Analytical vectors and new criterion of regularity for representation of canonical commutation relations algebra

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New criterion of regularity for representation of canonical commutation relations algebra is given on the basis of concept of an analytical vector.

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1. Introduction

The algebra of canonical commutation relations (CCR) is a quantum mechanics core, and its regular representations play a dominant role in a science (various definitions of regularity for representation of CCR algebras will be viewed further). The review of CCR is given in a paper [1]. In the present article we make new definition of a regularity for representation of CCR algebras, using concept of an analytical vector, introdused in [2].

In the most simple case of one dimension, CCR are defined as follows:

$$[\hat{p}, \hat{q}] = -i\hat{I}, \tag{1.1}$$

where $\hat{p}$ and $\hat{q}$ are self-adjoint operators (in a quantum mechanics they are impulse and coordinate operators accordingly).

As is known in a case of finite number of operators, i.e. in a case

$$[\hat{p}_i, \hat{q}_k] = -i\delta_{ik}; \qquad 1 \le i, k \le n.$$

all conclusions are similar to the results in a case of two operators $\hat{p}$ and $\hat{q}$. Therefore we consider only the case when equality (1.1) is fulfilled. We note that if there is infinite number of operators (the quantum field theory case), the situation is more complicated and its viewing goes out for a framework of the present article.

The Schrodinger representation is the most known of CCR representations. It is realised in space $L_2(-\infty, +\infty)$, where functions $f(x)$ are such that $\int_{-\infty}^{+\infty} |f(x)|^2 dx < \infty$. Operators $\hat{p}$ and $\hat{q}$ in the given representation are defined as follows:

$$\hat{q}f(q) = qf(q), \qquad \hat{p}f(q) = -i\frac{d}{dq}f(q). \tag{1.2}$$

The following definition of regularity for representation of CCR algebras is widely used:

Definition 1. *Any representation of CCR algebras which is unitary equivalent to Schrodinger one is regular.*

There is one important circumstance: CCR algebras cannot be realised by bounded operators [1], at least one of operators $\hat{p}$ or $\hat{q}$ must be unbounded. We remind that in the closed space unbounded operators are defined in a dense domain.

In most papers CCR were investigated in Hilbert space. However, it is possible to study CCR in spaces, which have an indefinite metric [3]. Let's note that covariant gauge field theory demands transition from Hilbert space to a space with an indefinite metric [4],[5].

The Rellich-Dixmier's theorem is very important for the description of the regular representations of CCR algebra. It shows that representations of CCR algebra are regular for very wide class of operators (see [1]).

Theorem 1. Rellich-Dixmier's theorem. *Operators $\hat{q}$ and $\hat{p}$ form regular representation of CCR algebras if:*
1. there exist dense domain $D \in D_q \bigcap D_p$ invariant under the action of $\hat{q}$ and $\hat{p}$ such that CCR hold on D;
2. the operator $(\hat{q}^2 + \hat{p}^2)$ is essentially self-adjoint on D.

Let's note that Fuglede has constructed an example of the irregular representation when only requirement 1 is fulfilled [6].

The representation of CCR algebras, realized by operators (1.2) in space $L_2(a,b)$, is an example of the irregular representation.

It is possible to define CCR in the following form:

$$[\hat{a}, \hat{a}^*] = \hat{I}, \tag{1.3}$$

where operator $\hat{a}$ and adjoint operator $\hat{a}^*$ are defined as

$$\hat{a} = \frac{1}{\sqrt{2}}(\hat{q} - i\hat{p}) \quad \text{and} \quad \hat{a}^* = \frac{1}{\sqrt{2}}(\hat{q} + i\hat{p}). \tag{1.4}$$

It is easy to check that in Hilbert space the spectrum of the operator $\hat{N} = \hat{a}^*\hat{a}$ is $Sp\hat{N} = \mathbb{N}$ in the regular representation.

