(\frac{9}{64} - 5\zeta_3 + \frac{5}{2}\zeta_5\right)\mathrm{C_F C_A}; \tag{2.12}$$

$$d_3[0,1] = \left(\frac{101}{16} - 6\zeta_3\right)\mathrm{C_F};\ d_3[0] = -\frac{69}{128}\mathrm{C_F^3} + \frac{71}{64}\mathrm{C_F^2 C_A} + \left(\frac{523}{768} - \frac{27}{8}\zeta_3\right)\mathrm{C_F C_A^2}\,. \tag{2.13}$$

which differ from the ones, originally obtained in Ref. [39], by the renormalization factor only. Let us emphasize that gluinos are used here as a pure technical device to reconstruct the β-function expansion of the perturbative coefficients.

Using the relation (2.9) for $n = 2$ and $n = 3$ and the already fixed $d_2[0]$ and $d_3[0]$-terms we get the expression for the $c_2[0]$ and $c_3[0]$ coefficients. Their knowledge allowed us to fix other elements in the $c_2[\ldots]$ and $c_3[\ldots]$ terms [38], without attracting additional gluino degrees of freedom. The results obtained in [38] read:

$$c_1 = -\frac{3}{4}\mathrm{C_F};\ c_2[1] = -\frac{3}{2}\mathrm{C_F};\ c_2[0] = \frac{21}{32}\mathrm{C_F^2} - \frac{1}{16}\mathrm{C_F C_A}; \tag{2.14}$$

$$c_3[2] = -\frac{115}{24}\mathrm{C_F};\ c_3[1] = \left(\frac{83}{24} - \zeta_3\right)\mathrm{C_F^2} + \left(\frac{215}{192} - 6\zeta_3 + \frac{5}{2}\zeta_5\right)\mathrm{C_F C_A}; \tag{2.15}$$

$$c_3[0,1] = \left(-\frac{59}{16} + 3\zeta_3\right)\mathrm{C_F};\ c_3[0] = -\frac{3}{128}\mathrm{C_F^3} - \frac{65}{64}\mathrm{C_F^2 C_A} - \left(\frac{523}{768} - \frac{27}{8}\zeta_3\right)\mathrm{C_F C_A^2}. \tag{2.16}$$

Note that the approximations for the coefficients of the Adler function and of the Bjorken polarized sum rule, similar to those of Eqs.(2.4–2.5) but with the terms proportional to the powers of β_0

[2] The 3-loop contribution of light gluinos coincide with the numerical result in [42], while at the 4-loop analytical result for gluino contribution, evaluated in [19], was confirmed in [43].

only, were studied in [45]. These expressions from Ref.[45] can be compared with the results of Eqs.(2.11-2.16).

Consider now one of the applications of the β-expansion formalism [39]. Substituting the β-expanded expressions for d_i and c_i into the general relations of Eq.(1.8) we found the following identities [38]:

$$\begin{aligned}
\mathscr{P}_1(a_s) &= a_s\Big\{P_1^{(1)}+a_sP_1^{(2)}+a_s^2P_1^{(3)}\Big\} \\
&= -a_s\Big\{c_2[1]+d_2[1]+a_s\Big(c_3[1]+d_3[1]+d_1\left(c_2[1]-d_2[1]\right)\Big) \\
&\quad +a_s^2\Big(c_4[1]+d_4[1]+d_1\left(c_3[1]-d_3[1]\right)+d_2[0]c_2[1]+d_2[1]c_2[0]\Big)\Big\} && (2.17) \\
\mathscr{P}_2(a_s) &= a_s\{P_2^{(1)}+a_sP_2^{(2)}\} \\
&= a_s\Big\{c_3[2]+d_3[2]+a_s\Big(c_4[2]+d_4[2]-d_1(c_3[2]-d_3[2])\Big)\Big\} && (2.18) \\
\mathscr{P}_3(a_s) &= a_s\Big\{P_3^{(1)}\Big\} \\
&= -a_s\{c_4[3]+d_4[3]\} = a_s\mathrm{C_F}\left(\frac{307}{2}-\frac{203}{2}\zeta_3-45\zeta_5\right) && (2.19) \\
\mathscr{P}_n(a_s) &= a_sP_n^{(1)} = (-1)^n a_s\{c_n[n-1]+d_n[n-1]\} && (2.20)
\end{aligned}$$

The elements $d_n[n-1]$ ($c_n[n-1]$) can be obtained from the results in [13] for the leading renormalon chain insertions. The elements $d_n[l],(l<n-1)$ stem from the subleading renormalon chains. Different relations between the elements d_4 (d_n) and c_4 (c_n) can be derived from the expression (1.8) which is definitely true at the 5-loop level [38]. Its coefficients were obtained by us analytically from the requirement of their $\mathrm{n_f}$ independence [38]. We also used the property of universality of the weight function $\mathscr{P}_1$ ($\mathscr{P}_n$) of the different β_i-terms. This property implies that the first term $\mathscr{P}_1(a_s)$ in Eq.(2.17) generates the following chain of equations:

