PROCEEDINGS OF SCIENCE

One rigorous negative result in quantum field theory

Melita Mnatsakanova[*]
Skobeltsyn Institute of Nuclear Physics, Moscow, Russia
E-mail: mnatsak@theory.sinp.msu.ru

Yu. S. Vernov
Institute for Nuclear Research of Russian Academy of Sciences, Moscow, Russia
E-mail: vernov@inr.ac.ru

It has been proved that in commutative and space-space noncommutative quantum field theory commutator of interactive field cannot be a function of linear combination of differences of time and one spatial coordinate in corresponding points.

The XIXth International Workshop on High Energy Physics and Quantum Field Theory
8-15 September 2010
Golitsyno, Moscow, Russia

[*]Speaker.

© Copyright owned by the author(s) under the terms of the Creative Commons Attribution-NonCommercial-ShareAlike Licence. http://pos.sissa.it/

PoS(QFTHEP2010)042

1. Introduction

Negative results play important role in axiomatic QFT. They show that relation between asymptotic and interacting fields is very nontrivial. The most important example of such results is Haag's theorem.

Here we consider another well-known result and show that it is possible to obtain the stronger result at the same assumptions. Let us recall original statement [1]:

Theorem 1. *If any local field $\varphi(x)$ is irreducible and*

$$[\varphi(x), \varphi(y)] = A(x-y), \tag{1.1}$$

then operator $A(x)$ is multiple of unit operator, that is $\varphi(x)$ is an asymptotic field.

First we show that, actually, the commutator in question can not be an operator depending on the difference between one spatial coordinate in points x and y. Our result is most interesting in the case of noncommutative theory [2], [3], precisely, in the case of space-space noncommutativity, in which time commutes with spatial variables and, as a consequence, one spatial variable, say x_3, commutes with others.

Let us recall that noncommutative quantum field theory (NC QFT) is defined by the Heisenberg-like commutation relations between coordinates

$$[x^\mu, x^\nu] = i\,\theta^{\mu\nu}, \tag{1.2}$$

where $\theta^{\mu\nu}$ is a constant antisymmetric matrix.

In our proof we use the following general principles of axiomatic field theory:

i) Local commutativity condition (LCC);

ii) Irreducibility of the set of field operators.

For simplicity we consider the case of neutral scalar fields.

Local commutativity means that

$$[\varphi(x), \varphi(y)] = 0, \quad \text{if} \quad x \sim y. \tag{1.3}$$

The condition $x \sim y$ in usual (commutative) theory means that

$$(x-y)^2 < 0.$$

In noncommutative quantum field theory (NC QFT) LCC can be fulfilled with respect to commutative variables only. The reason is that test functions, corresponding to noncommutative variables, belong to the one of Gelfand-Shilov spaces S^β with $\beta < 1/2$ [4], which does not contain functions with finite support and so corresponding field operators can not satisfy LCC. Thus in NC QFT we have the following LCC:

$$[\varphi(x), \varphi(y)] = 0, \quad \text{if} \quad (x_0 - y_0)^2 - (x_3 - y_3)^2 < 0. \tag{1.4}$$

PoS(QFTHEP2010)042

Let us stress that our result is valid in any theory, where this condition is fulfilled.

Now let us recall the condition of irreducibility.

The set of field operators $\varphi(x)$ is irreducible if the bounded operator, which commutes with all field operators, has to be $C\mathbb{I}$, where $\mathbb{I}$ is identical operator and C is some function.

Our proof is the modification of the classical proof given in the book [1].

2. Generalization of the Theorem 1

Let us prove that

Theorem 2. *If*

$$[\varphi(x), \varphi(y)] = A(x_3 - y_3, X, Y), \tag{2.1}$$

where we denote all other variables as X, Y, then

$$A(x_3 - y_3, X, Y) = C\mathbb{I},$$

where C is some function.

Proof Let us remind the Jacobi identity:

$$[\varphi(x), [\varphi(y), \varphi(z)]] + [\varphi(y), [\varphi(z), \varphi(x)]] + [\varphi(z), [\varphi(x), \varphi(y)]] = 0 \tag{2.2}$$

If

$$(z_0 - y_0)^2 - (z_3 - y_3)^2 < 0, \qquad (z_0 - x_0)^2 - (z_3 - x_3)^2 < 0, \tag{2.3}$$

then in accordance with LCC from Jacobi identity it follows that

$$[\varphi(z), [\varphi(x), \varphi(y)]] = 0. \tag{2.4}$$

The conditions (2.3) are fulfilled if

$$x_3 = \lambda + x'_3, \quad y_3 = \lambda + y'_3, \quad x'_3,\ y'_3 \quad \text{are arbitrary}, \quad \lambda = (0, 0, \lambda_3, 0) \quad \lambda^2 \to -\infty.$$

So, $A(x_3 - y_3, X, Y)$, which we have in (3.1), is:

$$A(x_3 - y_3, X, Y) = A(x'_3 - y'_3, X, Y).$$

In accordance with eq. (2.4):

$$[\varphi(z), [\varphi(x), \varphi(y)]] = [\varphi(z), A(x_3 - y_3, X, Y)] = [\varphi(z), A(x'_3 - y'_3, X, Y)] = 0, \tag{2.5}$$

where z, x' and y' are arbitrary. So we see that $A(x'_3 - y'_3, X, Y)]$ commutes with $\varphi(z)$ at arbitrary z.

Owing to irreducibility of the set of quantum field operators, $[\varphi(x'), \varphi(y')] = C\mathbb{I}$, where C is some function.