The existence of a vacuum vector is a key feature of the regular representations of CCR algebra in Hilbert space. These representations are known as Fock representations. It is obvious that, cause of (1.1) and (1.4), a vector $\psi_0 = Ce^{-\frac{q^2}{2}}$ satisfies the requirement: $\hat{a}\psi_0 = 0$, and hence $\hat{N}\psi_0 = 0$. Though all definitions of regular representations are equivalent, some representations are more convenient for research of CCR in spaces which differ from Hilbert space. The Krein space is an example of such space which has an indefinite metric [7], [8]. For example, the requirement of existence of an eigenvector for the operator $\hat{N}$:

$$\hat{N}\psi_\alpha = \alpha\psi_\alpha \tag{1.5}$$

is one of definitions of regularity of representations in Krein space [3].

In view of that operators $\hat{p}$ and $\hat{q}$, and naturally $\hat{a}$ and $\hat{a}^*$, are unbounded there are some difficulties related to definition of domains, in which they can be determined. Use of the representation of CCR in a Weyl form eliminates this difficulty:

$$e^{it\hat{p}}e^{is\hat{q}} = e^{ist}e^{is\hat{q}}e^{it\hat{p}} \tag{1.6}$$

It is well-known from the Stone's theorem that operators $e^{it\hat{p}}$ and $e^{is\hat{q}}$ are bounded as operators $\hat{p}$ and $\hat{q}$ are self-adjoint [9].

CCR in a Weyl form are widely used in a quantum mechanics (see, for example, [10]). Until now CCR in this form were considered in Hilbert space, but it is natural to study a problem of existence of a Weyl representation in a space with an indefinite metric.

We hope what for these purposes, and probably more, a new definition of regularity for representation of CCR algebra will be very useful.

2. Analytical vectors and their connection with CCR representation in a Weyl form

Let's remember a definition of an analytical vector [2].

Definition 2. *Let $\hat{A}$ be a linear operator on Hilbert space H. A vector $\xi \in H$ is called analytic for $\hat{A}$, if ξ is in the domain of $\hat{A}^k$ for every $k \in \mathbb{N}$ and for every $t > 0$*

$$\sum_{k=0}^{\infty} \frac{t^k}{k!}\|\hat{A}^k\xi\| < +\infty \tag{2.1}$$

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In this case we can define the operator $\exp(t\hat{A})$ as its Taylor series

$$e^{t\hat{A}}\xi = \sum_{k=0}^{\infty} \frac{t^k}{k!} \hat{A}^k \xi \tag{2.2}$$

for all ξ at which our series (2.2) converges.

So, now we can formulate the main theorem.

Theorem 2. *Let's prove that a representation of CCR algebras is regular, if there is a dense domain D, in witch any vector $\xi \in D$ obeys conditions*

$$\sum_{k=0}^{\infty} \frac{t^k}{k!} \|\hat{q}^k \xi\| < \infty, \quad \forall t > 0; \quad \sum_{k=0}^{\infty} \frac{s^k}{k!} \|\hat{p}^k \xi\| < \infty, \quad \forall s > 0 \tag{2.3}$$

and also any regular representation obeys conditions (2.3).

At first we will prove what this representation is regular. Our proof is similar to the proof in [1], but without the assumption of boundedness of operators $\hat{p}$ and $\hat{q}$. From the relation (1.1) it immediately follows that

$$\hat{p}\hat{q}^n - \hat{q}^n\hat{p} = -i(\hat{q}^n)', \quad ('=d/dq). \tag{2.4}$$

Hence, with (2.4) and (2.3),

$$\hat{p}e^{is\hat{q}} - e^{it\hat{p}}\hat{p} = -i(e^{it\hat{q}})'. \tag{2.5}$$

From (2.5) it directly follows

$$e^{-it\hat{q}}\hat{p}e^{it\hat{q}} = (\hat{p}+t\hat{I})$$

and,

$$e^{-it\hat{q}}\hat{p}^n e^{it\hat{q}} = (\hat{p}+t\hat{I})^n. \tag{2.6}$$

With condition (2.3) we have what

$$U_t V_s = e^{its} V_s U_t, \qquad U_t \equiv e^{it\hat{p}}, \ V_s \equiv e^{is\hat{q}}. \tag{2.7}$$

Thereby we have proved an existence of a Weyl relation (2.7) in domain D.