$$\begin{aligned}
P_1^{(1)} &= -c_2[1]-d_2[1] = -c_3[0,1]-d_3[0,1] = -c_4[0,0,1]-d_4[0,0,1] = \dots \\
&= -c_n[\underbrace{0,0,\dots,1}_{n-1}]-d_n[\underbrace{0,0,\dots,1}_{n-1}] = \left(-\frac{21}{8}+3\zeta_3\right)\mathrm{C_F} && (2.21)
\end{aligned}$$

which fixes the corresponding sums of c_n and d_n in any order by the universal first term $P_1^{(1)}$ in the polynomial $\mathscr{P}_1$. The second term of $\mathscr{P}_1$ in Eq.(2.17) defines a similar chain of equations

$$\begin{aligned}
P_1^{(2)} &= -c_3[1]-d_3[1]-d_1\left(c_2[1]-d_2[1]\right) = -c_4[0,1]-d_4[0,1]-d_1\left(c_3[0,1]-d_3[0,1]\right) = \dots \\
&= -c_n[\underbrace{0,\dots,1}_{n-2}]-d_n[\underbrace{0,\dots,1}_{n-2}]-d_1\Big(c_{n-1}[\underbrace{0,\dots,1}_{n-2}]-d_{n-1}[\underbrace{0,\dots,1}_{n-2}]\Big) = \dots \\
&= \left(\frac{397}{96}+\frac{17}{2}\zeta_3-15\zeta_5\right)\mathrm{C_F^2}-\left(\frac{47}{48}-\zeta_3\right)\mathrm{C_FC_A} && (2.22)
\end{aligned}$$

where the explicit expression for $P_1^{(2)}$ is already known (see [38]). The expressions for $P_1^{(3)}$ and $P_2^{(2)}$ are determined by Eqs.(2.17) and (2.18) respectively, and read

$$P_1^{(3)} = -c_4[1] - d_4[1] - d_1\big(c_3[1] - d_3[1]\big) + d_2[0]c_2[1] + d_2[1]c_2[0] \tag{2.23}$$
$$P_2^{(2)} = c_4[2] + d_4[2] - d_1(c_3[2] - d_3[2]) \tag{2.24}$$

where their concrete form is known from the studies of Ref. [38].

Thus, $P_1^{(3)}$ and $P_2^{(2)}$ depend on still unknown contributions $d_4[1]$, $d_4[2]$ and $c_4[1]$, $c_4[2]$ for the β-expanded form of the general $SU(N_c)$-group expressions for the 5-loop terms d_4 and c_4 obtained in [26]. Therefore, to reformulate the considerations of [38] within the β-expansion approach, it is necessary to determine these unknown contributions to d_4 and c_4.

The theoretical problem mentioned above may be solved after extra analytical evaluation of the gluino contributions to the 5-loop perturbative coefficients d_4 and c_4 and applications of the results of evaluation of the gluino contributions to the 3-loop QCD β-function [46]. The knowledge of these inputs from the $N = 1$ SUSY model may also allow a better understanding of special features and constraints on the elements of the sum of $d_5 + c_5$ coefficients which can be useful for study of the different forms of QCD generalizations of the Crewther relation in high orders of perturbation theory.

3. The constraints for the structure of the 5-loop analytical results

In this section, we explain how the relations discussed above and, in particular, the ones of Eqs.(2.11-2.16) allow one to get an additional constraint which confirms the correctness of the analytical results of Ref. [26].

We first apply the β-expansion approach to the sum $d_4 + c_4$. As the next step we consider the concrete number of flavours which nullify the β-function coefficients β_i. This is done by fixing n_i as a roots of the coefficient $\beta_i(\mathrm{n_f} = \mathrm{n_i})$. The condition $\beta_0(\mathrm{n_f} = n_0) = 0$, which gives $\mathrm{T_F}n_0 = (11/4)\mathrm{C_A}$, was studied by Banks–Zaks some time ago [47]. Applying this ansatz to the sum of the β-expanded forms for the sum of the analysed 5-loop terms, we get [38]

$$c_4(n_0)+d_4(n_0)=c_4[0]+d_4[0]+\beta_2(n_0)\,(c_4[0,0,1]+d_4[0,0,1])+\beta_1(n_0)\,(c_4[0,1]+d_4[0,1])\,. \tag{3.1}$$

The terms in the r.h.s. of Eq.(3.1) are already known from the r.h.s. of Eq.(2.10) for $\mathrm{c}_4[0] + \mathrm{d}_4[0]$, r.h.s. of Eq.(2.21) for the $P_1^{(1)}$-coefficient, and r.h.s. of Eq.(2.22) for the $P_1^{(2)}$- term. Substituting the concrete analytical results into these expressions, we obtain [38]

$$\begin{aligned} \mathrm{d}_4(n_0) + \mathrm{c}_4(n_0) = & -\frac{333}{1024}\mathrm{C_F^4} - \mathrm{C_A C_F^3}\left(\frac{1661}{3072} - \frac{1309}{128}\zeta_3 + \frac{165}{16}\zeta_5\right) \\ & -\mathrm{C_A^2 C_F^2}\left(\frac{3337}{1536} + \frac{7}{2}\zeta_3 + \frac{105}{16}\zeta_5\right) - \mathrm{C_A^3 C_F}\left(\frac{28931}{12288} - \frac{1351}{512}\zeta_3\right). \end{aligned} \tag{3.2}$$