Thus we have proved that commutator $[\varphi(x'), \varphi(y')]$ has to be a function. It is known that in this case any Wightman function $\langle \psi_0, \varphi(x_1), \ldots \varphi(x_n)\, \psi_0 \rangle$ has to be some superposition of two-point Wightman functions or one-point ones and so in this case the set of Wightman functions cannot define any nontrivial theory.

Let us stress that our result is valid in a space of arbitrary dimensions.

PoS(QFTHEP2010)042

3. Generalization of the Theorem 2

Theorem 3. *If*

$$[\varphi(x), \varphi(y)] = A(x_0 - y_0 + \alpha(x_3 - y_3), X, Y), \quad \alpha \in \mathbb{R} \quad \textit{is arbitrary}, \tag{3.1}$$

where we denote all other variables as X, Y, *then*

$$A(x_0 - y_0 + \alpha(x_3 - y_3), X, Y) = C\mathbb{I},$$

C is some function.

Proof If $|\alpha| \geq 1$, then from LCC it follows that $[\varphi(x_\alpha), \varphi(y_\alpha)] = 0$, $x_\alpha = x_0 + \alpha\vec{x}$, $y_\alpha = y_0 + \alpha\vec{y}$. Thus we can use the previous proof.

If $|\alpha| < 1$, we have to use analytical properties of Wightman functions. Let us recall that Wightman functions are analytical functions in tubes $T_n^- : \nu \in T_n^-$ if $\nu = \xi - i\eta$, ξ is arbitrary, $\eta \in V^+$, which means that $\eta_0^2 - \vec{\eta}^2 > 0$ [1], [5], [6].

Now let us point out that if $x_j \sim x_{j+1}$, that is if $(x_j - x_{j+1})^2 < 0$, then

$$W(x_1, \ldots x_j, x_{j+1}, \ldots x_n) = W(x_1, \ldots x_{j+1}, x_j, \ldots x_n). \tag{3.2}$$

Taking into account that the condition $x_j \sim x_{j+1}$ exists in some vicinity of points x_j, x_{j+1}, we obtain that

$$W(\nu_1, \ldots \nu_j, \nu_{j+1}, \ldots \nu_n) = W(\nu_1, \ldots \nu_{j+1}, \nu_j, \ldots \nu_n) \quad \text{if} \quad \nu_j \sim \nu_{j+1}. \tag{3.3}$$

Let us consider points $\xi_j = 0$, $\xi_{j+1} = 0$. Then $(\nu_i - \nu_{i+1})^2 = -\eta_0^2 - \vec{\eta}^2 < 0$ as $\eta \in V^+$. It is evident that also $\eta_i^\alpha \sim \eta_{i+1}^\alpha$, where $\eta_{i,i+1}^\alpha = i(\eta_0, \alpha\vec{\eta})$. Now we can use the previous proof, only considering Jacobi identity in imaginary points.

Now let us proof that irreducibility of the set of operators $\varphi(x)$ implies that the set $\varphi(ix)$ is irreducible as well.

Indeed, we have one-to-one correspondence between the set of operators $\varphi(x)$ and $\varphi(ix)$.

Let bounded operator A commutes with $\varphi(x)$. We show that also $[A\varphi(ix)] = 0 \forall x$. In fact, if for some x there exists vector Φ such that

$$(A\varphi(ix) - \varphi(ix)A)\,\Phi = \Psi \neq 0,$$

then using map $T\varphi(ix) = \varphi(x)$, we obtain that $T\Psi = 0, \Psi \neq 0$. In accordance with one-to-one correspondence between the sets $\varphi(x)$ and $\varphi(ix)$ only $T\,0 = 0$. We come to the contradiction. Thus $[A\varphi(ix)] = 0 \forall x$. As the set of operators $\varphi(x)$ is irreducible $A = C\mathbb{I}, C \in \mathbb{C}$.

It is easy to show that as $[\varphi(ix), \varphi(iy)] = C\mathbb{I}$, $\forall x, y$, C is some function, then $W(ix_1, \ldots ix_n)$ is some linear combination of two- or one-point Wightman functions in imaginary points. Using the analytical properties of Wightman functions, we come to the conclusion that $W(x_1, \ldots x_n)$ is also a linear combination of two- or one-point Wightman functions. Thus the field in question can be only a trivial one. The proof is completed.

If we consider space-space NC QFT, the same result is valid in respect with commutative coordinates.

PoS(QFTHEP2010)042

4. Conclusion

It has been proved that commutator $[\varphi(x),\varphi(y)]$ cannot be a function of linear combination $(x_0 - y_0 + \alpha(x_3 - y_3), X, Y)$. This result is valid in space-space noncommutative quantum field theory as well.

References

[1] N. N. Bogoliubov, A. A. Logunov and I. T. Todorov, *Introduction to Axiomatic Quantum Field Theory*, Benjamin, Reading, Mass 1975.

[2] M. R. Douglas and N. A. Nekrasov, *Noncommutative field theory*, *Rev. Mod. Phys.*, **73** 977 (2001), [hep-th/0106048].

[3] R. J. Szabo, *Quantum field theory on noncommutative spaces*, *Phys. Rept.*, **378** 207 (2003), [hep-th/0109162].

[4] M. Chaichian, M. N. Mnatsakanova, A. Tureanu and Yu. S. Vernov, *Test Functions Space in Noncommutative Quantum Field Theory*, *JHEP* **09** (2008) 125, [hep-th/0706.1712v1].

[5] R. F. Streater and A. S. Wightman, *PCT, Spin and Statistics and All That*, Benjamin, NewYork 1964.

[6] R. Jost, *The general theory of quantized fields*, A.M.S., Providence, RI 1965.

PoS(QFTHEP2010)042