The next step is a Weyl representations extension on a full space H. For this purpose it is enough to note that D is a dense domain, and U_t and V_s are bounded operators that follows from the Stone's theorem. We will note that the Stone's theorem make some requirements on groups U_t and V_s, but they are weak ([11]).

Now we will prove that if the relation (2.7) is fulfilled in H then our representation is regular. But that part of task is made for us by the von Neumann's theorem (see [1]). We will note, as any regular representation is the direct sum of irreducible representations, it is enough to view only an irreducible representation.

Now we will show that any regular representation contains analytical vectors in dense domain and, hence, satisfies a Weyl relation (2.7) in H. For this purpose it is convenient to use a CCR relation in the form (1.3). In this case:

$$\hat{q} = \frac{1}{\sqrt{2}}(\hat{a}+\hat{a}^*), \qquad \hat{p} = \frac{1}{i\sqrt{2}}(\hat{a}-\hat{a}^*). \tag{2.8}$$

Let us show that the operator $\hat{q}$, for example, has an analytical vector in dense domain D. In the regular representation we can construct an orthogonal basis, which consists of eigenvectors ψ_n of the operator $\hat{N} = \hat{a}^* \hat{a}$. It is obviously from (1.3) and (1.5) that $Sp\hat{N} = \mathbb{N}$. It is easy to show that:

$$(\psi_n, \psi_n) = n!, \text{ where } \psi_n = (\hat{a}^*)^n \psi_0, \ \psi_0 \text{ is a vacuum vector}, \ (\psi_0, \psi_0) = 1. \tag{2.9}$$

and

$$(\hat{a}^* \psi_n, \hat{a}^* \psi_n) = (n+1)!, \quad (\hat{a} \psi_n, \hat{a} \psi_n) = n!. \tag{2.10}$$

Let us view a domain D, which consists of all finite linear combinations of vectors ψ_n. As H consists of all finite or converging linear combinations of vectors ψ_n, then D is a dense domain. The norm of a vector ψ_n can be set by the formula $\|\psi_n\| = \sqrt{(\psi_n, \psi_n)}$. According to (2.9)

$$\|\psi_n\| = \sqrt{n!}. \tag{2.11}$$

Accordingly, for a vector

$$\psi = \sum_{m}^{m+n} C_k \psi_k \tag{2.12}$$

from (2.9) - (2.11) we have the following restriction:

$$\|\hat{q}\psi\| \leq \sum_{m}^{m+n} \|C_k \hat{q} \psi_k\| \leq \sqrt{2}C \sqrt{(m+n+1)!},$$
$$C = \max |C_k|, \quad m \leq k \leq m+n. \tag{2.13}$$

For obtaining (2.13) we have used that

$$(\hat{a}^* \psi_n, \hat{a}^* \psi_n) = (\psi_n, (n+1) \psi_n) = (n+1)(\psi_n, \psi_n),$$
$$(\hat{a}\psi_n, \hat{a}\psi_n) = (\psi_n, n\psi_n) = n(\psi_n, \psi_n).$$

With (2.13) at the end we have

$$\sum_{k=0}^{\infty} \frac{t^k}{k!} \|\hat{q}^k \psi\| \leq \sum_{k=0}^{\infty} \frac{t^k}{k!} C^k (2(m+n+1)!)^{k/2}. \tag{2.14}$$

It is obvious that the series (2.14) converges for any finite m and n.

Thus, it is proved that the vector $\psi, \forall \psi \in D$ is analytical for the operator $\hat{q}$. The proof of that fact that any vector $\psi \in D$ is analytical for the operator $\hat{p}$ can be made by a similar way.

3. Conclusion

In the present article a new criterion of regularity for representation of canonical commutation relations algebras is given on the basis of concept of an analytical vector. We hope that new definition will be useful for study a Weyl representation in the indefinite metrics space.

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