Then applying the Banks-Zaks way of fixation of $\mathrm{n_f}$ to the concrete analytical expression for $c_4(\mathrm{n_f}) + d_4(\mathrm{n_f})$, which follows from the work of Ref.[26], we reproduced the r.h.s. of Eq.(3.2) [38]. This agreement gave us an extra argument in favour of the correctness of the results of distinguished computer analytical calculations of the INR-Karlsruhe-SINP group [26]. Moreover, having a look at the r.h.s. of Eq.(3.2), we observe the absence of the ζ_7 and ζ_3^2-terms which exist in analytical expressions of both d_4 and c_4 (see Ref.[26]). This feature observed in Ref. [38] confirms the foundation of the proportionality of two transcendentalities mentioned above to the first coefficient β_0 of the QCD β-function [26].

In the same way, taking $\mathrm{n_f} = n_1, (\beta_1(n_1) = 0)$ one can get the following constraint:

$$
\begin{aligned}
c_4(n_1) + d_4(n_1) \; = \; & c_4[0] + d_4[0] + \beta_0(n_1)\,(c_4[1] + d_4[1]) + \beta_0^2(n_1)\,(c_4[2] + d_4[2]) \\
& + \beta_2(n_0)\,(c_4[0,0,1] + d_4[0,0,1]) + \beta_0^3(n_1)\,(c_4[3] + d_4[3])\,. \qquad (3.3)
\end{aligned}
$$

The terms $c_4[0] + d_4[0]$ and $c_4[0,0,1] + d_4[0,0,1]$ in the r.h.s. of Eq. (3.3) can be obtained like in the previous case, while $c_4[3] + d_4[3]$ follows from Eq.(2.19). The term $c_4[1] + d_4[1]$ can be extracted from $P_1^{(3)}$ in Eq.(2.23) and the term $c_4[2] + d_4[2]$ can be extracted from $P_2^{(2)}$ in Eq.(2.24). The requirement $\beta_1(n_1) = 0$ leads to the following expression for $\mathrm{n_f}$:

$$
\mathrm{n_f} = \frac{17\mathrm{C_A}^2}{(10\mathrm{C_A} + 6\mathrm{C_F})\mathrm{T_F}}\,. \qquad (3.4)
$$

For this expression we get a more complicated analytical structure of the r.h.s. of Eq.(3.3) which does not reveal any special cancellations of the transcendental functions. In view of this we do not present it here in the explicit form.

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Appendix

Throughout these studies we used the following expressions for the coefficients of the QCD β-function:

$$
\begin{aligned}
\beta_0 &= \frac{11}{12}\mathrm{C_A} - \frac{1}{3}\mathrm{T_F n_f}, \qquad (3.5) \\
\beta_1 &= \frac{17}{24}\mathrm{C_A^2} - \frac{5}{12}\mathrm{C_A T_F n_f} - \frac{1}{4}\mathrm{C_F T_F n_f} \qquad (3.6) \\
\beta_2 &= \frac{2857}{3456}\mathrm{C_A^3} - \frac{1415}{1728}\mathrm{C_A^2 T_F n_f} - \frac{205}{576}\mathrm{C_F C_A T_F n_f} \\
&\quad + \frac{79}{864}\mathrm{C_A T_F^2 n_f^2} + \frac{1}{32}\mathrm{C_F^2 T_F n_f} + \frac{11}{144}\mathrm{C_F T_F^2 n_f^2}\,, \qquad (3.7)
\end{aligned}
$$

They are known from the 3-loop analytical calculations performed in [28] and confirmed later on in [29]. In the case of QCD, supplemented by $\mathrm{n_g}$ number of flavours of Majorano gluinos, the coefficients of the β-function receive additional $\overline{MS}$-scheme contributions obtained at 3-loops in

[46], namely,

$$\overline{\beta}_0 = \beta_0 - \frac{1}{6}\mathrm{C_A n_g} \tag{3.8}$$

$$\overline{\beta}_1 = \beta_1 - \frac{5}{6}\mathrm{C_A^2 n_g} - \frac{1}{8}\mathrm{C_A^2 n_g} \tag{3.9}$$

$$\overline{\beta}_2 = \beta_2 - \frac{247}{432}\mathrm{C_A^3 n_g} + \frac{7}{54}\mathrm{C_A^3 T_F n_f n_g} + \frac{11}{288}\mathrm{C_F C_A^2 T_F n_f n_g} + \frac{145}{3456}\mathrm{C_A^3 n_g^2} \tag{3.10}$$

Up to the 2-loop level they are scheme-independent. When SUSY is not violated, the 3-loop calculations should be performed in the dimensional reduction and $\overline{DR}$-scheme which preserve supersymmetry at the 3-loop level. However, since we are interested in the contributions of $\mathrm{n_g}$ number of gluiono flavours only, we limit ourselves to the considerations of the part of the $N = 1$ SUSY 3-loop contributions evaluated in the $\overline{MS}$-scheme.

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